7

Self-force

The inhomogeneous Maxwell equations have been solved in (2.16), (2.17). Thus it is natural to insert them into the Lorentz force in order to obtain a closed, albeit memory equation for the position of the particle.

According to (2.16), (2.17) the Maxwell fields are a sum of initial and retarded terms. We discuss first the contribution from the initial fields. By our specific choice of initial conditions they have the representation, for $t \geq 0$,

$$\begin{aligned} \boldsymbol{E}_{\text{ini}}(\boldsymbol{x}, t) = &- \int_{-\infty}^{0} \mathrm{d}s \int \mathrm{d}^3 y \, \big(\nabla G_{t-s}(\boldsymbol{x} - \boldsymbol{y}) \, e\varphi(\boldsymbol{y} - \boldsymbol{q}^0 - \boldsymbol{v}^0 s) \\ &+ \partial_t G_{t-s}(\boldsymbol{x} - \boldsymbol{y}) \boldsymbol{v}^0 \, e\varphi(\boldsymbol{y} - \boldsymbol{q}^0 - \boldsymbol{v}^0 s) \big) \, , \end{aligned} \tag{7.1}$$

$$\boldsymbol{B}_{\text{ini}}(\boldsymbol{x}, t) = \int_{-\infty}^{0} \mathrm{d}s \int \mathrm{d}^3 y \, \nabla \times G_{t-s}(\boldsymbol{x} - \boldsymbol{y}) \, \boldsymbol{v}^0 e\varphi(\boldsymbol{y} - \boldsymbol{q}^0 - \boldsymbol{v}^0 s) \, ; \tag{7.2}$$

compare with (4.31), (4.32). Since G_t is concentrated on the light cone, one concludes from (7.1), (7.2) that $\boldsymbol{E}_{\text{ini}}(\boldsymbol{x}, t) = 0$, $\boldsymbol{B}_{\text{ini}}(\boldsymbol{x}, t) = 0$ for $|\boldsymbol{q}^0 - \boldsymbol{x}| \leq t - R_\varphi$. If we had allowed for more general initial data, such a property would hold only asymptotically for large t.

Next we note that constrained by energy conservation the particle cannot travel too far. Using the bound on the potential, one can find a $\bar{v} < 1$ such that

$$\sup_{t \in \mathbb{R}} |\boldsymbol{v}(t)| < \bar{v} < 1 \, , \tag{7.3}$$

cf. Eq. (7.26). The charge distribution vanishes for $|\boldsymbol{x} - \boldsymbol{q}(t)| \geq R_\varphi$. Therefore, once

$$t \geq \bar{t}_\varphi = 2R_\varphi / (1 - \bar{v}) \, , \tag{7.4}$$

https://doi.org/10.1017/9781009402286.008 Published online by Cambridge University Press

the initial fields and the charge distribution have no overlap. We conclude that for $t > \bar{t}_\varphi$ the initial fields make no contribution to the self-force and it remains to discuss the effect of the retarded fields.

We insert (2.12), (2.13) into the Lorentz force for which purpose it is convenient to use the scaled version (6.11). The external potentials are set equal to zero for a while. Then on the macroscopic scale, for $t \geq \varepsilon \bar{t}_\varphi$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(m_\mathrm{b}\gamma\, \boldsymbol{v}^\varepsilon(t)\big) = \boldsymbol{F}^\varepsilon_{\mathrm{self}}(t) \tag{7.5}$$

with the self-force

$$\boldsymbol{F}^\varepsilon_{\mathrm{self}}(t) = e^2 \int_0^t \mathrm{d}s\, \varepsilon \int \mathrm{d}^3k\, |\widehat{\varphi}(\varepsilon \boldsymbol{k})|^2\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}^\varepsilon(t)-\boldsymbol{q}^\varepsilon(s))}\big((|\boldsymbol{k}|^{-1}\ \sin|\boldsymbol{k}|(t-s))\mathrm{i}\boldsymbol{k} \\ -(\cos|\boldsymbol{k}|(t-s))\boldsymbol{v}^\varepsilon(s) - (|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|(t-s))\, \boldsymbol{v}^\varepsilon(t)\times(\mathrm{i}\boldsymbol{k}\times\boldsymbol{v}^\varepsilon(s))\big)\,, \tag{7.6}$$

which in position space for $\varepsilon = 1$ was already written down in Eq. (2.57).

Equation (7.5) is exact under the stated conditions on the initial fields. No information has been discarded. The interaction with the field has been merely transcribed into a memory term. To make further progress we have to use a suitable approximation which exploits the assumption that the external forces are slowly varying. Since this corresponds to small ε, we just have to Taylor-expand $\boldsymbol{F}^\varepsilon_{\mathrm{self}}(t)$, which is carried out in section 7.2 with the proper justification left for section 7.3. But before that, and to make contact with previous work, we take a closer look at the memory term.

