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BIFLATNESS AND BIPROJECTIVITY OF BANACH ALGEBRAS GRADED OVER A SEMILATTICE

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Abstract. We give sufficient conditions and necessary conditions for a Banach algebra, which is ℓ_1-graded over a semi-lattice, to be biflat or biprojective. As an application we characterise biflat and biprojective discrete convolution algebras for commutative semi-groups.

1. Introduction. The concepts *biprojective* and *biflat*, to be defined in the next section, are instances of homological triviality. One of the basic issues of (Banach-algebraic) homology is to measure obstructions to lifting and extension problems. Let $\mathfrak{A}$ be a Banach algebra. An injection, respectively surjection,

$$0 \to X \to Y \text{ or } Y \to Z \to 0$$

of Banach $\mathfrak{A}-$bimodules X, Y, Z is *admissible* if it splits as Banach spaces. Biprojectivity of $\mathfrak{A}$ is the property that all lifting problems

$$\begin{tikzcd} & \mathfrak{A} \arrow[dl, dashed] \arrow[d] & \\ Y \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

can be solved when $Y \to Z \to 0$ is admissible. Biflatness is the property that all lifting problems

$$\begin{tikzcd} & \mathfrak{A} \arrow[dl, dashed] \arrow[d] & \\ Y^* \arrow[r] & X^* \arrow[r] & 0 \end{tikzcd}$$

can be solved when $0 \to X \to Y$ is admissible. The most important instance of homological triviality is that of *amenability*, the concept introduced in [**11**]. Recall that $\mathfrak{A}$ is amenable, if and only if $\mathfrak{A}$ is biflat and has a bounded approximate identity (see [**9**]).

These notions of homological triviality have been explored in various classes of Banach algebras. At focus in this paper is the class of discrete convolution algebras. Let S be a semi-group. The discrete convolution algebra $\ell^1(S)$ is the Banach algebra with the universal property that for each Banach algebra $\mathfrak{A}$ and each bounded multiplicative map $\phi\colon S \to \mathfrak{A}$ there is a unique Banach algebra homomorphism $\widetilde{\phi}\colon \ell^1(S) \to \mathfrak{A}$ completing the diagram

$$\begin{tikzcd} \ell^1(S) \arrow[dr, "\widetilde{\phi}"] & \\ S \arrow[u, "\iota"] \arrow[r, "\phi"] & \mathfrak{A} \end{tikzcd}$$

where $\iota\colon S \to \ell_1(S)$ is the canonical embedding of S into the set of point masses in $\ell_1(S)$. Inasmuch as the unit ball in a Banach algebra is a semi-group w.r.t the algebra multiplication, discrete convolution algebras is perhaps the most basic class of Banach algebras to be considered in the endeavour to understand general Banach algebra properties. With this perspective it is also important that results about algebras $\ell^1(S)$ are obtained with a minimal appeal to specific semi-group properties. In a recent treatise ([**4**]), a rather encompassing account of Banach algebraic properties of semi-group algebras is given, in particular, the authors conclude the description of amenability in terms of algebraic properties of the semi-group ([**4**, Theorem 10.12]).

Recently, the other notions of homological triviality, biprojectivity and biflatness, have been investigated. In [**1**] Choi characterises biflatness of $\ell_1(S)$ when S is a Clifford semi-group. In particular when S is a semi-lattice he shows that $\ell_1(S)$ is biflat if and only if it is biprojective, if and only if $\sup\{\#(sS) \mid s \in S\} < +\infty$ (*uniform local finiteness*), where # denotes cardinality.

In this paper, we investigate biflatness and biprojectivity of Banach algebras which are ℓ_1-graded over a semi-lattice. Such algebras, with a slightly more restrictive notion than ours, were introduced in [**2**] as a Banach-algebraic version of strong semi-lattice diagrams of semi-groups, cf. [**10**, Cpt. IV]. They form a framework incorporating many examples of semi-group algebras, notably Clifford semi-group algebras. Finite semi-lattice graded algebras have also been studied by Ghandehari et al. in their work on amenability constants ([**7**]). Our main result characterises biflatness and biprojectivity of certain semi-lattice ℓ_1-graded Banach algebras in terms of the constituents (Theorems 4.4 and 4.6). Our techniques require a condition that facilitates passing from the constituents to the full ℓ_1-graded algebra. With this condition biflatness and biprojectivity can be viewed as local amenability respectively local contractibility. The results of [**7**] on amenability constants for finite semi-lattices of Banach algebras are instrumental in this.

As an application, we prove that if S is commutative, then $\ell_1(S)$ is biflat if and only S is a Clifford semi-group on a uniformly locally finite semi-lattice, and biprojective if and only if in addition each (maximal) subgroup of S is finite.

Note: After this work was completed the paper [**13**] has come to our knowledge. In this Ramsden establishes the above mentioned uniform local finiteness as a general necessary condition for biflatness of discrete convolution algebras and gives a complete characterisation of biflatness in the case of inverse semi-groups.

2. Preliminaries. In this section, we establish notation and define basic concepts. For Banach spaces X and a subset $M \subseteq X$ the closed linear span of M is $\mathrm{cl}(M)$.

For a Banach space Y, the Banach space of bounded operators $X \to Y$ is $\mathcal{B}(X, Y)$ with the uniform norm. We use X^* for $\mathcal{B}(X, \mathbb{C})$, the dual space, and use $\langle x, x^* \rangle$, $x \in X, x^* \in X^*$ to denote the duality.

A (closed) subspace $E \subseteq X$ is *weakly complemented* if $E^\perp = \{x^* \in X^* \mid \langle e, x^* \rangle = 0 \, \forall e \in E\}$ is complemented in X^*. This is equivalent to the existence of $r \in \mathcal{B}(E^*, X^*)$ such that $\iota^* \circ r = \mathrm{id}_{E^*}$, where $\iota\colon E \to X$ is the inclusion.

The projective tensor product is denoted $X \widehat{\otimes} Y$ and there are isometric identifications $(X \widehat{\otimes} Y)^* \cong \mathcal{B}(X, Y^*) \cong \mathcal{B}^2(X, Y)$, the latter being the bounded bilinear forms on $X \times Y$. According to this identification, we write

$$\langle x \otimes y, \Phi \rangle = \langle y, \Phi(x) \rangle = \Phi(x, y), \quad x \in X, y \in Y$$

for $\Phi \in (X \widehat{\otimes} Y)^*$.

If $(f_\gamma)_\Gamma$ is a bounded net in a dual Banach space it has a w*-cluster point. To avoid tedious repetitions of the phrase 'by passing to a subnet, if necessary, we may assume that $(f_\gamma)_\Gamma$ is w*-convergent', we shall make the tacit assumption that all bounded nets have been chosen to be w*-convergent, and thus without further comment write $\text{w*-}\lim_\gamma f_\gamma$. For example, for a bounded net $\phi_\gamma \in \mathcal{B}(X, Y^*)$ we have

$$\Big(\text{w*-}\lim_\gamma \phi_\gamma\Big)(x) = \text{w*-}\lim_\gamma(\phi_\gamma(x)), \quad x \in X$$

where $\text{w*-}\lim_\gamma$ on the left refers to the $X \widehat{\otimes} Y$-topology on $\mathcal{B}(X, Y^*)$ and on the right to the Y-topology on Y^*. Thus we may without ambiguity write $\text{w*-}\lim_\gamma \phi_\gamma(x)$. This will also be used without further comment.

We shall use κ for the canonical embedding $X \to X^{**}$ given by $\langle x^*, \kappa(x) \rangle = \langle x, x^* \rangle$, $x \in X, x^* \in X^*$ and, if necessary for emphasis, κ_X.

For a Banach algebra $\mathfrak{A}$ we denote the category of left Banach $\mathfrak{A}$-modules and bounded module homomorphisms by $\mathfrak{A}\mathbf{mod}$. The Banach space of bounded left module homomorphisms $N \to M$, $N, M \in \mathfrak{A}\mathbf{mod}$ is ${}_{\mathfrak{A}}\mathbf{h}(N, M)$. The corresponding right and bi- module versions are $\mathbf{mod}\mathfrak{A}$, $\mathbf{h}_{\mathfrak{A}}(N, M)$ and $\mathfrak{A}\mathbf{mod}\mathfrak{A}$, ${}_{\mathfrak{A}}\mathbf{h}_{\mathfrak{A}}(N, M)$, respectively. If $X \in \mathfrak{A}\mathbf{mod}$ the dual action of $\mathfrak{A}$ gives $X^* \in \mathbf{mod}\mathfrak{A}$ and similarly for right and bi-modules.

The multiplication on a Banach algebra $\mathfrak{A}$ is denoted $\Pi\colon \mathfrak{A} \widehat{\otimes} \mathfrak{A} \to \mathfrak{A}$ or, if needed for emphasis, $\Pi_{\mathfrak{A}}$.

We recall the basic homological concepts and facts needed for the paper. For details we refer to **[9]**.

