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SOME DETERMINANTS THAT SHOULD BE BETTER KNOWN

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1.

Since it is no longer fashionable to publish *per se* results which depend only on algebraic manipulation, many useful and complicated results of quite general interest languish unseen hidden as lemmata in specialist papers. In particular, the theory of transcendental numbers is rich in ingenious techniques for evaluating determinants. These techniques are apparently not well known even to workers in the field of transcendental numbers, let alone to researchers on other areas where the results might find application. This note accordingly discusses a variety of interesting results on determinants and is to be viewed as an appendix to the encyclopaedic volumes of Muir (1911, 1933) which the researcher might approach in order to obtain information in this area. For reasons of motivation a brief mention is made of the context in which the determinants arise.

2. A Wronskian Determinant

We suppose that $f(z)$ is a sufficiently often differentiable function and consider the Wronskian of the functions $1, f(z), (f(z))^2, \cdots (f(z))^{m-1}$; that is, the $m \times m$ determinant

$$\Delta = \left| \left(\frac{d}{dz}\right)^{h-1} (f(z))^{k-1} \right|_{h,k},$$

where we display the typical, (h,k) element. In general this determinant appears quite intractable and it is surprising indeed that it is easily given in the rather simple closed form:

$$\Delta = \left(\prod_{h=1}^{m} (h-1)!\right) (f'(z))^{m(m-1)/2}.$$

So, in particular, Δ vanishes if and only if $f'(z)$ vanishes. To see the assertion we observe that explicitly:

$$\left(\frac{d}{dz}\right)^{h-1}(f(z))^{k-1} = \sum_{\substack{a_1+\cdots+a_{k-1}=h-1\\ a_i\geqq 0}} (h-1)!\frac{f^{(a_1)}(z)}{a_1!}\frac{f^{(a_2)}(z)}{a_2!}\cdots\frac{f^{(a_{k-1})}(z)}{a_{k-1}!}$$

(where the sum is over all sets of $k-1$ non-negative integers a_i with sum $h-1$).

$$= \sum_{s=1}^{k}\binom{k-1}{s-1}(f(z))^{k-s}A_{s,h},$$

where

$$A_{s,h} = \sum_{\substack{b_1+\cdots+b_{s-1}=h-1\\ b_i>0}} (h-1)!\frac{f^{(b_1)}(z)}{b_1!}\cdots\frac{f^{(b_{s-1})}(z)}{b_{s-1}!}$$

(and the sum is over all sets of $s-1$ positive integers b_i with sum $h-1$). The essential feature is that the $A_{s,h}$ are independent of k.

Now by sequentially subtracting the 1-st column of Δ, $\binom{k-1}{1}(f(z))^{k-1}$ times from the k-th column, $k=2,3,\cdots,m$; then the 2-nd column $\binom{k-1}{2}(f(z))^{k-2}$ times from the k-th column, $k=3,4,\cdots,m$; and so on....

We do not affect the value of the determinant, but simplify Δ to find that

$$\Delta = |A_{k,h}|_{h,k}.$$

But $A_{k,h}=0$ whenever $k>h$ so the surviving determinant has no entries below the diagonal. Since

$$A_{k,h} = (h-1)!(f'(z))^{h-1},$$

we immediately obtain the asserted evaluation of Δ.

For a slightly more detailed description of this evaluation see Fel'dman (1951).

3. Determinants of Systems of Approximating Polynomials

Let $f_1(z),\cdots,f_m(z)$ be functions analytic on some domain G and not vanishing simultaneously on any point of a sequence Ω: $z_1,z_2,\cdots$ of points of G (not necessarily distinct). Write

$$\psi_0(z)=1 \qquad \psi_\lambda(z)=\prod_{\mu=1}^{\lambda}(z-z_\mu) \qquad (\lambda=1,2,\cdots)$$

and say that a function $F(z)$ analytic on G has order at least λ if $F(z)/\psi_\lambda(z)$ is analytic on G.

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Finally, let $\rho(1), \cdots, \rho(m)$ be non-negative integers with sum σ. Then it is easy to see (by counting the number of coefficients to be determined, and the number of conditions to be satisfied) that for each $h = 1, 2, \cdots, m$ we can construct polynomials $A_{h1}(z), \cdots, A_{hm}(z)$ satisfying (not all $A_{hk}(z) \equiv 0$)

$$\text{degree} \quad A_{hk}(z) \leqq \rho(k) + \delta_{hk} - 1$$

$$\text{order} \quad R_h(z) = \sum_{k=1}^{m} A_{hk}(z) f_k(z) \geqq \sigma$$

$$\text{and} \quad A_{hh}(z) \quad \text{monic (or } \equiv 0)$$

Denote by $A(z)$ the $m \times m$ matrix:

$$A(z) = (A_{hk}(z))_{h,k}$$

Then *either* $\det A(z) \equiv 0$ *or* degree $A_{hh}(z) = \rho(h)$ $(h = 1, \cdots, m)$ *and* $\det A(z) = \psi_\sigma(z)$.