7.1 Memory equation

Equation (7.6) can be simplified, for which it is convenient to set $\varepsilon = 1$. By partial integration

$$\int_0^t \mathrm{d}s \int \mathrm{d}^3k\, |\widehat{\varphi}(\boldsymbol{k})|^2\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}(t)-\boldsymbol{q}(s))}\boldsymbol{v}(s)\, \frac{\mathrm{d}}{\mathrm{d}s}\, |\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|(t-s) \\ = -\int \mathrm{d}^3k\, |\widehat{\varphi}(\boldsymbol{k})|^2\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}(t)-\boldsymbol{q}(0))}\boldsymbol{v}(0)|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|t \\ -\int_0^t \mathrm{d}s \int \mathrm{d}^3k |\widehat{\varphi}(\boldsymbol{k})|^2\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}(t)-\boldsymbol{q}(s))}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|(t-s))\big(\dot{\boldsymbol{v}}(s) \\ +\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v}(s))\boldsymbol{v}(s)\big)\,. \tag{7.7}$$

https://doi.org/10.1017/9781009402286.008 Published online by Cambridge University Press

Since $t \geq \bar{t}_\varphi$, the boundary term vanishes. Inserting (7.7) into (7.6), returning to physical space, and setting $t - s = \tau$, one has for $t \geq \bar{t}_\varphi$

$$\boldsymbol{F}_{\text{self}}(t) = -e^2 \int_0^\infty \mathrm{d}\tau \Big[\dot{\boldsymbol{v}}(t-\tau) + (1 - \boldsymbol{v}(t)\cdot\boldsymbol{v}(t-\tau))\nabla_{\boldsymbol{x}} + \boldsymbol{v}(t-\tau)(\boldsymbol{v}(t) - \boldsymbol{v}(t-\tau))\cdot\nabla_{\boldsymbol{x}}\Big] W_t(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{q}(t)-\boldsymbol{q}(t-\tau)}\,, \quad (7.8)$$

where

$$W_t(\boldsymbol{x}) = \int \mathrm{d}^3k\, |\widehat{\varphi}(\boldsymbol{k})|^2\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} |\boldsymbol{k}|^{-1} \sin|\boldsymbol{k}|t\,. \quad (7.9)$$

In (7.8) we have extended the integration to ∞, since the integrand vanishes anyway for $\tau \geq \bar{t}_\varphi$. Carrying out the integrations on the angles in (7.9) one obtains

$$W_t(\boldsymbol{x}) = |\boldsymbol{x}|^{-1}\big(h(|\boldsymbol{x}|+t) - h(|\boldsymbol{x}|-t)\big)\,, \quad (7.10)$$

$$h(w) = 2\pi \int_0^\infty \mathrm{d}k\, g(k)\cos kw \quad (7.11)$$

with $g(|\boldsymbol{k}|) = |\widehat{\varphi}(\boldsymbol{k})|^2$. Since φ vanishes for $|\boldsymbol{x}| \geq R_\varphi$, $h(w) = 0$ for $|w| \geq 2R_\varphi$. Note that $|\boldsymbol{q}(t) - \boldsymbol{q}(t-\tau)| \leq \bar{v}\,\tau$. Thus for $t \geq \bar{t}_\varphi$ we indeed have $W_t(\boldsymbol{q}(t) - \boldsymbol{q}(t-\tau)) = 0$, as claimed before. $\boldsymbol{F}_{\text{self}}(t)$ has a finite memory extending backwards in time up to $t - \bar{t}_\varphi$.

To go beyond (7.10) one has to use a specific form factor $\widehat{\varphi}$. Two choices, popular at the time, are $\varphi_{\text{s}}(\boldsymbol{x}) = (4\pi R_\varphi^2)^{-1}\,\delta(|\boldsymbol{x}| - R_\varphi)$ and $\varphi_{\text{b}}(\boldsymbol{x}) = e\,(4\pi R_\varphi^3/3)^{-1}$ for $|\boldsymbol{x}| \leq R_\varphi$, $\varphi_{\text{b}}(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| \geq R_\varphi$. For the uniformly charged sphere one finds

$$h(R_\varphi w) = \begin{cases} (8\pi R_\varphi)^{-1}(1 - |w|/2) & \text{for } |w| \leq 2\,, \\ 0 & \text{for } |w| \geq 2\,, \end{cases} \quad (7.12)$$

and for the uniformly charged ball

$$h(R_\varphi w) = \begin{cases} (8\pi R_\varphi)^{-1}\,\frac{9}{8}\,\widetilde{h} * \widetilde{h}(w) & \text{for } |w| \leq 2\,, \\ 0 & \text{for } |w| \geq 2\,, \end{cases} \quad (7.13)$$

with $\widetilde{h}(w) = (1 - w^2)$ for $|w| \leq 1$ and $\widetilde{h}(w) = 0$ otherwise.

