TRACES OF MATRICES OF ZEROS AND ONES

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1. Introduction. This paper continues the study appearing in **(9)** and **(10)** of the combinatorial properties of a matrix A of m rows and n columns, all of whose entries are 0's and 1's. Let the sum of row i of A be denoted by r_i and let the sum of column j of A be denoted by s_j. We call $R = (r_1, \ldots, r_m)$ the *row sum vector* and $S = (s_1, \ldots, s_n)$ the *column sum vector* of A. The vectors R and S determine a class

$$\mathfrak{A} = \mathfrak{A}(R, S) \tag{1.1}$$

consisting of all $(0, 1)$-matrices of m rows and n columns, with row sum vector R and column sum vector S. The majorization concept yields simple necessary and sufficient conditions on R and S in order that the class $\mathfrak{A}$ be non-empty **(4; 9)**. Generalizations of this result and a critical survey of a wide variety of related problems are available in **(6)**.

Consider the 2 by 2 submatrices of A of the types

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

An *interchange* is a transformation of the elements of A which changes a minor of type A_1 into type A_2, or vice versa, and leaves all other elements of A unaltered. The interchange theorem (9) asserts that if A and A^* belong to A, then A is transformable into A^* by interchanges.

The *term rank* ρ of A is the order of the greatest minor of A with a non-zero term in its determinant expansion **(8)**. This integer equals the minimal number of rows and columns which contain collectively all of the non-zero elements of A **(7)**. Now let $\bar{\rho}$ be the maximal and $\tilde{\rho}$ the minimal term rank for the matrices in $\mathfrak{A}$. The interchange theorem implies the existence of a matrix A in $\mathfrak{A}$ of term rank ρ **(9)**. Here ρ is an arbitrary integer in the interval

$$\tilde{\rho} \leqslant \rho \leqslant \bar{\rho}. \tag{1.2}$$

Let $\delta_i = (1, \ldots, 1, 0, \ldots, 0)$ be a vector of n components, with 1's in the first r_i positions and 0's elsewhere. The matrix

$$\bar{A} = \begin{bmatrix} \delta_1 \\ \cdots \\ \delta_m \end{bmatrix} \tag{1.3}$$

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is called *maximal*, and $\bar{A}$ is the *maximal form* of A. Suppose that the components of R and S are positive. Define

$$R' = (r_1 - 1, \ldots, r_m - 1)$$

and let $\bar{A}'$ be the maximal matrix of m rows and n columns with row sum vector R'. Let the column sum vector of $\bar{A}'$ equal

$$\bar{S}' = (\bar{s}'_1, \ldots, \bar{s}'_n).$$

Renumber the subscripts of the column sum vector $S = (s_1, \ldots, s_n)$ so that

$$s_1 \geqslant \ldots \geqslant s_n.$$

Define

$$s'_i = s_i - 1 \qquad (i = 1, \ldots, n),$$
$$\bar{s}'_0 = s'_0 = 0,$$

and let

$$M = \max\left(\sum_{i=0}^{k} (s'_i - \bar{s}'_i)\right) \qquad (k = 0, \ldots, n). \tag{1.4}$$

One may prove **(10)**

$$\bar{\rho} = m - M. \tag{1.5}$$

A simple formula for $\tilde{\rho}$ analogous to (1.5) for $\bar{\rho}$ does not appear to exist. However, Haber in a forthcoming paper obtains an algorithm that yields an effective procedure for the determination of $\tilde{\rho}$ (5).

Throughout the discussion we suppose that $\mathfrak{A}$ is non-empty and that $R = (r_1, \ldots, r_m)$ and $S = (s_1, \ldots, s_n)$ satisfy

$$r_1 \geqslant \ldots \geqslant r_m > 0, \tag{1.6}$$

$$s_1 \geqslant \ldots \geqslant s_n > 0. \tag{1.7}$$

We call the above R and S and the associated $\mathfrak{A}$ of (0, 1)-matrices *normalized*. Term rank is invariant under permutations of rows and columns. Thus normalization does not restrict this concept. Indeed, formula (1.5) for $\bar{\rho}$ actually requires a normalized S.

For $A = [a_{rs}]$ in $\mathfrak{A}$ we define the *trace* of A by

$$\operatorname{tr}(A) = \sum_{i=1}^{\epsilon} a_{ii}, \tag{1.8}$$

where

$$\epsilon = \min(m, n). \tag{1.9}$$

Fulkerson has recently investigated feasibility conditions for the existence of a (0, 1)-matrix of order n with specified row and column sums and 0 trace **(3)**. He utilizes the theory of network flows **(1; 2; 4)** and obtains an especially simple criterion for the case in which $\mathfrak{A}$ is normalized **(3)**. Let $\tilde{\sigma}$ be the maximal and $\tilde{\sigma}$ the minimal trace for the matrices in the normalized $\mathfrak{A}$. In the present paper we develop a trace theory for $\bar{\sigma}$ and $\tilde{\sigma}$ analogous to the term rank theory for $\bar{\rho}$ and $\tilde{\rho}$. The requirement of positive components on R and S is without loss of generality. The ordering of the components in accordance with (1.6) and (1.7) does impose a restriction. But this is necessary in order to obtain conclusions of the uncomplicated type to be described.