Definition 2.1. A Banach $\mathfrak{A}$ algebra is *biprojective* if $\Pi_{\mathfrak{A}}$ is a retraction in $\mathfrak{A}\mathbf{mod}\mathfrak{A}$, i.e. if there is $\rho \in {}_{\mathfrak{A}}\mathbf{h}_{\mathfrak{A}}(\mathfrak{A}, \mathfrak{A} \widehat{\otimes} \mathfrak{A})$ such that $\Pi_{\mathfrak{A}} \circ \rho = \mathrm{id}_{\mathfrak{A}}$. Such a map will be termed a *splitting of the multiplication* on $\mathfrak{A}$.

A Banach $\mathfrak{A}$ algebra is *biflat* if $\Pi_{\mathfrak{A}}^*$ is a coretraction in $\mathfrak{A}\mathbf{mod}\mathfrak{A}$, i.e. if there is $\rho \in {}_{\mathfrak{A}}\mathbf{h}_{\mathfrak{A}}((\mathfrak{A} \widehat{\otimes} \mathfrak{A})^*, \mathfrak{A}^*)$ such that $\rho \circ \Pi_{\mathfrak{A}}^* = \mathrm{id}_{\mathfrak{A}^*}$. This is equivalent to the existence of $\rho \in {}_{\mathfrak{A}}\mathbf{h}_{\mathfrak{A}}(\mathfrak{A}, (\mathfrak{A} \widehat{\otimes} \mathfrak{A})^{**})$ such that $\Pi_{\mathfrak{A}}^{**} \circ \rho = \kappa_{\mathfrak{A}}$. In either version we shall refer to ρ as a *weak splitting of the multiplication* on $\mathfrak{A}$.

For $X \in \mathfrak{A}\mathbf{mod}\mathfrak{A}$ the closed subspace of $\mathcal{B}(\mathfrak{A}, X)$ consisting of module derivations is denoted $\mathcal{Z}^1(\mathfrak{A}, X)$. The *Hochschild coboundary map* $\delta\colon X \to \mathcal{Z}^1(\mathfrak{A}, X)$ is $x \mapsto (\delta_x\colon a \mapsto a.x - x.a)$. The image of δ is precisely the subspace of inner derivations.

The Banach algebra $\mathfrak{A}$ is *amenable* if $\delta(X^*) = \mathcal{Z}^1(\mathfrak{A}, X^*)$ for all $X \in \mathfrak{A}\mathbf{mod}\mathfrak{A}$. This is equivalent to '$\mathfrak{A}$ has a *virtual diagonal*', i.e. an element $\Delta \in (\mathfrak{A} \widehat{\otimes} \mathfrak{A})^{**}$ such that

$$a.\Delta - \Delta.a = 0, \; a.\Pi^{**}(\Delta) = \kappa(a), \quad a \in \mathfrak{A}.$$

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It is also equivalent to '$\mathfrak{A}$ is biflat and has a bounded approximate identity'.

These concepts will in particular be studied in the context of semi-group algebras. We recall the well-known definition.

DEFINITION 2.2. Let S be a semi-group. We consider elements of the Banach space $\ell^1(S)$ as formal power series with exponents in S

$$\ell^1(S) = \left\{ \sum_{s\in S} a_s X^s \mid \sum_{s\in S} |a_s| < \infty \right\},$$

and define multiplication as power series multiplication

$$\sum_{s\in S} a_s X^s \cdot \sum_{s\in S} b_s X^s = \sum_{s\in S} \left(\sum_{ut=s} a_u b_t \right) X^s.$$

Very often the semi-group structure gives rise to a grading of the semi-group algebra. Such a grading is of some interest in itself and has been considered by several, see for instance Y. Choi [**2**] and Ghandehari et al. [**7**]. Recall that a *semi-lattice* is an abelian semi-group in which each element is idempotent.

DEFINITION 2.3. Let $\mathfrak{A}$ be a Banach algebra and assume that $\mathfrak{A}$ as a Banach space is an ℓ^1-direct sum of Banach subalgebras $\mathfrak{A} = \bigoplus_{\alpha\in\Lambda} A_\alpha$. If

$$\forall \alpha, \beta \in \Lambda \exists \nu \in \Lambda \colon A_\alpha A_\beta + A_\beta A_\alpha \subseteq A_\nu,$$

then obviously for given $\alpha, \beta \in \Lambda$ the corresponding ν is uniquely determined. It follows that the mapping $(\alpha, \beta) \mapsto \nu$ defines a semi-lattice multiplication on Λ. In this situation we say that $\mathfrak{A}$ is *ℓ^1-graded over the semi-lattice* Λ.

The grading thus defined has a *universal property*.

PROPOSITION 2.4. *Let $\mathfrak{A} = \bigoplus_{\alpha\in\Lambda} A_\alpha$ be an ℓ^1-graded Banach algebra, and let $\iota_\alpha \colon A_\alpha \to \mathfrak{A}$, $\alpha \in \Lambda$ be the natural inclusions of the subalgebras A_α. Let $\mathfrak{B}$ be a Banach algebra. For each uniformly bounded family of Banach algebra homomorphisms $\varphi_\alpha \colon A_\alpha \to \mathfrak{B}$ such that*

$$\varphi_\alpha(a)\varphi_\beta(b) = \varphi_{\alpha\beta}(ab), \quad a \in A_\alpha, b \in A_\beta,\ \alpha, \beta \in \Lambda \tag{†}$$

there is a unique Banach algebra homomorphism $\Phi \colon \mathfrak{A} \to \mathfrak{B}$ such that $\Phi \circ \iota_\alpha = \varphi_\alpha$ for all $\alpha \in \Lambda$.

Proof. Existence and uniqueness as a bounded linear map satisfying $\Phi \circ \iota_\alpha = \varphi_\alpha$ for all $\alpha \in \Lambda$ follow from the universal property of ℓ^1- direct sums, and (†) ensures that Φ is a homomorphism. □

As indicated, our main examples of ℓ^1-graded Banach algebras come from semi-group algebras. Other examples are

EXAMPLE 2.5. Let $\mathfrak{A}$ be a Banach algebra and let Λ be a family of closed 2-sided ideals of $\mathfrak{A}$. If Λ has the property $I, J \in \Lambda \implies I \cap J \in \Lambda$, then the ℓ^1-sum $\bigoplus_{I\in\Lambda} I$ is naturally an ℓ^1-graded Banach algebra.

If Λ has the property $I, J \in \Lambda \implies \mathrm{cl}(I+J) \in \Lambda$, then the ℓ^1-sum $\bigoplus_{I\in\Lambda} \mathfrak{A}/I$ is an ℓ^1-graded Banach algebra, when the multiplication is given by $(a+I)(b+J) = ab + \mathrm{cl}(I+J),\ a, b \in \mathfrak{A},\ I, J \in \Lambda$.

3. Hereditary properties. In this section we examine hereditary properties of biprojectivity and biflatness.

In order to synthesise biflatness of a given algebra from that of its parts it is necessary to have control of norms of weak splittings of multiplication. For this we make the following definitions.

DEFINITION 3.1. Let $\mathfrak{A}$ be a Banach algebra. The *amenability constant* $\mathrm{ac}(\mathfrak{A})$ for $\mathfrak{A}$ is

$$\inf\{\|M\| \mid M \text{ is a virtual diagonal for } \mathfrak{A}\}.$$

The *biflatness constant* $\mathrm{bc}(\mathfrak{A})$ is

$$\inf\{\|\rho\| \mid \rho\colon (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^* \to \mathfrak{A}^* \text{ is a weak splitting of multiplication on } \mathfrak{A}\}.$$

The *generator constant* $\mathrm{gc}(\mathfrak{A})$ is

$$\sup \mathrm{inv}(\delta(X^*)),$$

where the supremum is with respect to all contractive $X \in \mathfrak{A}\mathbf{mod}\mathfrak{A}$. Here $\mathrm{inv}(\delta(X^*))$ is the *inversion constant* of the Hochschild coboundary map $\delta\colon X^* \to \mathcal{Z}^1(\mathfrak{A}, X^*)$. i.e. the norm of the inverse of the the map $X^*\big/\ker\delta \to \mathcal{Z}^1(\mathfrak{A}, X^*)$ (or $+\infty$ if δ is not surjective).

In other words, the *generator constant* is the infimum of numbers $C > 0$ so that for all $X \in \mathfrak{A}\mathbf{mod}\mathfrak{A}$ with $\|a.x\|, \|x.a\| \le \|a\|\|x\|,\ a \in \mathfrak{A}, x \in X$ and all derivations $D\colon \mathfrak{A} \to X^*$ there is $x^* \in X^*$ with $\|x^*\| \le C\|D\|$ and $D(a) = a.x^* - x^*.a,\ a \in \mathfrak{A}$.

We have adopted the usual convention that $\inf\emptyset = +\infty$, so that $\mathrm{ac}(\mathfrak{A}) = +\infty$ means that $\mathfrak{A}$ is not amenable etc.

We have the following relations between these numbers.