For the charged sphere $W_t(\boldsymbol{x})$ is piecewise linear and, by first taking the gradient of W, the time integrations simplify. In the approximation of small velocities the motion of the charged particle is then governed by the differential–difference

https://doi.org/10.1017/9781009402286.008 Published online by Cambridge University Press

equation

$$m_{\mathrm{b}}\dot{\boldsymbol{v}}(t) = e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}(t)) + \boldsymbol{v}(t) \times \boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q}(t))\big) + \frac{e^2}{12\pi R_\varphi^2}\big(\boldsymbol{v}(t-2R_\varphi) - \boldsymbol{v}(t)\big)\,, \tag{7.14}$$

where we have reintroduced the external fields.

The memory equation (7.14) is of suggestive simplicity. To have a well-defined dynamics one has to prescribe $\boldsymbol{q}(0)$ and $\boldsymbol{v}(t)$ for $-2R_\varphi \le t \le 0$ as initial data. Of course, the coupled system determines these data completely. However, the supporters of differential–difference equations regard (7.14) as the starting point with no instruction for the choice of initial data. Their claim is that solutions to (7.14) are not very sensitive to this choice. While there is some evidence on the linearized level, the dependence on the initial data for the full nonlinear problem remains to be studied.

7.2 Taylor expansion

We return to Eq. (7.5). As will be explained in section 7.3, one knows that there exists a constant C, independent of ε for $\varepsilon < \varepsilon_0$, such that

$$\begin{aligned} &|\ddot{\boldsymbol{q}}^\varepsilon(t)| \le C\,, \quad |\dddot{\boldsymbol{q}}^\varepsilon(t)| \le C\big(1+\varepsilon(\varepsilon+|t|)^{-2}\big)\,,\\ &|\ddddot{\boldsymbol{q}}^\varepsilon(t)| \le C\big(1+\varepsilon(\varepsilon+|t|)^{-2}+\varepsilon(\varepsilon+|t|)^{-3}\big) \end{aligned} \tag{7.15}$$

for all t, provided the total charge e is sufficiently small. This smallness condition merely reflects the fact that at present we do not know how to do better mathematically. Physically we expect (7.15) to hold no matter how large e.

Note that in higher time derivatives the mismatch of the initial conditions becomes visible. Only if the charge is allowed to move for a time span of order $\varepsilon^{1/3}$, which is short on the macroscopic scale but long as $\mathcal{O}(\varepsilon^{-2/3})$ on the microscopic scale, do the derivatives become uniformly bounded.

Because of (7.15) we are allowed to Taylor-expand in (7.6). To simplify notation we set $\boldsymbol{v}^\varepsilon(t) = \boldsymbol{v}$ and $t - s = \tau$. Then

$$\boldsymbol{v}^\varepsilon(s) = \boldsymbol{v}^\varepsilon(t-\tau) = \boldsymbol{v} - \dot{\boldsymbol{v}}\tau + \frac{1}{2}\ddot{\boldsymbol{v}}\tau^2 + \mathcal{O}(\tau^3)\,, \tag{7.16}$$

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}^\varepsilon(t)-\boldsymbol{q}^\varepsilon(s))} &= \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}^\varepsilon(t)-\boldsymbol{q}^\varepsilon(t-\tau))} = \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v})\tau}\Big(1+\frac{1}{2}\tau^2\mathrm{i}(\boldsymbol{k}\cdot\dot{\boldsymbol{v}}) - \frac{1}{6}\tau^3\,\mathrm{i}(\boldsymbol{k}\cdot\ddot{\boldsymbol{v}})\\ &\quad - \frac{1}{2}\Big(\frac{1}{2}\tau^2(\boldsymbol{k}\cdot\dot{\boldsymbol{v}}) - \frac{1}{6}\tau^3(\boldsymbol{k}\cdot\ddot{\boldsymbol{v}})\Big)^2 + \mathcal{O}((|\boldsymbol{k}|\tau^2)^3)\Big)\,. \end{aligned} \tag{7.17}$$

Inserting in (7.6) and substituting $s' = \varepsilon^{-1}s,\ \boldsymbol{k}' = \varepsilon\boldsymbol{k}$ yields