Let A be in the normalized $\mathfrak{A}$ and write

$$A = \begin{bmatrix} \mathrm{W} & \mathrm{X} \\ \mathrm{Y} & Z \end{bmatrix}, \tag{1.10}$$

where W is of size e by f $(0 \leqslant e \leqslant m; 0 \leqslant f \leqslant n)$. For an arbitrary (0, 1)-matrix Q, let $N_0(Q)$ denote the number of 0's in Q and $N_1(Q)$ the number of 1's in Q. Let

$$t_{ef} = N_0(W) + N_1(Z) \qquad (e = 0, \ldots, m; f = 0, \ldots, n) \tag{1.11}$$

and define

$$T = [t_{ef}] \qquad (e = 0, \ldots, m;\ f = 0, \ldots, n). \tag{1.12}$$

T is called the *structure matrix of the class* $\mathfrak{A}$. Its elementary properties are developed in § 2. Section 3 yields explicit formulae for $\bar{\sigma}$ and $\tilde{\sigma}$ in terms of the entries of T:

$$\bar{\sigma} = \min_{e,f} \{t_{ef} + \max(e, f)\} \tag{1.13}$$

$$\tilde{\sigma} = \max_{e,f} \{\min(e, f) - t_{ef}\} \qquad (e = 0, \ldots, m; f = 0, \ldots, n). \tag{1.14}$$

Matrices with an unusually simple block decomposition are shown to exist for the case of maximal and minimal trace, and these matrices play an essential role in the derivations of (1.13) and (1.14). Section 4 stresses similarities and differences in the behaviour of trace and term rank. The paper concludes with the determination of the domain of intermediate values for the traces σ of the matrices in $\mathfrak{A}$. This usually consists of all integers in the interval

$$\tilde{\sigma} \leqslant \sigma \leqslant \bar{\sigma}. \tag{1.15}$$

But certain classes $\mathfrak{A}$ exclude $\tilde{\sigma} + 1$ and others exclude $\bar{\sigma} - 1$.

2. The structure matrix. Let A belong to the normalized class $\mathfrak{A} = \mathfrak{A}$ (R, S) described in § 1 and write

$$A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}, \tag{2.1}$$

where W is of size e by f $(0 \leqslant e \leqslant m; 0 \leqslant f \leqslant n)$. Let

$$T = [t_{ef}] \qquad (e = 0, \ldots, m;\ f = 0, \ldots, n) \tag{2.2}$$

denote the structure matrix of $\mathfrak{A}$. This means that

$$t_{ef} = N_0(W) + N_1(Z) \quad (e = 0, \ldots, m;\ f = 0, \ldots, n). \tag{2.3}$$

where $N_0(W)$ denotes the number of 0's in W and $N_1(Z)$ the number of 1's in Z. It follows at once from (2.3) that

$$t_{ef} = ef + (r_{e+1} + \ldots + r_m) - (s_1 + \ldots + s_f) \quad (e = 0, \ldots, m;\ f = 0, \ldots, n). \tag{2.4}$$

Thus the structure matrix is independent of the particular choice of A in $\mathfrak{A}$. Note that if

$$\tau = N_1(A) = r_1 + \ldots + r_m, \tag{2.5}$$

then the first row and column of T are given by

$$\begin{aligned} t_{0f} &= \tau - (s_1 + \ldots + s_f), \\ t_{s0} &= \tau - (r_1 + \ldots + r_e). \end{aligned} \tag{2.6}$$

The structure matrix has a number of interesting properties that give insight into the combinatorial behaviour of $\mathfrak{A}$. Its entries are, of course, non-negative integers and its size is $m + 1$ by $n + 1$. For notational convenience we number the rows of a matrix of these dimensions from 0 through m and its columns from 0 through n. Let E_k be the triangular matrix of order $k + 1$, with 1's on and below the main diagonal and 0's elsewhere. Let $E_k{}^T$ denote the transpose of E_k, and number the rows and columns of E_k and $E_k{}^T$ from 0 through k. Let S be the m by n matrix of 1's. Then

$$E_m \begin{bmatrix} \tau & -s_1 & \ldots & -s_n \\ -r_1 & & & \\ \vdots & & S & \\ -r_m & & & \end{bmatrix} E_n^T = T. \tag{2.7}$$

For the eth row vector of the product of the first two matrices on the left side of equation (2.7) is

$$(r_{e+1} + \ldots + r_m, - s_1 + e, \ldots, - s_n + e).$$

If this row is multiplied by the fth column of $E_n{}^T$, then we obtain

$$ef + (r_{e+1} + \ldots + r_m) - (s_1 + \ldots + s_f). \tag{2.8}$$

But by (2.4) this is t_{ef}.