LEMMA 3.2. *Let $\mathfrak{A}$ be a Banach algebra with an approximate identity of bound $\beta \in [1, +\infty]$ (with $\beta = +\infty$ meaning that $\mathfrak{A}$ does not have a bounded approximate identity). Then*

$\mathrm{bc}(\mathfrak{A}) \le \mathrm{ac}(\mathfrak{A}) \le \beta\,\mathrm{bc}(\mathfrak{A}),\ \ \mathrm{gc}(\mathfrak{A}) \le \mathrm{ac}(\mathfrak{A}) + (1 + 2\,\mathrm{ac}(\mathfrak{A}))(\beta^2 + 2\beta),$ *and*
$\mathrm{bc}(\mathfrak{A}) \le 1 + 2\,\mathrm{gc}(\mathfrak{A})$.

Proof. Let $\rho\colon \mathfrak{A} \to (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$ be a weak splitting of multiplication and let e_λ be a bounded approximate identity for $\mathfrak{A}$. Any w*-cluster point of the net $\rho(e_\lambda)$ will be a virtual diagonal for $\mathfrak{A}$, so $\mathrm{ac}(\mathfrak{A}) \le \beta\,\mathrm{bc}(\mathfrak{A})$. Conversely, if Δ is a virtual diagonal for $\mathfrak{A}$, then $a \mapsto a.\Delta$ is a weak splitting of multiplication, so that $\mathrm{bc}(\mathfrak{A}) \le \mathrm{ac}(\mathfrak{A})$. If X is neo-unital and $D\colon \mathfrak{A} \to X^*$ is a derivation, then D is generated by the functional

$$x \mapsto \langle (a, b) \mapsto \langle x.a, D(b)\rangle, \Delta\rangle$$

(cf. **[12]**), so that we have a generator of norm not exceeding $\mathrm{ac}(\mathfrak{A})\|D\|$. In general, by looking at successive restrictions of $D(a) \in X^*,\ a \in \mathfrak{A}$ to the modules $\mathfrak{A}X\mathfrak{A} \subseteq \mathfrak{A}X \subseteq X$ as in Proposition 1.8 of **[11]** we get $\mathrm{gc}(\mathfrak{A}) \le \mathrm{ac}(\mathfrak{A}) + (1 + 2\,\mathrm{ac}(\mathfrak{A}))(\beta^2 + 2\beta)$.

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Finally, consider the derivation $D(a) = a \otimes \mathbf{1} - \mathbf{1} \otimes a : \mathfrak{A} \to \ker \Pi$ and let $\Lambda \in \ker \Pi^{**}$ be a generator. Then $\rho(a) = a.(\mathbf{1} \otimes \mathbf{1} - \Lambda)$, $a \in \mathfrak{A}$ defines a w*-splitting of the multiplication. Since $\|D\| = 2$, we get $\mathrm{bc}(\mathfrak{A}) \leq 1 + 2\,\mathrm{gc}(\mathfrak{A})$. □

We now consider a short exact sequence

$$0 \to I \xrightarrow{\iota} \mathfrak{A} \xrightarrow{q} \mathfrak{B} \to 0, \tag{E}$$

where I is a closed 2-sided ideal of the Banach algebra $\mathfrak{A}$.

PROPOSITION 3.3. *Assume that* $I = \mathrm{cl}(\mathfrak{A}I + I\mathfrak{A})$. *If* $\mathfrak{A}$ *is biflat (biprojective), then* $\mathfrak{B}$ *is biflat (biprojective). Precisely,* $\mathrm{bc}(\mathfrak{B}) \leq \|q\|\,\mathrm{bc}(\mathfrak{A})$.

Proof. The argument for the biprojective case is given in [**14**]. A similar argument gives the biflat case as follows: Let $\rho : (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^* \to \mathfrak{A}^*$ be a weak splitting of the multiplication on $\mathfrak{A}$. We want to complete the diagram

$$\begin{array}{ccc} (\mathfrak{B}\widehat{\otimes}\mathfrak{B})^* & \xrightarrow{(q\widehat{\otimes}q)^*} & (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^* \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\rho} \\ \mathfrak{B}^* & \xrightarrow{q^*} & \mathfrak{A}^* \end{array}$$

so that $\tau \circ \Pi^*_{\mathfrak{B}} = \mathbf{1}_{\mathfrak{B}^*}$. Let $\phi \in (\mathfrak{B}\widehat{\otimes}\mathfrak{B})^*$ and put $\psi = \phi \circ q\widehat{\otimes}q$. In order to define $\tau(\phi)$ we must show that $\rho(\psi)(I) = \{0\}$. Let $i = \alpha' i' + i''\alpha''$ where $i', i'' \in I$ and $\alpha', \alpha'' \in \mathfrak{A}$. Then

$$\begin{aligned} \rho(\psi)(i) &= \rho(\psi)(\alpha' i' + i''\alpha'') \\ &= \rho(i'.\psi)(\alpha') + \rho(\psi.i'')(\alpha''). \end{aligned}$$

But $i'.\psi(a', a'') = \psi(a', a''i') = \phi(q(a'), q(a''i')) = 0$, $a', a'' \in \mathfrak{A}$ so $i'.\psi = 0$. Similarly $\psi.i'' = 0$. Since $\mathrm{cl}(\mathfrak{A}I + I\mathfrak{A}) = I$ we get $\rho(\psi)(I) = \{0\}$ as desired. Hence there is a map $\tau : (\mathfrak{B}\widehat{\otimes}\mathfrak{B})^* \to \mathfrak{B}^*$ making the diagram commutative. By injectivity of the maps q^*, $(q\widehat{\otimes}q)^*$ and the closed graph theorem τ is a bounded $\mathfrak{B}$-bimodule homomorphism. Finally

$$\begin{aligned} q^* \circ \tau \circ \Pi^*_{\mathfrak{B}} &= \rho \circ (q\widehat{\otimes}q)^* \circ \Pi^*_{\mathfrak{B}} \\ &= \rho \circ \Pi^*_{\mathfrak{A}} \circ q^* \\ &= q^*, \end{aligned}$$

so, since q^* is injective, we get $\tau \circ \Pi^*_{\mathfrak{B}} = \mathbf{1}_{\mathfrak{B}^*}$. □

Amenability is inherited by weakly complemented ideals. The situation is similar for biflatness. In order to deal with this we first need

LEMMA 3.4. *Let* Y *be a Banach space and let* X *be a closed subspace. If* X *is weakly complemented in* Y, *then* $X\widehat{\otimes}X$ *is weakly complemented in* $Y\widehat{\otimes}Y$.

Proof. Let λ be a right inverse to the dual of the inclusion $X \hookrightarrow Y$, i.e. $\langle x, \lambda(x')\rangle = \langle x, x'\rangle$, $\forall x \in X \forall x' \in X^*$. For $F \in (X\widehat{\otimes}X)^*$ define $\Lambda(F) \in (Y\widehat{\otimes}Y)^*$ by

$$\Lambda(F)(y, y') = \langle y', \lambda(\xi \mapsto \langle y, \lambda(F(\cdot, \xi))\rangle)\rangle, \quad y, y' \in Y.$$

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Then

$$\Lambda(F)(x, x') = \langle x', \lambda(\xi \mapsto \langle x, \lambda(F(\cdot, \xi))\rangle)\rangle = \langle x, \lambda(F(\cdot, x'))\rangle = F(x, x')$$

for all $x, x' \in X$. □

PROPOSITION 3.5. *Suppose in addition to* (E) *that I is weakly complemented in $\mathfrak{A}$, say $\iota^* \circ r = \mathrm{id}_{I^*}$. If $\mathfrak{A}$ is biflat, then* $\mathrm{cl}(\mathfrak{A}I\mathfrak{A})$ *is a weak retract of $\mathfrak{A}$. In particular it is biflat, and we have the estimate* $\mathrm{bc}(\mathrm{cl}(\mathfrak{A}I\mathfrak{A})) \leq \|r\|^2(\mathrm{bc}(\mathfrak{A}))^3$.

Proof. Consider the inclusion $\iota_{|\,\mathrm{cl}(I\mathfrak{A})}\colon \mathrm{cl}(I\mathfrak{A}) \to \mathfrak{A}$. First we prove that $\mathrm{cl}(I\mathfrak{A})$ is a weak retract of $\mathfrak{A}$ as right modules, i.e. $\iota^*_{|\,\mathrm{cl}(I\mathfrak{A})}$ is a retraction in $\mathfrak{A}\mathbf{mod}$. Define $R\colon \mathrm{cl}(I\mathfrak{A})^* \to (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^*$ by

$$\langle a \otimes b, R(m)\rangle = \langle a, r(b.\tilde{m})\rangle, \quad a, b \in \mathfrak{A},\ m \in \mathrm{cl}(I\mathfrak{A})^*.$$

where $\tilde{m} \in I^*$ is some Hahn-Banach extension of $m \in \mathrm{cl}(I\mathfrak{A})^*$. Since $\langle \xi, b.\tilde{m}\rangle = \langle \xi b, m\rangle$, for all $\xi \in I$, $b \in \mathfrak{A}$, this definition unambigously defines a bounded linear map. Actually, R is a left-module homomorphism:

$$\begin{aligned}\langle a \otimes b, R(c.m)\rangle &= \langle a, r(b.\widetilde{c.m}\rangle = \langle a, r(bc.\tilde{m}\rangle \\ &= \langle a \otimes bc, R(m)\rangle = \langle a \otimes b, c.R(m)\rangle, \quad a, b, c \in \mathfrak{A},\end{aligned}$$

since we may choose $\widetilde{c.m} = c.\tilde{m}$.