$$\begin{aligned}
\boldsymbol{F}^{\varepsilon}_{\text{self}}(t) = e^2 \int_0^{\varepsilon^{-1}t} \mathrm{d}\tau \varepsilon^{-1} \int \mathrm{d}^3k |\widehat{\varphi}(\boldsymbol{k})|^2 \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v})\tau} \Big\{ &(|\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|\tau)\mathrm{i}\boldsymbol{k} \\
&- (\cos |\boldsymbol{k}|\tau)\Big(\boldsymbol{v} - \varepsilon\tau\dot{\boldsymbol{v}} + \frac{1}{2}\varepsilon^2\tau^2\ddot{\boldsymbol{v}}\Big) - (|\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|\tau)(\boldsymbol{v} \times (\mathrm{i}\boldsymbol{k} \times \boldsymbol{v}) \\
&- \boldsymbol{v} \times (\mathrm{i}\boldsymbol{k} \times \varepsilon\tau\dot{\boldsymbol{v}}) + \frac{1}{2}\,\boldsymbol{v} \times (\mathrm{i}\boldsymbol{k} \times \varepsilon^2\tau^2\ddot{\boldsymbol{v}})) + \frac{1}{2}\,\varepsilon\tau^2\mathrm{i}(\boldsymbol{k}\cdot\dot{\boldsymbol{v}}) \\
&\times \big((|\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|\tau)\mathrm{i}\boldsymbol{k} - (\cos |\boldsymbol{k}|\tau)(\boldsymbol{v} - \varepsilon\tau\dot{\boldsymbol{v}}) - (|\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|\tau)(\boldsymbol{v} \times (\mathrm{i}\boldsymbol{k} \times \boldsymbol{v}) \\
&- \boldsymbol{v} \times (\mathrm{i}\boldsymbol{k} \times \varepsilon\tau\dot{\boldsymbol{v}}))\big) + \Big(- \frac{1}{6}\,\varepsilon^2\tau^3\mathrm{i}(\boldsymbol{k}\cdot\ddot{\boldsymbol{v}}) - \frac{1}{8}\,\varepsilon^2\tau^4(\boldsymbol{k}\cdot\dot{\boldsymbol{v}})^2\Big) \\
&\times \big((|\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|\tau)\mathrm{i}\boldsymbol{k} - (\cos |\boldsymbol{k}|\tau)\boldsymbol{v} - (|\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|\tau)(\boldsymbol{v} \times (\mathrm{i}\boldsymbol{k} \times \boldsymbol{v}))\big)\Big\} \\
&+ \mathcal{O}(\varepsilon^2)\,. \qquad (7.18)
\end{aligned}$$

The terms proportional to ε^{-1} cancel by symmetry. We sort all other terms,

$$\begin{aligned}
\boldsymbol{F}^{\varepsilon}_{\text{self}}(t) = e^2\!\int \mathrm{d}^3k\, |\widehat{\varphi}(\boldsymbol{k})|^2 \Big\{ &\big(- (\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\nabla_{\boldsymbol{v}} + \dot{\boldsymbol{v}}(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})\big) \int_0^{\varepsilon^{-1}t} \mathrm{d}\tau \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v})\tau} (|\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|\tau) \\
&+ \big(\dot{\boldsymbol{v}} + \frac{1}{2}\,\boldsymbol{v}(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\big) \int_0^{\varepsilon^{-1}t} \mathrm{d}\tau\, \tau \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v})\tau} (\cos |\boldsymbol{k}|\tau) + \varepsilon\Big(\frac{1}{2}\big[- (\boldsymbol{v}^2 - 1) \\
&\times (\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\nabla_{\boldsymbol{v}} + \boldsymbol{v}(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}}) + (\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\nabla_{\boldsymbol{v}} - \ddot{\boldsymbol{v}}(\boldsymbol{v}\cdot\nabla_{v})\big] \\
&+ \frac{1}{6}\big[- (1 - \boldsymbol{v}^2)(\ddot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\nabla_{\boldsymbol{v}} - \boldsymbol{v}(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}}) + 3(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\nabla_{\boldsymbol{v}} \\
&- 3\dot{\boldsymbol{v}}(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\big] + \frac{1}{8}\big[(\boldsymbol{v}^2 - 1)(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})^2\nabla_{\boldsymbol{v}} \\
&- \boldsymbol{v}(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})^2\big]\Big) \int_0^{\varepsilon^{-1}t} \mathrm{d}\tau\, \tau \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v})\tau} (|\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|\tau) \\
&+ \varepsilon\Big(- \ddot{\boldsymbol{v}} - \frac{1}{6}\big[\boldsymbol{v}(\ddot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}}) + 3\dot{\boldsymbol{v}}(\dot{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{v}})\big]\Big) \int_0^{\varepsilon^{-1}t} \mathrm{d}\tau\, \tau^2 \mathrm{e}^{-i(\boldsymbol{k}\cdot\boldsymbol{v})\tau} \cos |\boldsymbol{k}|\tau \Big\} \\
&+ \mathcal{O}(\varepsilon^2)\,. \qquad (7.19)
\end{aligned}$$

https://doi.org/10.1017/9781009402286.008 Published online by Cambridge University Press

To take the limit $\varepsilon \to 0$ we go back to position space and use the fundamental solution of the wave equation. Then

$$\lim_{\varepsilon\to 0}\int_0^{\varepsilon^{-1}t} d\tau \int d^3k |\widehat{\varphi}(\boldsymbol{k})|^2 e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})\tau} (|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\,\tau^p$$
$$= \int_0^\infty dt \int d^3x \int d^3y\, \varphi(\boldsymbol{x})\varphi(\boldsymbol{y}) \frac{1}{4\pi t}\delta(|\boldsymbol{x}+\boldsymbol{v}t-\boldsymbol{y}|-t)\,t^p$$
$$= \begin{cases} \int d^3k\, |\widehat{\varphi}(\boldsymbol{k})|^2[\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-1} & \text{for} \quad p=0\,, \\ \int d^3x\, \varphi(\boldsymbol{x}) \int d^3y\varphi(\boldsymbol{y})\,(\gamma^2/4\pi) & \text{for} \quad p=1\,. \end{cases} \tag{7.20}$$