If in (2.4) we replace f by $f + 1$, then

$$t_{e,f+1} = ef + e + (r_{e+1} + \ldots + r_m) - (s_1 + \ldots + s_{f+1}) \quad (e = 0, \ldots, m;\ f = 0, \ldots, n - 1). \tag{2.9}$$

By (2.4) and (2.9),

$$t_{e,f+1} = t_{ef} + e - s_{f+1} \qquad (e = 0, \ldots, m;\ f = 0, \ldots, n-1). \tag{2.10}$$

Similarly, we may deduce

$$t_{e+1,f} = t_{ef} + f - r_{e+1} \qquad (e = 0, \ldots, m-1;\ f = 0, \ldots, n). \tag{2.11}$$

The recursions (2.10) and (2.11) are useful in constructing T from a given R and S.

From (2.10) we see that the eth row of T may be written in the form

$$(t_{e0}, t_{e0} + e - s_1, t_{e1} + e - s_2, \ldots, t_{e,n-1} + e - s_n). \tag{2.12}$$

S is normalized so that by (2.12), if $e \leqslant s_n$, then

$$t_{e0} \geqslant t_{e1} \geqslant \ldots \geqslant t_{en}, \tag{2.13}$$

and if $e \geqslant s_1$, then

$$t_{e0} \leqslant t_{e1} \leqslant \ldots \leqslant t_{en}. \tag{2.14}$$

On the other hand, if $s_n < e < s_1$, then there must exist an integer $f (0 < f < n)$ such that

$$t_{e0} \geqslant t_{e1} \geqslant \ldots \geqslant t_{ef} \leqslant t_{e,f+1} \leqslant \ldots \leqslant t_{en}. \tag{2.15}$$

The columns of T have an analogous monotonic behaviour.

The following numerical example affords a simple illustration of the preceding remarks:

$$R = (4, 3, 2, 2, 1), \qquad S = (4, 4, 2, 1, 1),$$

$$T = \begin{bmatrix} 12 & 8 & 4 & 2 & 1 & 0 \\ 8 & 5 & 2 & 1 & 1 & 1 \\ 5 & 3 & 1 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 & 4 & 6 \\ 1 & 1 & 1 & 3 & 6 & 9 \\ 0 & 1 & 2 & 5 & 9 & 13 \end{bmatrix}.$$

3. Maximal and minimal traces. Let $\bar{\sigma}$ be the maximal and $\tilde{\sigma}$ the minimal trace for the matrices in the normalized class $\mathfrak{A}$. In this section we develop simple block decompositions for the matrices of maximal and minimal trace and use these decompositions to derive (1.13) and (1.14). We begin with an elementary property of the trace function for the class $\mathfrak{A}$.

THEOREM 3.1. *Suppose the normalized $\mathfrak{A}$ contains a matrix of trace σ. Then there exists an $A = [a_{rs}]$ in $\mathfrak{A}$ of trace σ with the 1's in the initial positions on the main diagonal*

$$a_{11} = \ldots = a_{\sigma\sigma} = 1, \quad a_{\sigma+1,\sigma+1} = \ldots = a_{\epsilon\epsilon} = 0 \qquad (\epsilon = \min(m, n)). \tag{3.1}$$

For suppose that we have an A in $\mathfrak{A}$ of trace σ with

$$a_{11} = \ldots = a_{e-1,e-1} = 1 \quad (e - 1 < \sigma), \qquad a_{ee} = 0. \tag{3.2}$$

It suffices to show that it is possible to transform A by interchanges into a matrix of trace σ with the e leading diagonal elements equal to 1. Now there must exist an integer $t > \sigma$ such that $a_{tt} = 1$. Suppose that

$$a_{et} = a_{te} = 0. \tag{3.3}$$

Since $r_e \geqslant r_t$ and $s_e \geqslant s_t$, there exist integers u and v such that $a_{ue} = 1$, $a_{ut} = 0$ and $a_{ev} = 1$, $a_{tv} = 0$. We apply an interchange involving positions (e, t), (e, v), (t, v), (t, t), and follow this by an interchange involving positions (e, e), (e, t), (u, t), (u, e). This gives a 0 in the (t, t) position and a 1 in the (e, e) position. The other diagonal elements are not altered. The remaining cases

$$a_{et} = a_{te} = 1, \tag{3.4}$$

$$a_{et} = 1, \qquad a_{te} = 0, \tag{3.5}$$

$$a_{et} = 0, \qquad a_{te} = 1, \tag{3.6}$$

are disposed of by similar arguments.

We turn now to a study of the maximal trace $\bar{\sigma}$ for matrices in the class $\mathfrak{A}$.

THEOREM 3.2. *Let $\bar{\sigma} \neq \min(m, n)$. Then there exists a matrix $A_{\bar{\sigma}}$ of trace $\bar{\sigma}$ in the normalized $\mathfrak{A}$ of the form*

$$A_{\bar{\sigma}} = \begin{bmatrix} S & * & * \\ * & \tilde{0} & 0 \\ * & 0 & 0 \end{bmatrix}. \tag{3.7}$$

Here S is a matrix of 1's of specified size e by f $(0 < e \leqslant \bar{\sigma};\ 0 < f \leqslant \bar{\sigma})$. The matrix $\tilde{0}$ is of size g by h and has 1's in the main diagonal positions of $A_{\bar{\sigma}}$ and 0's in all other positions. Moreover,

$$e + g = f + h = \bar{\sigma}. \tag{3.8}$$

The 0's denote zero matrices.