Put $\hat{r} = \rho \circ R$, where ρ is a weak splitting of the multiplication on $\mathfrak{A}$. Then $\hat{r} \in {}_{\mathfrak{A}}\mathbf{h}(\mathrm{cl}(I\mathfrak{A})^*, \mathfrak{A}^*)$ since R and ρ both are left-module homomorphisms. For $a, a' \in \mathfrak{A}$, $\xi \in I$, $m \in \mathrm{cl}(I\mathfrak{A})^*$ we have in turn

$$\begin{aligned}\langle a \otimes a', R(m).\iota(\xi)\rangle &= \langle \iota_{|\,\mathrm{cl}(I\mathfrak{A})}(\xi a) \otimes a', R(m)\rangle \\ &= \langle \iota_{|\,\mathrm{cl}(I\mathfrak{A})}(\xi a), r(a'.\tilde{m})\rangle \\ &= \langle \xi a, a'.\tilde{m}\rangle \\ &= \langle a \otimes a', \Pi^*(\tilde{m}.\iota(\xi))\rangle\end{aligned}$$

and

$$\begin{aligned}\langle \xi a, \iota^*_{|\,\mathrm{cl}(I\mathfrak{A})}(\hat{r}(m))\rangle &= \langle \iota_{|\,\mathrm{cl}(I\mathfrak{A})}(\xi a), \rho(R(m))\rangle \\ &= \langle a, \rho(R(m).\iota(\xi))\rangle \\ &= \langle a, \rho(\Pi^*(\tilde{m}.\iota(\xi)))\rangle \\ &= \langle a, \tilde{m}.\iota(\xi)\rangle \\ &= \langle \xi a, m\rangle.\end{aligned}$$

It follows that $\iota^*_{|\,\mathrm{cl}(I\mathfrak{A})} \circ \hat{r} = \mathbf{1}_{\mathrm{cl}(I\mathfrak{A})^*}$.

Either by repeating the construction with respect to the module multiplication on the right with $\hat{r}$ replacing r, or by using that $\mathfrak{A}\widehat{\otimes}\mathfrak{A}$ is biflat when $\mathfrak{A}$ is, we obtain $\lambda \in {}_{\mathfrak{A}}\mathbf{h}_{\mathfrak{A}}(\mathrm{cl}(\mathfrak{A}I\mathfrak{A})^*, \mathfrak{A}^*)$ so that $(\iota_{|\,\mathrm{cl}(\mathfrak{A}I\mathfrak{A})})^* \circ \lambda = \mathbf{1}_{\mathrm{cl}(\mathfrak{A}I\mathfrak{A})^*}$. Let Λ be as in the proof of Lemma 3.4 with $X = \mathrm{cl}(\mathfrak{A}I\mathfrak{A})$ and $Y = \mathfrak{A}$. Then it is easy to check that Λ is a bimodule homomorphism and that $(\iota_{|\,\mathrm{cl}(\mathfrak{A}I\mathfrak{A})})^* \circ \rho \circ \Lambda$ is a weak splitting of the multiplication on $\mathrm{cl}(\mathfrak{A}I\mathfrak{A})$. The estimate of $\mathrm{bc}(\mathrm{cl}(\mathfrak{A}I\mathfrak{A}))$ follows directly from the construction of the weak splitting as a composition of linear maps. □

We have the following strengthening of hereditarity of amenability.

COROLLARY 3.6. *If I is a closed ideal of a biflat Banach algebra $\mathfrak{A}$, then I is amenable if (and only if) I has a bounded approximate identity.*

Proof. Let $(e_\gamma)_\Gamma$ be a bounded approximate identity for I. We must prove that I is biflat. For each $\gamma \in \Gamma$ let $r_\gamma \in \mathcal{B}(I^*, \mathfrak{A}^*)$ be defined as

$$r_\gamma(f)(a) = f(e_\gamma a), \quad f \in I^*, \; a \in \mathfrak{A}.$$

Then $r = \text{w*-}\lim_\gamma r_\gamma$ satisfies $\iota^* \circ r = \mathrm{id}_{I^*}$ as required for I to be weakly complemented. Clearly $\mathrm{cl}(\mathfrak{A} I \mathfrak{A}) = I$, so we may invoke 3.5. □

Under certain conditions biflatness is actually biprojectivity.

PROPOSITION 3.7. *Suppose that multiplication on $\mathfrak{A}$ is approximable, i.e. the linear maps $\mathfrak{A} \to \mathfrak{A}$*

$$a \mapsto ab, \quad a \mapsto ba$$

are approximable operators for all $b \in \mathfrak{A}$. If $\mathfrak{A}$ is biflat, then it is in fact biprojective.

Proof. Let $\rho \colon (\mathfrak{A} \widehat{\otimes} \mathfrak{A})^* \to \mathfrak{A}^*$ be a weak splitting of the multiplication. Since $\mathrm{cl}(\mathfrak{A}^2) = \mathfrak{A}$ it follows from compactness that $\rho^* \circ \kappa$ maps $\mathfrak{A}$ into $\kappa(\mathfrak{A} \widehat{\otimes} \mathfrak{A})$. Specifically, since the operators $\tau \mapsto a.\tau.b \colon \mathfrak{A} \widehat{\otimes} \mathfrak{A} \to \mathfrak{A} \widehat{\otimes} \mathfrak{A}$, $a, b \in \mathfrak{A}$ are all approximable and hence compact, we have $\mathfrak{A}.(\mathfrak{A} \widehat{\otimes} \mathfrak{A})^{**}.\mathfrak{A} \subseteq \kappa(\mathfrak{A} \widehat{\otimes} \mathfrak{A})$ by Theorem VI.4.2 of [**6**]. Identifying $\mathfrak{A} \widehat{\otimes} \mathfrak{A}$ with $\kappa(\mathfrak{A} \widehat{\otimes} \mathfrak{A})$ the co-restriction of $\rho^* \circ \kappa$ is a splitting of the multiplication on $\mathfrak{A}$. □

REMARK 3.8. One may suspect that weak compactness of multiplication is sufficient for the conclusion. However, it is not true that weak compactness or even compactness is preserved with respect to the projective tensor product.

PROPOSITION 3.9. *Let $0 \to I \xrightarrow{\iota} \mathfrak{A} \xrightarrow{q} \mathfrak{A}/I \to 0$ be an extension of Banach algebras as in* (E)*, and suppose that $\mathfrak{A}/I$ is amenable. Let $D \colon \mathfrak{A} \to X^*$ be a derivation. For each $m \in X^*$ such that $D_{|I} = \delta_m$ there is $k \in \mathrm{ann}_I(X^*)$ such that $D = \delta_m + \delta_k$. In particular* $\mathrm{gc}(\mathfrak{A}) \le \mathrm{gc}(I) + \mathrm{gc}(\mathfrak{A}/I) + \mathrm{gc}(I)\,\mathrm{gc}(\mathfrak{A}/I)$.

Proof. The decomposition $D = \delta_m + \delta_k$ follows from the proof of [**11**, Proposition 5.1]. Since $\|D_{|I}\| \le \|D\|$ and $D - \delta_m$ drops to a derivation on $\mathfrak{A}/I$, we obtain the estimate on $\mathrm{gc}(\mathfrak{A})$. □

4. Banach algebras graded over a semi-lattice. In order to study algebras graded over a semi-lattice we shall use the result [**1**, Theorem 5.6]. Choi exploits the Schützenberger representation in his proof. We give a proof without specific reference to combinatorial results thus keeping a more Banach-algebraic approach. This is at the cost of an estimate on the supremum in 4.1 below.

PROPOSITION 4.1 (Y. Choi). *Let Λ be a semilatice. If $\ell^1(\Lambda)$ is biflat, then*

$$\sup\{\#(s\Lambda) \mid s \in \Lambda\} < \infty.$$

Proof. Clearly $\Lambda^2 = \Lambda$. For each $s \in \Lambda$ we may view $\ell^1(s\Lambda)$ as a complemented ideal in $\ell^1(\Lambda)$ and since, with appropriate identifications, $\mathrm{cl}(\ell^1(\Lambda)\ell^1(s\Lambda)) = \ell^1(s\Lambda^2) = \ell^1(s\Lambda)$ it follows from 3.5 that $\mathrm{bc}(\ell^1(s\Lambda)) \le \mathrm{bc}(\ell^1(\Lambda))$. As $\ell^1(s\Lambda)$ is unital with unit s, we further get $\mathrm{ac}(\ell^1(s\Lambda)) \leq \mathrm{bc}(\ell^1(\Lambda))$. From the main result of **[5]** we first conclude that $s\Lambda$ is finite for each $s \in \Lambda$ and next that $\sup(\#(s\Lambda)) < \infty$. In fact, by the details of the proof in **[5]**, for finite semi-lattices E we have $\mathrm{ac}(\ell^1(E)) \to \infty$ as $\#E \to \infty$. Alternatively we may refer to Corollary 1.8 of **[7]** for a more precise estimate. $\square$

We recall some concepts and definitions for semi-lattices.