By the same method

$$\lim_{\varepsilon\to 0}\int_0^{\varepsilon^{-1}t} d\tau \int d^3k\, |\widehat{\varphi}(\boldsymbol{k})|^2 e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})\tau}\, \tau^{1+p}\frac{d}{d\tau}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)$$
$$= -\big(1+p+(\boldsymbol{v}\cdot\nabla_{\boldsymbol{v}})\big)\int_0^\infty dt \int d^3k\, |\widehat{\varphi}(\boldsymbol{k})|^2 e^{-i(\boldsymbol{k}\cdot\boldsymbol{v})t}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|t)t^p$$
$$= \begin{cases} -\int d^3k\, |\widehat{\varphi}(\boldsymbol{k})|^2(\boldsymbol{k}^2+(\boldsymbol{k}\cdot\boldsymbol{v})^2)[\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{v})^2]^{-2} & \text{for} \quad p=0\,, \\ -\int d^3x\, \varphi(\boldsymbol{x}) \int d^3y\varphi(\boldsymbol{y})\,(2\gamma^4/4\pi) & \text{for} \quad p=1\,. \end{cases} \tag{7.21}$$

Collecting all terms the final result reads

$$\begin{aligned} \boldsymbol{F}^{\varepsilon}_{\text{self}}(t) = &-m_{\text{f}}(\boldsymbol{v})\dot{\boldsymbol{v}} + \varepsilon(e^2/6\pi)\big[\gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 3\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{v} \\ &+3\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \gamma^2\ddot{\boldsymbol{v}}\big] + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{7.22}$$

for $t > 0$ with

$$\begin{aligned} m_{\text{f}}(\boldsymbol{v}) = \; m_{\text{e}}\Big[&\big(|\boldsymbol{v}|^{-2}\gamma^2(3-\boldsymbol{v}^2) - |\boldsymbol{v}|^{-3}(3+\boldsymbol{v}^2)\text{arctanh}|\boldsymbol{v}|\big)\widehat{\boldsymbol{v}}\otimes\widehat{\boldsymbol{v}} \\ &+ \big(-|\boldsymbol{v}|^{-2} + |\boldsymbol{v}|^{-3}(1+\boldsymbol{v}^2)\,\text{arctanh}|\boldsymbol{v}|\big)\mathbb{1}\Big]\,. \end{aligned} \tag{7.23}$$

Note that $m_{\text{f}}(\boldsymbol{v}) = d(\boldsymbol{P}_{\text{s}} - m_{\text{b}}\gamma\boldsymbol{v})/d\boldsymbol{v}$ as a 3×3 matrix.

Up to order ε, $\boldsymbol{F}^{\varepsilon}_{\text{self}}(t)$ consists of two parts of a rather different character. The term $-m_{\text{f}}(\boldsymbol{v})\dot{\boldsymbol{v}}$ is the contribution from the electromagnetic field to the change in total momentum. We computed this term already in section 4.1 via a completely different route. As emphasized there, since the Abraham model is semirelativistic, the velocity dependence of m_{f} has no reason to be of relativistic form and indeed it is not. The term proportional to ε in (7.22) is the *radiation reaction*. Again there

https://doi.org/10.1017/9781009402286.008 Published online by Cambridge University Press

is no a priori reason to expect it to be relativistic, but in fact it is. Using the four-vector notation of section 2.5, the radiation reaction can be rewritten as

$$\varepsilon(e^2/6\pi)(\ddot{\mathbf{u}} - (\dot{\mathbf{u}}\cdot\dot{\mathbf{u}})\mathbf{u}) = \varepsilon(e^2/6\pi)(\mathbf{g}+\mathbf{u}\otimes\mathbf{u})\cdot\ddot{\mathbf{u}}\,. \tag{7.24}$$

The space part is the term proportional to e^2 of (7.22), i.e. the radiation reaction force, and the time part is the work done by this force per unit time.