Consider a matrix A in $\mathfrak{A}$ with the $\bar{\sigma}$ 1's in the initial positions on the main diagonal. We have $\bar{\sigma} > 0$. The block in the lower right corner of size $m - \bar{\sigma}$ by $n - \bar{\sigma}$ must be a zero block. For the row and column sum vectors are normalized and if the block contained a 1, then a suitable interchange would increase $\bar{\sigma}$. Now the matrix A may be selected to be of the following form:

$$A = \begin{bmatrix} S_1 & * & R_1 \\ * & * & 0_1 \\ C_1 & 0_2 & 0 \end{bmatrix}. \tag{3.9}$$

Here 0 is the zero block of size $m - \bar{\sigma}$ by $n - \bar{\sigma}$, 0_1 and 0_2 are zero blocks, R_1 has at least one 1 in each row, and C_1 has at least one 1 in each column. For let us consider two vectors X_1 and X_2 from among the first $\bar{\sigma}$ rows of A and let the entries of these vectors total x_1 and x_2, resepctively. Let X_1 have 0's in its last $n - \bar{\sigma}$ positions and let X_2 have a 1 in at least one of these positions. Suppose X_1 is above X_2 in A. If $x_1 > x_2$, then we may apply an interchange involving X_1 and X_2 that does not shift a 1 on the main diagonal and places a 1 in one of the last $n - \bar{\sigma}$ positions of X_1. If $x_1 = x_2$, we may still apply the interchange and place a 1 in one of the last $n - \bar{\sigma}$ positions of X_1. However, in exceptional cases a single interchange may be available for this purpose and this may force a reduction in trace to $\bar{\sigma} - 1$. But if this is the case, then a second interchange confined to the first $\bar{\sigma}$ columns restores trace $\bar{\sigma}$. This procedure yields the blocks R_1 and 0_1 of (3.9). Next we work on the columns and in the same way. Again a single interchange may be available and force a reduction in trace to $\bar{\sigma} - 1$. But under these circumstances there is always available a second interchange confined to the first $\bar{\sigma}$ rows and columns that regains trace $\bar{\sigma}$. This is the case since otherwise an interchange exists that restores the 1 to the main diagonal and places a 1 in the block 0, contradicting the maximality of $\bar{\sigma}$. This gives us a matrix of the form (3.9).

Now let S_1 of (3.9) be of size $\bar{e}$ by $\bar{f}$. S_1 must be a block of 1's, for otherwise we could increase $\bar{\sigma}$. An A of the form (3.9) with fixed $\bar{e}$ and $\bar{f}$ we call *reduced*. The $\bar{\sigma}$ 1's on its main diagonal we call *essential* 1's. All other 1's are called *unessential*. Without loss of generality we may assume $\bar{e} \leqslant \bar{f}$.

We now consider a reduced A^* in $\mathfrak{A}$ of the form

$$A^* = \left[\begin{array}{cc|c|c} S_1 & S_2 & Y & R_1 \\ \hline \multicolumn{2}{c|}{X} & Z & 0_1 \\ \hline C_1 & \multicolumn{2}{c|}{0_2} & 0 \end{array}\right]. \tag{3.10}$$

Here S_2 is a block of 1's of size $\bar{e}$ by $f^* - \bar{f}$ with $f^* - \bar{f}$ maximal in A^*. Among all reduced A in $\mathfrak{A}$ we select that A^* with its corresponding $f^* - \bar{f}$ minimal. We must allow the case $f^* - \bar{f} = 0$. But if the minimal $f^* - \bar{f} > 0$, then S_2 is a block of 1's that appears in all of the reduced A in $\mathfrak{A}$. If Y is not present, then (3.7) holds with $h = 0$. Suppose then that Y is present. Then our A^* has a 0 in the first column of Y. If the block Z contains no unessential 1, then (3.7) holds with $f = f^*$. Suppose, therefore, that an unessential 1 appears in the (s, t) position of Z, where s is maximal in Z. If $t = 1$, then there is a 0 in column t of Y. Suppose that $t \neq 1$ and let the $(s, 1)$ position of Z contain

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a 0 or an essential 1. If column t of Y contains only 1's, then we may perform an interchange using unessential 1's and the 0 in column 1 of Y to obtain a 0 in column t of Y. We henceforth require A^* to have a 0 in column t (but no longer column 1) of Y.