PROOF. We observe that $\det A(z)$ is a polynomial of degree at most σ and attains that degree if and only if degree $A_{hh}(z) = \rho(h)$ for all h. On the other hand for each k, $f_k(z) \cdot \det A(z)$ can be seen to have order at least σ for by appropriate column operations its k-th column becomes the functions $R_1(z), \cdots, R_m(z)$. Hence, since the functions $f_1(z), \cdots, f_m(z)$ do not vanish simultaneously on Ω, $\det A(z)$ has order at least σ and thus if it is a polynomial of degree less than σ it must vanish identically. In the alternative case $\det A(z)$ is monic and being of degree σ and of order at least σ we have the assertion.

The principal application of this result is that when degree $A_{hh}(z) = \rho(h)$ for all h, and $z_0 \in G$ is a point close to points of Ω but not a point of Ω we obtain m linearly independent linear forms,

$$A_{h1}(z_0) f_1(z_0) + A_{h2}(z_0) f_2(z_0) + \cdots + A_{hm}(z_0) f_m(z_0)$$

in the quantities $f_1(z_0), \cdots, f_m(z_0)$ and can expect that the forms have relatively small absolute value.

Discussion of particular cases and applications of the above construction can be found in Mahler (1931,2; 1931; 1953; 1968) Coates (1966), Baker (1966) and van der Poorten (1971).

Dually, we can for each $h = 1, 2, \cdots, m$ find polynomials

$$\mathfrak{A}_{h1}(z), \cdots, \mathfrak{A}_{hm}(z)$$

satisfying (not all $\mathfrak{A}_{hk}(z) \equiv 0$):

$$\text{degree} \quad \mathfrak{A}_{hk}(z) \leqq \sigma - 1 - \rho(k) + \delta_{hk}$$

$$\text{order} \quad \mathfrak{R}_{hjk}(z) = \mathfrak{A}_{hj}(z) f_k(z) - \mathfrak{A}_{hk}(z) f_j(z) \geqq \sigma$$

$$\text{and} \quad \mathfrak{A}_{hh}(z) \quad \text{monic} \quad (\text{or } \equiv 0)$$

If we denote by $\mathfrak{A}(z)$ the $m \times m$ matrix:

$$\mathfrak{A}(z) = (\mathfrak{A}_{hk}(z))_{h,k}$$

then by an argument quite analogous to that provided above we see that *either* $\det \mathfrak{A}(z) \equiv 0$ *or* degree $\mathfrak{A}_{hh}(z) = \sigma - \rho(h)$ $(h = 1, 2, \cdots, m)$ *and* $\det \mathfrak{A}(z) = (\psi_\sigma(z))^{m-1}$.

Observation upon a remarkable relationship between the matrices $A(z)$ and $\mathfrak{A}(z)$ dates back to Hermite (1873, 1893) and was developed in detail by Mahler (1931,2), (1968), and subsequently by Jager (1964) and Coates (1966). We find that *the following are equivalent*:

(i) $\det A(z) \not\equiv 0$ (*and thus* degree $A_{hh}(z) = \rho(h)$ *for all* h)

(ii) $\det \mathfrak{A}(z) \not\equiv 0$ (*and thus* degree $\mathfrak{A}_{hh}(z) = \sigma - \rho(h)$ *for all* h)

(iii) $A(z)\mathfrak{A}'(z) = \psi_\sigma(z)I$ (*thus* $\mathfrak{A}(z)$ *is the conjugate of the adjoint matrix of* $A(z)$)

(iv) *the given conditions uniquely determine the matrices* $A(z)$ *and* $\mathfrak{A}(z)$.