7.3 How can the acceleration be bounded?

We return to the microscopic time scale. From the conservation of energy together with condition (P), we have

$$\begin{aligned} E_{\mathrm{s}}(\boldsymbol{v}^0) + e\phi(\varepsilon\boldsymbol{q}^0) &= \mathcal{E}(\boldsymbol{E}^0, \boldsymbol{B}^0, \boldsymbol{q}^0, \boldsymbol{v}^0) = \mathcal{E}(\boldsymbol{E}(t), \boldsymbol{B}(t), \boldsymbol{q}(t), \boldsymbol{v}(t)) \\ &\geq m_{\mathrm{b}}\gamma(\boldsymbol{v}(t)) + e\,\bar{\phi} \end{aligned} \tag{7.25}$$

and therefore

$$\sup_{t\in\mathbb{R}} |\boldsymbol{v}(t)| \leq \bar{v} < 1\,. \tag{7.26}$$

In (6.4) the external forces are of order ε. Superficially the self-force is of order one. However for a Coulombic charge soliton field the self-force vanishes. Thus if we could show that the deviations from the appropriate local soliton field are of order ε, then the acceleration would satisfy

$$\sup_{t\in\mathbb{R}} |\dot{\boldsymbol{v}}(t)| \leq C\,\varepsilon \tag{7.27}$$

with C a suitable constant. This is what we want to prove. We will not keep track of the constants, and the value of C changes from equation to equation. We make sure, however, that the e-dependence is explicit and that C depends only on $\overline{v}$, and thus is determined by the initial conditions. Of course, to justify the Taylor expansion of section 7.2, we also need analogous estimates of higher derivatives, which can be obtained with considerably more effort through the same scheme. Here we want to explain how to get (7.27) and why we need e to be sufficiently small, at least for the moment.

From the equations of motion one has

$$\dot{\boldsymbol{v}} = m_0(\boldsymbol{v})^{-1}\left[\varepsilon e\big(\boldsymbol{E}_{\mathrm{ex}}(\varepsilon\boldsymbol{q}) + \boldsymbol{v}\times\boldsymbol{B}_{\mathrm{ex}}\,(\varepsilon\boldsymbol{q})\big) + e\big(\boldsymbol{E}_{\varphi}(\boldsymbol{q}) + \boldsymbol{v}\times\boldsymbol{B}_{\varphi}(\boldsymbol{q})\big)\right], \tag{7.28}$$

where $m_0^{-1}(\boldsymbol{v}) = (m_{\mathrm{b}}\gamma)^{-1}(\mathbb{1} - \widehat{\boldsymbol{v}}\otimes\widehat{\boldsymbol{v}})$ is the matrix inverse of $m_0(\boldsymbol{v})$. Clearly by (7.26) we have $\|m_0(\boldsymbol{v})^{-1}\| \leq C$ and, by condition (P), the first term is bounded as

$$\varepsilon\left|e\big(\boldsymbol{E}_{\mathrm{ex}}(\varepsilon\boldsymbol{q}) + \boldsymbol{v}\times\boldsymbol{B}_{\mathrm{ex}}(\varepsilon\boldsymbol{q})\big)\right| \leq C|e|\varepsilon\,. \tag{7.29}$$

https://doi.org/10.1017/9781009402286.008 Published online by Cambridge University Press

On the other hand, the self-force looks to be of order one. To reduce it in order we have to exploit the fact that $\boldsymbol{E}$, $\boldsymbol{B}$ deviate only slightly from $\boldsymbol{E}_v$, $\boldsymbol{B}_v$ close to the charge distribution, i.e. we subtract zero and rewrite the self-force as

$$e\big(\boldsymbol{E}_\varphi(\boldsymbol{q}) - \boldsymbol{E}_{v\varphi}(\boldsymbol{q}) + \boldsymbol{v} \times (\boldsymbol{B}_\varphi(\boldsymbol{q}) - \boldsymbol{B}_{v\varphi}(\boldsymbol{q}))\big) . \tag{7.30}$$

Our goal is to show that this difference is of order ε.

Let us define then

$$Z(\boldsymbol{x}, t) = \begin{pmatrix} \boldsymbol{E}(\boldsymbol{x}, t) - \boldsymbol{E}_{v(t)}(\boldsymbol{x} - \boldsymbol{q}(t)) \\ \boldsymbol{B}(\boldsymbol{x}, t) - \boldsymbol{B}_{v(t)}(\boldsymbol{x} - \boldsymbol{q}(t)) \end{pmatrix} . \tag{7.31}$$

Using Maxwell's equations and the relations $(\boldsymbol{v} \cdot \nabla)\, \boldsymbol{E}_v = -\nabla \times \boldsymbol{B}_v + e\varphi \boldsymbol{v}$, $(\boldsymbol{v} \cdot \nabla)\boldsymbol{B}_v = \nabla \times \boldsymbol{E}_v$ one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t} Z(t) = A Z(t) - g(t) , \tag{7.32}$$

where A is defined in (2.18) and

$$g(\boldsymbol{x}, t) = \begin{pmatrix} (\dot{\boldsymbol{v}}(t) \cdot \nabla_v)\boldsymbol{E}_v(\boldsymbol{x} - \boldsymbol{q}(t)) \\ (\dot{\boldsymbol{v}}(t) \cdot \nabla_v)\boldsymbol{B}_v(\boldsymbol{x} - \boldsymbol{q}(t)) \end{pmatrix} . \tag{7.33}$$