The preceding remarks imply that the entries in row s of X must be 1's. For if this is not the case then a single interchange gives a reduced A^* with a 0 in S_2, contradicting $f^* - \bar{f}$ minimal, or else two interchanges place a 1 in 0, contradicting $\bar{\sigma}$ maximal. Suppose that X has a 0 present in its (u, v) position, where $u < s$. Then we may apply an interchange involving this 0 and the 1 in the (s, v) position of X. If this interchange does not involve an essential 1, then a second interchange involving the unessential 1 in the (s, t) position of Z introduces a 0 into S_1 or S_2. This leads us to the same contradiction as before. Suppose then that the interchange involving the 0 in the (u, v) position and the 1 in the (s, v) position of X does involve an essential 1. Consider the case $s \leqslant f^* - \bar{e}$. Then a second interchange involving the unessential 1 in the (s, t) position of Z regains trace $\bar{\sigma}$ and introduces a 0 into S_1 or S_2. This is again a contradiction. If $s > f^* - \bar{e}$, then the second interchange involving the unessential 1 in the (s, t) position of Z introduces a 0 into S_1 or S_2. However, the trace of the matrix upon completion of this interchange remains at $\bar{\sigma} - 1$. But then we may apply a third interchange involving rows u and s of Z and regain trace $\bar{\sigma}$. Thus in all cases there is no 0 present in the (u, v) position of X, where $u < s$. This gives a matrix of the form (3.7) and completes the proof.

THEOREM 3.3. *The maximal trace $\bar{\sigma}$ for the matrices in the normalized $\mathfrak{A}$ is given by*

$$\bar{\sigma} = \min_{e,f} \{t_{ef} + \max(e, f)\} \qquad (e = 0, \ldots, m; f = 0, \ldots, n). \tag{3.11}$$

Let A be a matrix in $\mathfrak{A}$ of trace $\bar{\sigma}$ with the $\bar{\sigma}$ 1's in the initial positions on the main diagonal. Let A be subdivided into the four blocks W, X, Y, Z of (2.1) with W of size e by f. Now for the matrix A under consideration it is clear that

$$N_1(Z) \geqslant \bar{\sigma} - \max(e, f) \qquad (e = 0, \ldots, m; \ f = 0, \ldots, n). \tag{3.12}$$

But $N_0(W) \geqslant 0$ so that

$$t_{ef} + \max(e, f) = N_0(W) + N_1(Z) + \max(e, f) \geqslant \bar{\sigma} \qquad (e = 0, \ldots, m; \ f = 0, \ldots, n). \tag{3.13}$$

Suppose that $\bar{\sigma} \neq \min(m, n)$. Then we may specialize our A to the $A_{\bar{\sigma}}$ of Theorem 3.2. The submatrix S of $A_{\bar{\sigma}}$ is of size e by f. We may set $W = S$ and obtain $N_0(W) = 0$ and $N_1(Z) = \bar{\sigma} - \max(e, f)$. Thus if $\bar{\sigma} \neq \min(m, n)$, equality is attained in (3.13) for the dimension numbers e and f of the submatrix S of $A_{\bar{\sigma}}$. If $\bar{\sigma} = m$, equality is attained in (3.13) for $f = 0$ and $e = m$. If $\bar{\sigma} = n$, equality is attained in (3.13) for $e = 0$ and $f = n$. This proves Theorem 3.3.

We consider next the minimal trace $\tilde{\sigma}$ for the matrices in $\mathfrak{A}$.

THEOREM 3.4. *Let the matrices in the normalized* $\mathfrak{A}$ *have precisely* u *rows and* v *columns composed entirely of* 1's *and let* $\tilde{\sigma} \neq \max(u, v)$. *Then there exists a matrix* $A_{\tilde{\sigma}}$ *of trace* $\tilde{\sigma}$ *in* $\mathfrak{A}$ *of the form*

$$A_{\tilde{\sigma}} = \begin{bmatrix} S & S_1 & * \\ S_2 & \tilde{S} & * \\ * & * & 0 \end{bmatrix}. \tag{3.14}$$

Here S *is a matrix of* 1's *of order* $\tilde{\sigma}$. S_1 *of size* $\tilde{\sigma}$ *by* s *and* S_2 *of size* t *by* $\tilde{\sigma}$ *are matrices of* 1's. $\tilde{S}$ *is a matrix with* 0's *in the main diagonal positions of* $A_{\tilde{\sigma}}$ *and* 1's *in all other positions.* 0 *is a zero matrix.* (*The cases* $s = 0$ *and* $t = 0$ *are not excluded.*)

Let A be a matrix in $\mathfrak{A}$. If A is not square, then add zero rows at the bottom or zero columns at the right and obtain a square matrix $\tilde{A}$ of order $\max(m, n)$. In $\tilde{A}$ replace the 1's by 0's and the 0's by 1's. This yields a matrix $\tilde{C}$ called the *complement* of $\tilde{A}$. The matrix $\tilde{C}$ determines a class $\mathfrak{C}$. Let $\bar{C}$ be a matrix in $\mathfrak{C}$ of maximal trace $\bar{\sigma}_c$. Evidently

$$\tilde{\sigma} = \max(m, n) - \bar{\sigma}_c. \tag{3.15}$$

The matrix $\bar{C}$ has row sums and column sums in ascending order. Moreover, $\bar{\sigma}_c \neq \max(m, n) - \max(u, v)$, for otherwise $\tilde{\sigma} = \max(u, v)$. We now apply Theorem 3.2 to the block in the lower right corner of $\bar{C}$ of size $\max(m, n) - u$ by $\max(m, n) - v$. This tells us that $\bar{C}$ may be written in the form

$$\bar{C} = \left[\begin{array}{c|c|c} 0 & 0 & * \\ \hline 0 & \begin{array}{c|c} 0 & 0 \\ \hline 0 & \bar{0} \end{array} & * \\ \hline * & * & S \end{array}\right]. \tag{3.16}$$