Definition 4.2. Let Λ be a semi-lattice. The *partial order* on Λ is defined by $s \leq t$ if and only if $st = s$. We write $s < t$ when $s \leq t$ and $s \neq t$.

Following **[1]** Λ is *locally C-finite* if $\sup\{\#(s\Lambda) \mid s \in \Lambda\} = C$ and Λ is *uniformly locally finite* if Λ is locally C-finite for some constant $C > 0$. Note that $s\Lambda = \{t \in \Lambda \mid t \leq s\}$.

The *chain length* of Λ is

$$\mathbf{Chl}\,\Lambda = \sup\{n \in \mathbb{N} \mid e_1 < \cdots < e_n \text{ is a chain in } \Lambda\}.$$

We adopt a terminology from **[5]**. For $u, s \in \Lambda$

$$u \text{ covers } s \iff \{t \mid s \leq t \leq u\} = \{s, u\},$$

and we define the *covering number*

$$\mathrm{cov}\, u = \#\{s \mid u \text{ covers } s\}$$

For clarity we stipulate that $\mathbf{Chl}(\Lambda) = \infty$ and $\mathrm{cov}\, u = \infty$ if for each $n \in \mathbb{N}$ there are chains of length n respectively there are n distinct elements covered by u.

In **[7]** the authors study how amenability of a Banach algebra graded over a finite semi-lattice depends on the amenability of its summands and of the semi-lattice. It seems necessary to require some compatibility conditions in order to obtain results. We shall adapt the conditions of **[7]** to deal with infinite semi-lattices.

Definition 4.3. Let $\mathfrak{A} = \bigoplus_{\alpha\in\Lambda} A_\alpha$ be a semi-lattice graded Banach algebra.

(LA1) There are $C > 0$ and for each $\alpha \in \Lambda$ a bounded approximate identity $(e_\gamma^\alpha)_{\gamma\in\Gamma}$ for $\mathfrak{A}_\alpha$ of bound not exceeding C.

(LA2) For each $\alpha \in \Lambda$ there is a character $\chi_\alpha : A_\alpha \to \mathbb{C},\ \alpha \in \Lambda$ such that

$$\chi_{\alpha\beta}(a_\alpha a_\beta) = \chi_\alpha(a_\alpha)\chi_\beta(a_\beta) \quad \alpha, \beta \in \Lambda,\ a_\alpha \in A_\alpha,\ a_\beta \in A_\beta,$$

and $\mathrm{cl}(\mathfrak{A} \ker \chi_\alpha) \supseteq \ker \chi_\alpha$.

First we introduce notation: For $\beta \in \Lambda$

$$\mathfrak{A}_\beta = \bigoplus_{\alpha\in\beta\Lambda} A_\alpha \text{ and } \mathfrak{A}_{(\beta)} = \bigoplus_{\alpha\in\beta\Lambda\setminus\{\beta\}} A_\alpha.$$

We shall regard $\mathfrak{A}_\beta$ and $\mathfrak{A}_{(\beta)}$ as ideals of $\mathfrak{A}$, complemented as Banach subspaces. We shall also need the observation that if $\phi\colon \Lambda \to M$ is a semi-lattice homomorphism, then $B_m = \bigoplus_{\phi(\alpha)=m} A_\alpha$ is a Banach subalgebra of $\mathfrak{A}$ and there is a natural isometric isomorphism of semi-lattice graded Banach algebras

$$\bigoplus_{m\in M} B_m \cong \bigoplus_{\alpha\in\Lambda} A_\alpha.$$

We leave the details to the reader.

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THEOREM 4.4. *Let $\mathfrak{A}$ be as above and assume that* $\mathrm{cl}(\mathfrak{A}\mathfrak{A}_\alpha\mathfrak{A}) = \mathfrak{A}_\alpha$ *for all* $\alpha \in \Lambda$. *If $\mathfrak{A}$ is biflat (biprojective), then each A_α is biflat (biprojective). If $\mathfrak{A}$ further satisfies one of* (**LA1**) *or* (**LA2**), *then Λ is uniformly locally finite.*

Proof. By 3.5 $\mathfrak{A}_\alpha$ is biflat (biprojective), and by 3.3 applied to

$$0 \to \mathfrak{A}_{(\alpha)} \to \mathfrak{A}_\alpha \to A_\alpha \to 0$$

we obtain the first conclusion.

Assume that (**LA1**) holds. Then $\mathfrak{A}_\alpha$ is biflat and has a bounded approximate identity, hence is amenable for each $\alpha \in \mathfrak{A}$. We shall utilise the argument of [**5**, Lemma 9] by exploiting the sub-semilattices $E_n = \{e_1, \dots, e_n\}$ and $F_n = \{f_0, f_1, \dots, f_n, f\}$ of $\{0, 1\}^{\mathbb{N}}$ with pointwise multiplication, where

$$e_i = (\underbrace{1, \dots, 1}_{i}, 0, \dots) \quad i = 1, \dots, n, \dots$$

$$f_i = (\underbrace{1, 0, \dots, 0, 1}_{i+1}, 0, \dots), \quad i = 0, \dots, n, \dots$$

$$f = e_{n+2}.$$

First, assume that there is a chain $\alpha_1 < \cdots < \alpha_n$ in Λ. Then, as in [**5**], there is a semi-lattice homomorphism of $\alpha_n\Lambda$ onto E_n (if α_1 is minimal) or E_{n+1}. To be explicit let us assume onto E_n. By the observation above we may write

$$\mathfrak{A}_{\alpha_n} = \bigoplus_{e \in E_n} B_e$$

From [7, Theorem 2.2 and Example 1.3] we conclude that $\mathrm{ac}(\mathfrak{A}_{\alpha_n}) \geq \mathrm{ac}(\ell^1(E_n)) = 4n + 1$. Then $\mathrm{bc}(\mathfrak{A})^3 \geq \mathrm{bc}(\mathfrak{A}_{\alpha_n}) \geq \frac{4n+1}{C}$, where C is the constant in the definition of (**LA1**). It follows that $\mathbf{Chl}\,\Lambda < +\infty$.

Let $u \in \Lambda$ and choose $n \leq \mathrm{cov}\, u$, say u covers $s_1, \dots, s_n$. Since $\mathbf{Chl}\,\Lambda < +\infty$ there is a minimal element ω of $u\Lambda$, namely the first element of a chain in $u\Lambda$ of maximal length. The map $\phi\colon u\Lambda \to F_n$ given by

$$\phi(s) = \begin{cases} f & \text{if } s = u \\ f_i & \text{if } s = s_i,\ i = 1, \dots n \ , \\ f_0 & \text{else} \end{cases} \qquad s \in u\Lambda,$$

is a semi-lattice homomorphism. As above we get $\mathrm{bc}(\mathfrak{A})^3 \geq \frac{\mathrm{ac}(\ell_1(F_n))}{C}$. As $\lim_{n\to\infty} \mathrm{ac}(\ell^1(F_n)) = +\infty$ we obtain that $K = \sup\{\mathrm{cov}\, s \mid s \in \Lambda\} < \infty$. Setting $L = \mathbf{Chl}\,\Lambda$ an easy argument gives the estimate

$$\sup\{\# s\Lambda \mid s \in \Lambda\} \leq \frac{K^{L-1} - 1}{K - 1} + 1,$$

i.e. Λ is uniformly locally finite.

Assume that (**LA2**) holds. Since characters are bounded by 1, the universal property 2.4 of semi-lattice graded Banach algebras gives a Banach algebra epimorphism $\Xi\colon \mathfrak{A} \to \ell^1(\Lambda)$. By explicit assumption in (**LA2**) we have $\mathrm{cl}(\mathfrak{A} \ker \Xi\, \mathfrak{A}) = \ker \Xi$, so an appeal to 3.3 establishes biflatness of $\ell^1(\Lambda)$, from which the claim follows through 4.1. $\square$

It may seem unsatisfactory that amenability enters the picture, since amenability is biflatness *plus* bounded approximate identities. One might hope that biflatness of semi-lattice graded Banach algebras could be described solely in terms of biflatness of the constituents. The following simple example shows why this hope is too ambitious, the problem being that the hereditary properties of biflatness are effectively weaker than those of amenability.