Therefore (7.32) has again the structure of the inhomogeneous Maxwell equations. Since $Z(0) = 0$ by our assumption on the initial data, one has

$$Z(t) = -\int_0^t \mathrm{d}s\, U(t - s) g(s) . \tag{7.34}$$

In terms of $Z(t)$, using (7.28), (7.30), the acceleration is bounded through

$$|\dot{\boldsymbol{v}}(t)| \le C(\varepsilon + |e|) \int \mathrm{d}^3x \varphi(\boldsymbol{x}) |\boldsymbol{Z}_1(\boldsymbol{x} + \boldsymbol{q}(t), t) + \boldsymbol{v}(t) \times \boldsymbol{Z}_2(\boldsymbol{x} + \boldsymbol{q}(t), t)| . \tag{7.35}$$

Let us set $W(t, s) = U(t - s)g(s)$. Below we will prove that

$$|\boldsymbol{W}_1(t, s, \boldsymbol{q}(t) + \boldsymbol{x})| + |\boldsymbol{W}_2(t, s, \boldsymbol{q}(t) + \boldsymbol{x})| \le |e|C|\dot{\boldsymbol{v}}(s)|(1 + (t - s)^2)^{-1} \tag{7.36}$$

for $|\boldsymbol{x}| \le R_\varphi$. Therefore inserting (7.36) in (7.35) one obtains

$$|\dot{\boldsymbol{v}}(t)| \le |e|C\bigg(\varepsilon + |e| \int_0^t \mathrm{d}s\, (1 + (t - s)^2)^{-1}\, |\dot{\boldsymbol{v}}(s)|\bigg) . \tag{7.37}$$

https://doi.org/10.1017/9781009402286.008 Published online by Cambridge University Press

Let $\kappa = \sup_{t\geq 0} |\dot{\boldsymbol{v}}(t)|$. Then (7.37) reads

$$\kappa \leq |e|\, C\Big(\varepsilon + |e|\kappa \int_0^\infty \mathrm{d}s\, (1+s^2)^{-1}\Big), \quad \kappa \leq \frac{|e|\, C}{1-e^2\, C}\,\varepsilon\,. \tag{7.38}$$

From the computation below we will see that C depends on $\bar{v}$ (and on model parameters like the form factor $\widehat{\varphi}$), but not on e. Thus taking $|e|$ sufficiently small one can ensure $e^2\, C < 1$ and therefore $\kappa \leq C\varepsilon$ as claimed.

We still have to establish (7.36). $U(t)$ is given in Eqs. (2.12), (2.13). Since $\nabla\cdot \boldsymbol{g}_1(s) = 0 = \nabla\cdot\boldsymbol{g}_2(s)$, the term proportional to $\boldsymbol{k}\otimes\boldsymbol{k}$ drops out. In real space $|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|t$ becomes G_t from (2.15) and $\cos|\boldsymbol{k}|t$ becomes $\partial_t G_t$. Therefore

$$\begin{aligned}
\boldsymbol{W}_1(t,s,\boldsymbol{x}) &= \frac{1}{4\pi(t-s)^2}\int \mathrm{d}^3y\, \delta(|\boldsymbol{x}-\boldsymbol{y}|-(t-s)) \\
&\quad \times \big[(t-s)\nabla\times \boldsymbol{g}_2(\boldsymbol{y},s) + \boldsymbol{g}_1(\boldsymbol{y},s) - (\boldsymbol{x}-\boldsymbol{y})\cdot\nabla \boldsymbol{g}_1(\boldsymbol{y},s)\big], \\
\boldsymbol{W}_2(t,s,\boldsymbol{x}) &= \frac{1}{4\pi(t-s)^2}\int \mathrm{d}^3y\, \delta(|\boldsymbol{x}-\boldsymbol{y}|-(t-s)) \\
&\quad \times \big[-(t-s)\nabla\times \boldsymbol{g}_1(\boldsymbol{y},s) + \boldsymbol{g}_2(\boldsymbol{y},s) - (\boldsymbol{x}-\boldsymbol{y})\cdot\nabla \boldsymbol{g}_2(\boldsymbol{y},s)\big].
\end{aligned} \tag{7.39}$$

We insert g from (7.33). $\boldsymbol{E}_v$ and $\boldsymbol{B}_v$ are first-order derivatives of the function $\phi_{v\varphi}$ which according to (4.7) is given by

$$\phi_{v\varphi}(\boldsymbol{x}) = e\int \mathrm{d}^3y \varphi(\boldsymbol{x}-\boldsymbol{y})(4\pi)^{-1}\big[\big((1-\boldsymbol{v}^2)\boldsymbol{y}^2 + (\boldsymbol{v}\cdot\boldsymbol{y})^2\big)\big]^{-1/2}. \tag{7.40}$$