The 0's denote zero blocks. The 0 in the upper left corner of $\bar{C}$ is of size u by v. S is a block of 1's of size $\bar{e}$ by $\bar{f}$. $\bar{0}$ of size $\bar{g}$ by $\bar{h}$ has 1's in the main diagonal positions of $\bar{C}$ and 0's in all other positions. Moreover,

$$\bar{e} + \bar{g} = \bar{f} + \bar{h} = \bar{\sigma}_c. \tag{3.17}$$

Now take the complement of $\bar{C}$ and delete all zero rows or columns. This yields a matrix $A_{\tilde{\sigma}}$ of the type described in the theorem.

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THEOREM 3.5. *The minimal trace $\tilde{\sigma}$ for the matrices in the normalized $\mathfrak{A}$ is given by*

$$\tilde{\sigma} = \max_{e,f} \{\min(e,f) - t_{ef}\} \qquad (e = 0, \ldots, m; f = 0, \ldots, n). \tag{3.18}$$

Let A be a matrix in $\mathfrak{A}$ of trace $\tilde{\sigma}$ with the $\tilde{\sigma}$ 1's in the initial positions on the main diagonal. Let A be subdivided into the four blocks W, X, Y, Z of (2.1) with W of size e by f. Now for the matrix A under consideration, it is clear that

$$N_0(W) \geqslant \min(e,f) - \tilde{\sigma} \qquad (e = 0, \ldots, m;\ f = 0, \ldots, n). \tag{3.19}$$

But $N_1(Z) \geqslant 0$ so that

$$\min(e,f) - t_{ef} = \min(e,f) - (N_0(W) + N_1(Z)) \leqslant \tilde{\sigma} \qquad (e = 0, \ldots, m;\ f = 0, \ldots, n). \tag{3.20}$$

Suppose that $\tilde{\sigma} \neq \max(u, v)$, where the matrices in $\mathfrak{A}$ have precisely u rows and v columns composed entirely of 1's. Then we may specialize our A to the $A_{\tilde{\sigma}}$ of Theorem 3.4. The matrix $A_{\tilde{\sigma}}$ yields a W, X, Y, Z block subdivision with W of size e by f for which $N_0(W) = \min(e,f) - \tilde{\sigma}$ and $N_1(Z) = 0$. Thus if $\tilde{\sigma} \neq \max(u, v)$, then there exists an e and an f for which equality is attained in (3.20). If $\tilde{\sigma} = u$, equality is attained in (3.20) for $e = \tilde{\sigma}$ and $f = n$. If $\tilde{\sigma} = v$, equality is attained in (3.20) for $e = m$ and $f = \tilde{\sigma}$.

4. Trace and term rank. Let A belong to the normalized class $\mathfrak{A}$, and let $\bar{\rho}$ be the maximal term rank for the matrices in $\mathfrak{A}$. The integer $\bar{\rho}$ is given explicitly by (1.5). We derive a second formula for $\bar{\rho}$ analogous to (3.11) for $\tilde{\sigma}$.

THEOREM 4.1. *The maximal term rank $\bar{\rho}$ for the matrices in the normalized $\mathfrak{A}$ is given by*

$$\bar{\rho} = \min_{e,f} \{t_{ef} + (e + f)\} \qquad (e = 0, \ldots, m; f = 0, \ldots, n). \tag{4.1}$$

Let A be in the normalized $\mathfrak{A}$ and of maximal term rank $\bar{\rho}$. Let A be subdivided into the four blocks W, X, Y, Z of (2.1) with W of size e by f. Now the term rank of a matrix equals the minimal number of rows and columns which contain collectively all of the non-zero elements of the matrix. Hence for the matrix A under consideration, it is clear that

$$N_1(Z) + (e + f) \geqslant \bar{\rho}. \tag{4.2}$$

But $N_0(W) \geqslant 0$ so that

$$t_{ef} + (e + f) = N_0(W) + N_1(Z) + (e + f) \geqslant \bar{\rho}. \tag{4.3}$$

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Suppose that $\bar{\rho} \leqslant \min(m, n)$. Then by Theorem 3.2 of (10) we may specialize our A to a matrix $A_{\bar{\rho}}$ of term rank $\bar{\rho}$ with a W of size e by f for which $N_0(W) = 0$ and $N_1(Z) = \bar{\rho} - (e + f)$. Thus if $\bar{\rho} \neq \min(m, n)$, equality is attained in (4.3) for the dimension numbers e and f of the submatrix W of $A_{\bar{\rho}}$. If $\bar{\rho} = m$, equality is attained in (4.3) for $f = 0$ and $e = m$. If $\bar{\rho} = n$, equality is attained in (4.3) for $e = 0$ and $f = n$. This establishes (4.1).