EXAMPLE 4.5. Let $\mathfrak{A}$ be a biflat, non-amenable Banach algebra and consider the $\{0, 1\}$ graded Banach algebra $\mathfrak{A}_+ = \mathfrak{A} \oplus \mathbb{C}$. Then obviously the constituents of $\mathfrak{A}_+$ are biflat. However, were $\mathfrak{A}_+$ biflat, it would in fact be amenable, being unital. Then in turn its complemented ideal $\mathfrak{A}$ would be amenable, contrary to the choice of $\mathfrak{A}$. Note that $\mathfrak{A}_+$ does not satisfy (**LA1**).

Our next result gives sufficient conditions for biflatness. We remind the reader that for Banach algebras $\mathfrak{B} \subseteq \mathfrak{A}$ a bounded approximate identity $(e_\gamma)_\Gamma$ for $\mathfrak{B}$ is called *quasi-central for* $\mathfrak{A}$ if

$$\lim_\gamma \|e_\gamma a - a e_\gamma\| = 0, \quad a \in \mathfrak{A}.$$

THEOREM 4.6. *Let $\mathfrak{A} = \bigoplus_{\alpha\in\Lambda} A_\alpha$ and assume*
(i) *Λ is locally C-finite for some $C > 0$;*
(ii) *the A_α's are uniformly biflat;*
(iii) *the A_α's have uniformly bounded approximate identities, say $(e^\alpha_\gamma)_\gamma$, bounded by $D > 0$;*
(iv) *each $(e^\alpha_\gamma)_\gamma$ is quasi-central for $\mathfrak{A}$.*
Then $\mathfrak{A}$ is biflat.

Proof. We start by proving that for each finite sub-semilattice $F \subseteq \Lambda$ the Banach subalgebra $\bigoplus_{\alpha\in F} A_\alpha$ has a bounded approximate identity which is quasi-central for $\mathfrak{A}$. In particular $\mathfrak{A}$ satisfies (**LA1**). Let $F = \{\beta_1, \dots, \beta_k\}$ and put

$$E_{\gamma_1,\dots,\gamma_k} = \mathbf{1} - \big(\mathbf{1} - e^{\beta_1}_{\gamma_1}\big)\dots\big(\mathbf{1} - e^{\beta_k}_{\gamma_k}\big).$$

Then order $\gamma = (\gamma_1, \dots, \gamma_k)$ by the product order. It follows, using the quasi-central property of the $e^{\beta_i}_{\gamma_i}$'s, that $(E_\gamma)_\gamma$ is a quasi-central bounded approximate identity for $\bigoplus_{\alpha\in F} A_\alpha$. In the special case $F = \alpha\Lambda \setminus \{\alpha\}$ this bounded approximate identity will be denoted by $(E^\alpha_\gamma)_\gamma$. We have uniform boundedness

$$\|E^\alpha_\gamma\| \le 1 + (1 + D)^{C-1}$$

of these approximate identities.

We shall prove our statement by induction on $n = \mathbf{Chl}(\Lambda)$. For $n = 1$ there is nothing to prove, since in this case Λ is a singleton. Assume that the result is true for $\mathbf{Chl}(\Lambda) = n$ and let $\mathbf{Chl}(\Lambda) = n + 1$. Let Ω be the set of maximal elements in Λ. Then $\Lambda \setminus \Omega$ is an ideal of Λ with $\mathbf{Chl}(\Lambda \setminus \Omega) = n$, so $\bigoplus_{\beta\in\Lambda\setminus\Omega} A_\beta$ is biflat, say with weak splitting of multiplication ρ_0. To ease the notation in the remainder of the proof we set $\mathfrak{B} = \bigoplus_{\beta\in\Lambda\setminus\Omega} A_\beta$. For $\alpha \in \Omega$ choose a weak splitting ρ_α of the multiplication on A_α such that the family $(\rho_\alpha)_{\alpha\in\Omega}$ is uniformly bounded. Since $\mathfrak{B}$ and each A_α are (1-complemented) subspaces of $\mathfrak{A}$ we may regard the maps ρ_0 and $\rho_\alpha,\ \alpha \in \Omega$ as maps into $(\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$. We shall do so in the following. Now define $\tilde{\rho}_\alpha : A_\alpha \to (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$ by

$$\begin{aligned}\tilde{\rho}_\alpha(a_\alpha) &= \text{w*-}\lim_{\gamma} \rho_0\big(E^\alpha_\gamma a_\alpha\big)\\ &\quad + \text{w*-}\lim_{\gamma}\Big(\text{w*-}\lim_{\gamma'}\big(\mathbf{1}-E^\alpha_\gamma\big).\rho_\alpha(a_\alpha).\big(\mathbf{1}-E^\alpha_{\gamma'}\big)\Big)\end{aligned}$$

for $a_\alpha \in A_\alpha,\ \alpha \in \Omega$. By construction the $\tilde{\rho}_\alpha$'s are uniformly bounded and hence, by the universal property of ℓ^1-sums, together with ρ_0 define a bounded linear map $\rho : \mathfrak{A} \to (\mathfrak{A}\widehat{\otimes}\mathfrak{A})^{**}$ which extends ρ_0.

We next prove that

$$\rho(xy) = \rho(x)y \text{ and } \rho(yx) = y\rho(x) \tag{§}$$

for all $x \in A_\beta,\ y \in A_{\beta'},\ \beta, \beta' \in \Lambda$. It will then follow from linearity and continuity that ρ is a bimodule map.

It will be convenient to single out some of the arguments in

LEMMA 4.7. *Suppose that $\alpha \in \Omega$ and that $\beta \neq \alpha$. Then*

$$\text{w*-}\lim_{\gamma}\Big(\text{w*-}\lim_{\gamma'} y\big(\mathbf{1}-E^\alpha_\gamma\big)\rho_\alpha(x)\big(\mathbf{1}-E^\alpha_{\gamma'}\big)\Big) = 0 \tag{1}$$

$$\text{w*-}\lim_{\gamma}\Big(\text{w*-}\lim_{\gamma'}\big(\mathbf{1}-E^\alpha_\gamma\big)\rho_\alpha(x)\big(\mathbf{1}-E^\alpha_{\gamma'}\big)y\Big) = 0 \tag{2}$$

$$x\rho(u) = \rho(xu), \qquad \rho(u)x = \rho(ux) \tag{3}$$

for all $x \in A_\alpha,\ y \in A_\beta, u \in \mathfrak{B}$.

Proof of lemma. First note that module multiplication is w^*-continuous. We shall use this without further mentioning.

Since A_α has a bounded approximate identity we may write $x = x'x''$ for appropriate $x', x'' \in A_\alpha$. Then

$$\begin{aligned}&\text{w*-}\lim_{\gamma}\Big(\text{w*-}\lim_{\gamma'} y\big(\mathbf{1}-E^\alpha_\gamma\big)\rho_\alpha(x'x'')\big(\mathbf{1}-E^\alpha_{\gamma'}\big)\Big)\\ &\quad= \text{w*-}\lim_{\gamma}\Big(\text{w*-}\lim_{\gamma'} y\big(\mathbf{1}-E^\alpha_\gamma\big)x'\rho_\alpha(x'')\big(\mathbf{1}-E^\alpha_{\gamma'}\big)\Big)\\ &\quad= \text{w*-}\lim_{\gamma}\Big(y\big(\mathbf{1}-E^\alpha_\gamma\big)x'\ \text{w*-}\lim_{\gamma'}\rho_\alpha(x'')\big(\mathbf{1}-E^\alpha_{\gamma'}\big)\Big) = 0,\end{aligned}$$

since $yx' \in \mathfrak{A}_{(\alpha)}$ and (E^α_γ) is a quasi-central bounded approximate identity for $\mathfrak{A}_{(\alpha)}$ so that $\lim_\gamma \|y(\mathbf{1}-E^\alpha_\gamma)x'\| = 0$. This proves (1). The statement (2) is proved likewise.

To prove (3) it suffices, since $\mathrm{cl}(\mathfrak{B}^2) = \mathfrak{B}$, to prove it for $u = vw$; but in this case the result follows easily from the $\mathfrak{B}$-module property of ρ_0, viz.

$$\begin{aligned}x\rho_0(vw) &= xv\rho_0(w) = \rho_0(xvw),\\ \rho_0(vwx) &= \rho_0(v)wx = \rho_0(vw)x.\end{aligned}$$

□

We now proceed with the proof of (§). There are several cases.

Case $\beta \in \Lambda \setminus \Omega$: This case follows directly from (3).