Using (4.5) one has componentwise

$$\begin{aligned}
&|\nabla_v \boldsymbol{E}_v(\boldsymbol{x})| + |\nabla_v \boldsymbol{B}_v(\boldsymbol{x})| \leq C\,(\,|\nabla\phi_v(\boldsymbol{x})| + |\nabla\nabla_v\phi_v(\boldsymbol{x})|)\,, \\
&|\nabla\nabla_v \boldsymbol{E}_v(\boldsymbol{x})| + |\nabla\nabla_v \boldsymbol{B}_v(\boldsymbol{x})| \leq C\,(\,|\nabla\nabla_v\phi_v(\boldsymbol{x})| + |\nabla\nabla\nabla_v\phi_v(\boldsymbol{x})|)
\end{aligned} \tag{7.41}$$

and taking successive derivatives in (7.40) one obtains the bounds

$$\begin{aligned}
&|\nabla\phi_v(\boldsymbol{x})| + |\nabla\nabla_v\phi_v(\boldsymbol{x})| \leq e\,C\,(1+|\boldsymbol{x}|)^{-2}\,, \\
&|\nabla\nabla\phi_v(\boldsymbol{x})| + |\nabla\nabla\nabla_v\phi_v(\boldsymbol{x})| \leq e\,C\,(1+|\boldsymbol{x}|)^{-3}\,,
\end{aligned} \tag{7.42}$$

which imply

$$\begin{aligned}
&|\boldsymbol{g}_1(\boldsymbol{x},s)| + |\boldsymbol{g}_2(\boldsymbol{x},s)| \leq e\,C|\dot{\boldsymbol{v}}(s)|(1+|\boldsymbol{x}-\boldsymbol{q}(s)|^2)^{-1}\,, \\
&|\nabla\boldsymbol{g}_1(\boldsymbol{x},s)| + |\nabla\boldsymbol{g}_2(\boldsymbol{x},s)| \leq e\,C|\dot{\boldsymbol{v}}(s)|(1+|\boldsymbol{x}-\boldsymbol{q}(s)|^3)^{-1}\,.
\end{aligned} \tag{7.43}$$

We insert the bound (7.43) in (7.39) which results in an upper bound on $\mathcal{W}(t, s, \boldsymbol{q}(t) + \boldsymbol{x})$. Using the condition that $|\boldsymbol{x}| \leq R_\varphi$ and $|\boldsymbol{q}(t) - \boldsymbol{q}(s)| \leq \bar{v}|t - s|$ finally yields (7.36).

We summarize our findings as

Theorem 7.1 (Bounds on the velocity and its derivatives). *For the Abraham model satisfying conditions* (C), (P), *and* (I) *there exist constants* C, *depending through* $\bar{v}$ *only on the initial conditions, and* $\bar{e}$ *such that*

$$|\boldsymbol{v}(t)| \leq \bar{v} < 1\,, \quad |\dot{\boldsymbol{v}}(t)| \leq C\varepsilon\,, \quad |\ddot{\boldsymbol{v}}(t)| \leq C\big(\varepsilon^2 + \varepsilon(1 + |t|)^{-2}\big)\,,$$
$$|\dddot{\boldsymbol{v}}(t)| \leq C\big(\varepsilon^3 + \varepsilon^2(1 + |t|)^{-2} + \varepsilon(1 + |t|)^{-3}\big) \tag{7.44}$$

for all t *on the microscopic time scale, provided the charge is sufficiently small, i.e.* $|e| < \bar{e}$.

By keeping track of the constant C, one could get a bound on the charge admissible in Theorem 7.1. Since we believe this restriction to be an artifact of the method anyhow, there is no point in the effort.

Notes and references

Section 7.1

Sommerfeld (1904a, 1905) systematically uses memory equations. In fact he considers the Abraham model with the kinetic energy $m_\mathrm{b}\boldsymbol{v}^2/2$ for the particle and wants to understand what happens when $\boldsymbol{v}(0) > c$. He argues that the particle rapidly loses its energy to become slower than c by emitting what we now call Čerenkov radiation. The differential–difference equation (7.14) is derived by Page (1918) and its relativistic generalization by Caldirola (1956). For reviews we refer to Erber (1961) and Pearle (1982). Moniz and Sharp (1974, 1977) supply a linear stability analysis and show that the solutions to (7.14) are stable provided R_φ is not too small. For that reason Rohrlich (1997) regards (7.14) and its relativistic sister as the fundamental starting point for the classical dynamics of extended charges. We take the Abraham model as the basic dynamical theory. Memory equations are a useful tool in analyzing its properties.

Section 7.2

The Taylor expansion is taken from Kunze and Spohn (2000a). Such an expansion was already used in Sommerfeld (1904a, 1905), to be repeated in various disguises. The traditional expansion parameter is the size of the charge distribution, which in our context is replaced by the scaling parameter ε controlling the variation of the potentials.

Section 7.3

The contraction argument appears in Komech, Kunze and Spohn (1999). The bound on $\dot{v}(t)$ is taken from Kunze and Spohn (2000a), where also higher derivatives are discussed. It is claimed that $|\ddot{v}(t)| \leq C\varepsilon^2$ and $|\dddot{v}(t)| \leq C\varepsilon^3$. In the argument some initial terms are overlooked and the correct bounds are as given in (7.44).