Let A belong to the normalized class $\mathfrak{A}$. An element $a_{rs} = 1$ of A is an *invariant* 1 provided that no sequence of interchanges applied to A replaces $a_{rs} = 1$ by 0 (10). If $a_{rs} = 1$ is an invariant 1 of A, then the entries in the (r, s) position of all of the matrices in $\mathfrak{A}$ must be invariant 1's. Thus all or none of the matrices in $\mathfrak{A}$ contains an invariant 1, and we say $\mathfrak{A}$ is with or without an invariant 1. The normalized class $\mathfrak{A}$ is with an invariant 1 if and only if the matrices in $\mathfrak{A}$ are of the form

$$A = \begin{bmatrix} S & * \\ * & 0 \end{bmatrix}. \tag{4.4}$$

Here S is a matrix of 1's of size e by f $(0 \leqslant e < m; 0 < f \leqslant n)$ and 0 is a zero block (10). Now the entries of the structure matrix T of $\mathfrak{A}$ are non-negative integers. Moreover,

$$t_{ef} > 0 \qquad (e = 1, \ldots, m;\ f = 1, \ldots, n) \tag{4.5}$$

if and only if $\mathfrak{A}$ is without an invariant 1. Indeed, each

$$t_{ef} = 0 \qquad (e, f > 0) \tag{4.6}$$

yields the dimension numbers e and f for a block decomposition of the type displayed in (4.4).

Let $\bar{\rho}$ be the maximal and $\tilde{\rho}$ the minimal term rank for the matrices in $\mathfrak{A}$. If $\bar{\rho} < \min(m, n)$ and if $\mathfrak{A}$ is without an invariant 1, then $\tilde{\rho} < \bar{\rho}$ (10). But important classes do exist with $\tilde{\rho} = \bar{\rho}$, for example, the class of all (0, 1)-matrices of order $m = n$ with exactly k 1's in each row and column. An unsettled problem asks for a neat classification of all $\mathfrak{A}$ with $\tilde{\rho} = \bar{\rho}$. The corresponding problem for traces in a normalized $\mathfrak{A}$ has an easy solution. For let A in $\mathfrak{A}$ be of trace $\bar{\sigma}$ with 1's in the initial positions on the main diagonal. Then if $\tilde{\sigma} = \bar{\sigma}$, it follows readily that

$$A = \begin{bmatrix} S & * \\ * & 0 \end{bmatrix}. \tag{4.7}$$

Here S is the matrix of 1's of order $\bar{\sigma}$ and 0 is a zero block. Thus a normalized class $\mathfrak{A}$ has $\tilde{\sigma} = \bar{\sigma}$ if and only if its structure matrix contains a zero on the main diagonal.

A single interchange alters the term rank of a matrix by at most 1. It follows from this and the interchange theorem that there exists an A in $\mathfrak{A}$ of term rank ρ, where ρ is an arbitrary integer such that

$$\tilde{\rho} \leqslant \rho \leqslant \bar{\rho}. \tag{4.8}$$

However, a single interchange may alter the trace σ of a matrix in $\mathfrak{A}$ by 2. This causes a complication in finding the domain of intermediate values for σ

$$\tilde{\sigma} \leqslant \sigma \leqslant \bar{\sigma}. \tag{4.9}$$

The problem of intermediate values is settled by the following theorem.

THEOREM 4.2. *The traces of the matrices in the normalized $\mathfrak{A}$ take on all integral values in the interval $\tilde{\sigma} \leqslant \sigma \leqslant \bar{\sigma}$ unless $\mathfrak{A}$ contains a matrix of the form*

$$A = \begin{bmatrix} S & S^* & * \\ S^{*T} & I_c & 0 \\ * & 0 & 0 \end{bmatrix}. \tag{4.10}$$

Here S is a matrix of 1's *of order e, S^* is a rectangular matrix of* 1's, *S^{*T} is the transpose of S^*, I_c is the identity matrix or the complement of this matrix, and the* 0's *are zero matrices. The order of I_c is g with $g \geqslant 2$. (The cases $e = 0$, $e + g = m$, and $e + g = n$ are not excluded.)*

Two matrices in $\mathfrak{A}$ are transformable into each other by interchanges, and a single interchange applied to a matrix in $\mathfrak{A}$ may alter its trace by at most 2. Consecutive traces of mtarices in $\mathfrak{A}$ may differ by at most 2. Suppose then that σ and $\sigma - 2$ but not $\sigma - 1$ appear as the traces of matrices in $\mathfrak{A}$. Then there exists an A_σ in $\mathfrak{A}$ of trace σ with a principal minor of order 2 that is the identity.

Thus there exists an A_σ in $\mathfrak{A}$ with a principal minor of order g

$$M = [m_{ij}] \qquad (i, j = 1, \ldots, g) \tag{4.11}$$

composed of consecutive rows and columns of A_σ and such that

$$m_{11} = m_{gg} = 1, m_{1g} = m_{g1} = 0. \tag{4.12}$$

We let g be maximal among all matrices in $\mathfrak{A}$ of trace σ and write

$$A_\sigma = \begin{bmatrix} \bar{A} & \bar{B} & D \\ \bar{C} & M & \tilde{F} \\ E & \tilde{G} & \tilde{H} \end{bmatrix}. \tag{4.13}$$

The first row of A_σ passing through M must have the same sum as the last row of A_σ passing through M, for otherwise an interchange yields a trace of $\sigma - 1$. But $\mathfrak{A}$ is normalized so all rows of A_σ passing through M have the same sum. Similar remarks hold for the columns of A_σ passing through M.