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Case $\beta \in \Omega,\ \beta \neq \beta'$: Let $x \in A_\beta,\ y \in A_{\beta'}$. To be consistent with nomenclature above, set $\alpha = \beta$. Then $\beta\beta' < \alpha$ and

$$\begin{aligned}
\rho(yx) &= \rho_0(yx) \\
(\text{since } yx \in \mathfrak{A}_{(\alpha)}) &= \lim_\gamma \rho_0(E^\alpha_\gamma yx) \\
(\text{by quasi-centrality}) &= \lim_\gamma \rho_0(yE^\alpha_\gamma x) \\
(\text{by (3)}) &= y \lim_\gamma \rho_0(E^\alpha_\gamma x) \\
(\text{by (1)}) &= y\Big(\text{w*-}\lim_\gamma \rho_0(E^\alpha_\gamma x) \\
&\quad + \text{w*-}\lim_\gamma \Big(\text{w*-}\lim_{\gamma'}((\mathbf{1} - E^\alpha_\gamma)\rho_\alpha(x)(\mathbf{1} - E^\alpha_{\gamma'})\Big)\Big) \\
&= y\rho(x)
\end{aligned}$$

The statement about module multiplication on the right is proved by using (2) rather than (1).

Case $\beta = \beta' \in \Omega$: Let α be the common value of β and β'. Then for all $x, y \in A_\alpha$

$$\begin{aligned}
\rho(xy) = &\text{w*-}\lim_\gamma \rho_0(E^\alpha_\gamma xy) \\
&+ \text{w*-}\lim_\gamma \Big(\text{w*-}\lim_{\gamma'}(\mathbf{1} - E^\alpha_\gamma)\rho_\alpha(xy)(\mathbf{1} - E^\alpha_{\gamma'})\Big).
\end{aligned}$$

By quasi-centrality and (3) the first term equals $x\,\text{w*-}\lim_\gamma \rho_0(E^\alpha_\gamma y)$.

For the second term:

$$\begin{aligned}
&\text{w*-}\lim_\gamma \Big(\text{w*-}\lim_{\gamma'}(\mathbf{1} - E^\alpha_\gamma)x\rho_\alpha(y)(\mathbf{1} - E^\alpha_{\gamma'})\Big) \\
&= \text{w*-}\lim_\gamma \Big(\text{w*-}\lim_{\gamma'} x(\mathbf{1} - E^\alpha_\gamma)\rho_\alpha(y)(\mathbf{1} - E^\alpha_{\gamma'})\Big) \\
&= x\,\text{w*-}\lim_\gamma \Big(\text{w*-}\lim_{\gamma'}(\mathbf{1} - E^\alpha_\gamma)\rho_\alpha(y)(\mathbf{1} - E^\alpha_{\gamma'})\Big),
\end{aligned}$$

by quasi-centrality. Adding the two terms we get

$$\begin{aligned}
\rho(xy) = &x\,\text{w*-}\lim_\gamma \rho_0(E^\alpha_\gamma y) \\
&+ x\,\text{w*-}\lim_\gamma \Big(\text{w*-}\lim_{\gamma'}(\mathbf{1} - E^\alpha_\gamma)\rho_\alpha(y)(\mathbf{1} - E^\alpha_{\gamma'})\Big) \\
= &x\rho(y),
\end{aligned}$$

and, working similarly on the right, $\rho(xy) = \rho(x)y$.

Finally, let $\Pi\colon \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ be the multiplication. For $a \in \mathfrak{B}$ we clearly have $\Pi^{**}(\rho(a)) = \kappa(a)$, since ρ extends ρ_0. For $a \in A_\alpha,\ \alpha \in \Omega$ we have

$$\begin{aligned}
\Pi^{**}(\rho(a)) = &\Pi^{**}\Big(\text{w*-}\lim_\gamma(\rho_0(E^\alpha_\gamma a))\Big) \\
&+ \Pi^{**}\Big(\text{w*-}\lim_\gamma(\text{w*-}\lim_{\gamma'}(\mathbf{1} - E^\alpha_\gamma)\rho_\alpha(a)(\mathbf{1} - E^\alpha_{\gamma'}))\Big).
\end{aligned}$$

By w*-continuity of Π^{**} the first term equals $\text{w*-}\lim_\gamma E^\alpha_\gamma a$, since ρ_0 splits multiplication weakly. Using w*-continuity and that ρ_α splits multiplication weakly,

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we calculate the second term

$$\begin{aligned}
&\text{w*-}\lim_{\gamma}\left(\text{w*-}\lim_{\gamma'}(\mathbf{1}-E^{\alpha}_{\gamma})\Pi^{**}(\rho_{\alpha}(a))(\mathbf{1}-E^{\alpha}_{\gamma'})\right)\\
&\quad=\text{w*-}\lim_{\gamma}\left(\text{w*-}\lim_{\gamma'}(\mathbf{1}-E^{\alpha}_{\gamma})a(\mathbf{1}-E^{\alpha}_{\gamma'})\right)\\
&\quad=\text{w*-}\lim_{\gamma}\left(\text{w*-}\lim_{\gamma'}(a-E^{\alpha}_{\gamma}a-aE^{\alpha}_{\gamma'}+E^{\alpha}_{\gamma}aE^{\alpha}_{\gamma'})\right)\\
&\quad=a-\text{w*-}\lim_{\gamma}E^{\alpha}_{\gamma}a-\text{w*-}\lim_{\gamma'}aE^{\alpha}_{\gamma'}+\text{w*-}\lim_{\gamma}\left(\text{w*-}\lim_{\gamma'}E^{\alpha}_{\gamma}aE^{\alpha}_{\gamma'}\right) \qquad (4)\\
&\quad=a-\text{w*-}\lim_{\gamma}E^{\alpha}_{\gamma}a-\text{w*-}\lim_{\gamma'}aE^{\alpha}_{\gamma'}+\text{w*-}\lim_{\gamma}E^{\alpha}_{\gamma}a\\
&\quad=a-\text{w*-}\lim_{\gamma'}aE^{\alpha}_{\gamma'},
\end{aligned}$$

where for the identity (4) we have used that $E^{\alpha}_{\gamma}a \in \mathfrak{A}_{(\alpha)}$ and that $(E^{\alpha}_{\gamma'})$ is a bounded approximate identity for $\mathfrak{A}_{(\alpha)}$. Adding the two terms we obtain, using quasi-centrality

$$\begin{aligned}
\Pi^{**}\circ\rho(a)&=\text{w*-}\lim_{\gamma}E^{\alpha}_{\gamma}a+a-\text{w*-}\lim_{\gamma'}aE^{\alpha}_{\gamma'}\\
&=\kappa(a)
\end{aligned}$$

altogether proving that ρ splits multiplication weakly on $\mathfrak{A}$. □

5. Applications. We shall now apply the previous section to discrete convolution algebras. Our first application is to note that Choi's result ([**1**, Theorem 6.1]) for discrete convolution algebras on Clifford semi-groups is a special case of the general results in Section 4. Recall that a *Clifford semi-group* is a semi-group S which is a disjoint union

$$S=\bigvee_{\alpha\in\Lambda}G_{\alpha},$$

where Λ is a semi-lattice and the multiplication satisfies: each G_{α} is a group with the semi-group multiplication and $G_{\alpha}G_{\beta}\subseteq G_{\alpha\beta},\ \alpha,\beta\in\Lambda$. For details see [**3**]. It is clear that we may then view $\ell^1(S)$ as an ℓ^1-graded Banach algebra

$$\ell^1(S)=\bigoplus_{\alpha\in\Lambda}\ell^1(G_{\alpha}),$$

and we get

COROLLARY 5.1 (Y. Choi). *Let $S=\bigvee_{\alpha\in\Lambda}G_{\alpha}$ be a Clifford semi-group. Then $\ell^1(S)$ is biflat (biprojective) if and only if Λ is uniformly locally finite and each group $G_{\alpha},\ \alpha\in\Lambda$, is amenable (finite).*

Proof. For each $\alpha\in\Lambda$ the unit of G_{α} is denoted e_{α}. If $\beta<\alpha$, then $(e_{\alpha}e_{\beta})^2=e_{\alpha}e_{\beta}$ so that $e_{\alpha}e_{\beta}=e_{\beta}$. It further follows that the net of units $(e_{\alpha})_{\Lambda}$ is in the center of S and that $\ell^1(S)$ satisfies (**LA1**).

Since a discrete convolution algebra on a group is unital, it is amenable if and only if it is biflat. It is easy to show that $\mathrm{ac}(\ell^1(G))=1$ for an amenable group and hence by 3.2 we have $\mathrm{bc}(\ell^1(G))=1$.

It now follows by 4.4 that the condition is necessary for biflatness and by 4.6 that it is sufficient.

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If $\ell^1(S)$ is biprojective, it is in particular biflat, so Λ is uniformly locally finite. For each unit $e_\alpha \in G_\alpha,\ \alpha \in \Lambda$ we have $e_\alpha g_\beta = g_\beta e_\alpha,\ g_\beta \in G_\beta, \beta \in \Lambda$, so $\ell^1(e_\alpha S)$ is a unital biprojective Banach algebra possesing Grothendieck's approximation property, thus is finite dimensional ([**15**]). Hence each group G_α is finite.