Throughout the discussion we designate the submatrices of A_σ in (4.13) by $\bar{A} = [\bar{a}_{ij}]$, $\tilde{F} = [\tilde{f}_{ij}]$, etc. Suppose that in $\tilde{F}$ some $\tilde{f}_{uv} = 1$. If $\tilde{f}_{1v} = 0$, we may apply an interchange involving $\tilde{f}_{uv} = 1$ and $\tilde{f}_{1v} = 0$. This interchange cannot yield a trace of $\sigma - 1$. Nor can the interchange increase the trace to $\sigma + 1$, for then an interchange involving $m_{11} = m_{gg} = 1$ yieods a trace of $\sigma - 1$.

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Hence if some $\tilde{f}_{uv} = 1$, then there exists an A_σ of the form (4.13) with $\tilde{f}_{1v} = 1$. We may now apply an interchange involving $\tilde{f}_{1v} = 1$ and $m_{1g} = 0$. If the trace remains equal to σ, a second interchange involving $m_{11} = 1$ and $m_{g1} = 0$ yields a trace of $\sigma - 1$. Suppose then that the interchange involving $\tilde{f}_{1v} = 1$ and $m_{1g} = 0$ yields a trace of $\sigma + 1$. Let the 1 introduced on the main diagonal of A_σ be in the (t, t) position of $\tilde{H}$. Then $\tilde{g}_{t1} = 1$, for otherwise an interchange yields a trace of $\sigma - 1$. But now by an interchange involving $m_{gg} = \tilde{g}_{t1} = 1$, we regain trace σ and contradict the maximality of g. Hence $\tilde{F} = 0$. A similar argument gives $\tilde{G} = 0$. By the maximality of g, each $\tilde{h}_{uu} = 0$. If some $\tilde{h}_{uv} = 1$ with $u \neq v$, then an interchange involving $\tilde{h}_{uv} = m_{11} = 1$ yields a trace of $\sigma - 1$. Hence $\tilde{H} = 0$.

Suppose that some $\bar{c}_{uv} = 0$. The rows of A_σ passing through M have the same sum, so that if some $\bar{c}_{uv} = 0$, then there exists an A_σ of the form (4.13) with $\bar{c}_{1v} = 0$. But then $\bar{a}_{vv} = 1$, $\bar{b}_{v1} = 0$ and $\bar{b}_{vg} = \bar{c}_{gv} = 0$, for otherwise an interchange yields a trace of $\sigma - 1$. But this contradicts the maximality of g. Hence $\bar{C}$ is a matrix of 1's. Similarly, $\bar{B}$ is a matrix of 1's.

Suppose that some $\bar{a}_{uu} = 0$. Then an interchange involving $\bar{a}_{uu} = m_{1g} = 0$ yields a trace of $\sigma + 1$. Now apply an interchange involving $m_{11} = \bar{c}_{gu} = 1$. This regains trace σ and contradicts the maximality of g. Hence each $\bar{a}_{uu} = 1$. If some $\bar{a}_{uv} = 0$ with $u \neq v$, then an interchange involving $\bar{a}_{uv} = m_{1g} = 0$ retains trace σ. A second interchange yields a trace of $\sigma - 1$. Hence $\bar{A}$ is a matrix of 1's.

All row and column sums of $M = [m_{ij}]$ must be equal. Suppose there exist u and v such that

$$m_{uu} = 1, \qquad m_{vv} = 0 \qquad (1 < u,\ v < g). \tag{4.14}$$

If $m_{uv} = 0$ or if $m_{uv} = 1$, an interchange yields a trace of $\sigma - 1$. Hence M has trace g or trace 2. Suppose M has trace g. If $m_{1t} = 1$ with $t > 1$, then $m_{gt} = 1$. An interchange involving columns t and g followed by an interchange involving columns 1 and t yields a trace of $\sigma - 1$. Hence row 1 of M has sum 1 and $M = I$. Suppose M has trace 2. If $m_{1t} = 0$ with $t < g$, then an interchange involving columns 1 and t yields a trace of $\sigma - 1$. Hence row 1 of M has sum $g - 1$ and an interchange replaces M by the complement of I. This proves Theorem 4.2.

It is clear that Theorem 4.2 disposes of the problem of finding the domain of intermediate values for the traces σ of the matrices in $\mathfrak{A}$. For suppose that $\mathfrak{A}$ contains a matrix of the form (4.10) and that I_c is the identity matrix. Then $\tilde{\sigma} - 1$ is the single value excluded from the integers in the interval $\tilde{\sigma} \leqslant \sigma \leqslant \bar{\sigma}$. Suppose on the other hand that $\mathfrak{A}$ contains a matrix of the form (4.10) and that I_c is the complement of the identity. Then $\tilde{\sigma} + 1$ is the single value excluded from the integers in the interval $\tilde{\sigma} \leqslant \sigma \leqslant \bar{\sigma}$.

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