Conversely, if Λ is uniformly locally finite and each group is finite, then $\ell^1(S)$ is biflat by what we have already proved. The finiteness of the groups further implies that multiplication is approximable, so that biprojectivity now follows from 3.7. □

In the remainder of this section we will consider discrete convolution algebras on *abelian* semi-groups. As with the case of amenability ([**8**]), the structure theorem for abelian semi-groups will be the door of attack on the problem. We cite it in a form convenient to our purpose. First we recall that an abelian semi-group S is *archimedean* if

$$\forall s, t \in S \exists n \in \mathbb{N}\colon s^n \in tS.$$

THEOREM 5.2 (Structure theorem for abelian semi-groups). *Let S be an abelian semi-group. Then S is a disjoint union of archimedean semi-groups*

$$S = \bigvee_{\alpha\in\Lambda} S_\alpha$$

The index set Λ is equipped with a semi-lattice multiplication such that

$$S_\alpha S_\beta \subseteq S_{\alpha\beta} \quad \alpha, \beta \in \Lambda.$$

If $S = \bigvee_{\alpha'\in\Lambda'} S_{\alpha'}$ is another such decomposition, then Λ and Λ' are isomorphic, say via a semi-lattice isomorphism $\phi\colon \Lambda \to \Lambda'$ and S_α is isomorphic to $S_{\phi(\alpha)}$ for all $\alpha \in \Lambda$.

Proof. See [**3**, Theorem 4.13]. □

We shall refer to this result as the *archimedean decomposition* of the semi-group and S_α's as *archimedean components*. Note that if T is an archimedean sub-semigroup of S then there is $\alpha \in \Lambda$ such that $T \subseteq S_\alpha$.

We will prove

THEOREM 5.3. *Let S be an abelian semi-group with archimedean decomposition $S = \bigvee_{\alpha\in\Lambda} S_\alpha$. Then $\ell^1(S)$ is biflat if and only if Λ is uniformly locally finite and each S_α is a group.*

Proof. If $S = \bigvee_{\alpha\in\Lambda} S_\alpha$ and each S_α a group, then S is an abelian Clifford semi-group. Since abelian groups are amenable, we may appeal to 5.1 to establish sufficiency.

The proof of the converse consists of several steps. We start by doing away with the case $\#\Lambda = 1$:

LEMMA 5.4. *If S is archimedean and $\ell^1(S)$ is biflat, then S is a group.*

Proof. Using biflatness $\mathrm{cl}(\ell^1(S)^2) = \ell^1(S)$, so $S^2 = S$. Hence by 3.5 $\ell^1(sS)$ is biflat for each $s \in S$. So $s^2S = s^2S^2 = (sS)^2 = sS$ for each $s \in S$. Let $s, t \in S$. By the archimedean property there is $n \in \mathbb{N}$ such that $t^n \in sS$. So $tS = t^nS \subseteq sS$. It follows that $sS = tS$ for all $s, t \in S$. As $S^2 = \bigcup_{s\in S} sS$ we further conclude that $S = sS$ for all $s \in S$. For a given s we may thus write $s = se$ for appropriate $e \in S$. Then, given $t \in S = sS$ we may write $t = su$ for appropriate $u \in S$, whence $et = esu = su = t$; thus

e is a unit for S. One more application of $sS = S$ for all $s \in S$ gives that $e \in sS$ for all s, i.e. each element has an inverse. $\square$

The next lemma may be seen as doing away with the case $\#\Lambda = 2$.

LEMMA 5.5. *Suppose that Λ has a maximal element α_0. Then S_{α_0} is a group.*

Proof. Since α_0 is maximal the sub-semigroup $T = \bigvee_{\alpha \neq \alpha_0} S_\alpha$ is an ideal of S. Since $S^2 = S$ we obtain $ST = T$ and hence $\mathrm{cl}(\ell^1(S)\ell^1(T)) = \ell^1(T)$, so that $\ell^1(S)/\ell^1(T) = \ell^1(S_{\alpha_0})$ is biflat by 3.3. By 5.4 S_{α_0} is a group. $\square$

We now proceed with the proof of the theorem. First we produce a family of idempotents $e_\alpha \in S_\alpha$ so that $e_\alpha e_\beta = e_{\alpha\beta}$ for all $\alpha, \beta \in \Lambda$. Then we use this family to show that Λ is uniformly locally finite, so in particular $\mathbf{Chl}(\Lambda) < \infty$. Finally we prove inductively on $n = \mathbf{Chl}(\Lambda)$ that each archimedean component is a group.

Let $\alpha \in \Lambda$ be arbitrary and let $T = S(\bigvee_{\beta \leq \alpha} S_\beta)$. Then $\ell^1(T)$ is (isometrically isomorphic to) a closed ideal in $\ell^1(S)$ with $\mathrm{cl}(\ell^1(S)\ell^1(T)) = \ell^1(T)$, so $\ell^1(T)$ is biflat by 3.5. As T is a sub-semigroup of S, its archimedean components are subsets of the archimedean components of S. Let $T = \bigvee_{\beta \in M} T_\beta$ be the archimedean decomposition of T. We prove below that $T \cap S_\alpha$ is archimedean. It then follows that M has a maximal element β_0 and that $T \cap S_\alpha = T_{\beta_0}$. By 5.5 then $T_{\beta_0} = S(\bigvee_{\beta \leq \alpha} S_\beta) \cap S_\alpha$ is a group. In particular S_α contains an idempotent, e_α and $S(\bigvee_{\beta \leq \alpha} S_\beta) \cap S_\alpha = e_\alpha S_\alpha$. Note that, being archimedean, S_α contains at most one idempotent. Hence $e_\alpha e_\beta = e_{\alpha\beta}$ for all $\alpha, \beta \in \Lambda$.

To argue that $T \cap S_\alpha$ is archimedean, first observe that

$$T \cap S_\alpha = \{s_\alpha s_\beta \mid s_\alpha \in S_\alpha, s_\beta \in S_\beta,\ \beta \geq \alpha\}.$$

Let $s_\alpha s_\beta,\ u_\alpha u_{\beta'},\ \beta, \beta' \geq \alpha$ be arbitrary elements of $T \cap S_\alpha$. By the archimedean property of S_α there is $n \in \mathbb{N}$ such that $(s_\alpha s_\beta)^n \in u_\alpha u_{\beta'} S_\alpha$. But then $(s_\alpha s_\beta)^{n+1} \in u_\alpha u_{\beta'} s_\alpha s_\beta S_\alpha \subseteq u_\alpha u_{\beta'}(T \cap S_\alpha)$, thus establishing the defining property of being archimedean.

To see that Λ is uniformly locally finite, let $\phi_\alpha \colon \ell^1(S_\alpha) \to \mathbb{C}$ be the augmentation maps. Let $\Phi\colon \ell^1(S) \to \ell^1(\Lambda)$ be the corresponding surjection. Then $\ker \Phi = \bigoplus_{\alpha \in \Lambda} \ker \phi_\alpha$. Now $\ker \phi_\alpha$ is generated by elements of the form $s_\alpha - e_\alpha$. Since $S^2 = S$ we may write $s_\alpha = s_\beta s_\gamma$. Using $e_\alpha = e_\beta e_\gamma$ we get

$$s_\alpha - e_\alpha = (s_\beta - e_\beta)s_\gamma + e_\beta(s_\gamma - e_\gamma) \in \ell^1(S) \ker \phi_\beta + \ell^1(S) \ker \phi_\gamma,$$

so that $\mathrm{cl}(\ell(S) \ker \Phi) = \ker \Phi$. It follows from 3.3 that $\ell^1(\Lambda)$ is biflat and hence Λ is uniformly locally finite.

Finally, to see that each $S_\alpha,\ \alpha \in \Lambda$ is a group, we use induction on $n = \mathbf{Chl}(\Lambda)$. For $n = 1$ this is just 5.4. Assume that the result is true for $n,\ n \geq 1$ and suppose that $\mathbf{Chl}(\Lambda) = n + 1$. Let Ω be the set of maximal elements in Λ. By 5.5 S_α is a group for each $\alpha \in \Omega$. Let $T = \bigvee_{\alpha \notin \Omega} S_\alpha$. Then T is a sub-semigroup of S and has archimedean components $S_\alpha,\ \alpha \notin \Omega$. Since $\mathbf{Chl}(\Lambda \setminus \Omega) = n$ we finish the induction step by proving that $\ell^1(T)$ is biflat. Now T is an ideal in S, so we can conclude this from 3.5 provided $ST = T$. Since $S^2 = S$ each element $t \in T$ is a product $t = s_\alpha s_\beta$. The only possibility for $t \notin ST$ is if both $\alpha, \beta \in \Omega$. But in this case both S_α, S_β are groups. In particular, $s_\alpha = e_\alpha s_\alpha$ and $s_\beta = e_\beta s_\beta$, so that $t\ (= e_\alpha e_\beta t) \in ST$ also in this case. The induction step is hereby completed. $\square$

COROLLARY 5.6. *$\ell^1(S)$ is biprojective if and only if Λ is uniformly locally finite and each group S_α, $\alpha \in \Lambda$ is finite.*

Proof. If $\ell^1(S)$ is biprojective then it is biflat, so S is a Clifford semi-group. The rest follows from 5.1. □

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