Proof. Consider the identities

$$f_j(z) \sum_{k=1}^{m} \mathfrak{A}_{hk}(z) A_{gk}(z) = \mathfrak{A}_{hj}(z) R_g(z) - \sum_{k=1}^{m} A_{gk}(z) \mathfrak{R}_{hjk}(z)$$

$$(g, h, j = 1, 2, \cdots, m)$$

For each j all terms on the right have order at least σ, hence the polynomials

$$\sum_{k=1}^{m} \mathfrak{A}_{hk}(z) A_{gk}(z) \qquad (h, g = 1, 2, \cdots, m)$$

have order at least σ. On the other hand when $h \neq g$ the degree of the polynomial is at most $\sigma - 1$, hence the polynomial vanishes. When $h = g$ all terms are of degree $< \sigma$ except perhaps the term $\mathfrak{A}_{hh}(z)A_{hh}(z)$ which is of degree σ if and only if degree $\mathfrak{A}_{hh}(z) = \sigma - \rho(h)$ and degree $A_{hh}(z) = \rho(h)$.

Now suppose $\det A(z) \not\equiv 0$. Then if $\sum_{k=1}^{m} \mathfrak{A}_{hk}(z) A_{gk}(z)$ vanishes for all $g = 1, 2, \cdots, m$ it follows that $\mathfrak{A}_{hk}(z) \equiv 0$ for all $k = 1, 2, \cdots, m$ contradicting the definition. Thus necessarily $\sum_{k=1}^{m} \mathfrak{A}_{hk}(z) A_{hk}(z) \not\equiv 0$ whence it is equal to $\psi_\sigma(z)$ and degree $A_{hh}(z) = \rho(h)$ implies degree $\mathfrak{A}_{hh}(z) = \sigma - \rho(h)$ and so $\det A(z) \not\equiv 0$ implies $\det \mathfrak{A}(z) \not\equiv 0$. The converse argument is the same, and we have

$$\sum_{k=1}^{m} \mathfrak{A}_{hk}(z) A_{gk}(z) = \delta_{hg} \psi_\sigma(z)$$

as asserted by (iii). Finally, uniqueness follows by observing that $A(z)$ defines $\mathfrak{A}(z)$ and conversely. Observing that if, for any h, degree $A_{hh}(z) < \rho(h)$ then we can construct many matrices $A(z)$ satisfying the definition, shows that uniqueness implies $A(z) \not\equiv 0$ and completes the proof.

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4. Generalised Vandermonde Determinants

Let $\rho(1), \rho(2), \cdots, \rho(m)$ denote non-negative integers with sum $\rho(1)+\cdots+\rho(m)=\sigma$. If $\alpha_1, \alpha_2, \cdots, \alpha_m$ are distinct complex numbers one shows that an exponential polynomial of the shape

$$F(z)=\sum_{k=1}^{m} p_k(z)\, e^{\alpha_k z}=\sum_{k=1}^{m}\sum_{s=1}^{\rho(k)} a_{ks} z^{s-1} e^{\alpha_k z}$$

cannot vanish σ times (have a zero of order σ) at $z=0$ without vanishing identically, by showing that the $\sigma \times \sigma$ determinant

$$\Delta=\left|\left\{\left(\frac{d}{dz}\right)^{\lambda-1} z^{s-1} e^{\alpha_k z}\right\}_{z=0}\right|_{(k,s),\lambda}=\left|\frac{(\lambda-1)!}{(\lambda-s)!}\alpha_k^{\lambda-s}\right|_{(k,s),\lambda}$$

does not vanish. Here our notation is such as to display a typical term, rows being indexed by the σ pairs (k,s) arranged lexicographically, and columns by $\lambda=1,2,\cdots,\sigma$.

To evaluate Δ we employ the stratagem of introducing σ formally distinct quantities α_{ks} $(k=1,2,\cdots,m;\ s=1,2,\cdots,\rho(k))$ and introduce the large Vandermonde alternant

$$D=\left|\alpha_{ks}^{\lambda-1}\right|_{(k,s),\lambda}=\prod_{(k,s)<(h,t)}(\alpha_{ht}-\alpha_{ks}).$$

Rewriting the difference product in a more convenient shape, we have

$$D=\prod_{h=1}^{m}\prod_{t=1}^{\rho(h)}\left(\left\{\prod_{l=1}^{t-1}(\alpha_{ht}-\alpha_{hl})\right\}\prod_{k=1}^{h-1}\prod_{s=1}^{\rho(k)}(\alpha_{ht}-\alpha_{ks})\right).$$

Since

$$\{(\partial/\partial\alpha_{ks})^{s-1}\alpha_{ks}^{\lambda-1}\}_{\alpha_{ks}=\alpha_k}=\frac{(\lambda-1)!}{(\lambda-s)!}\alpha_k^{\lambda-s}$$

we can see that

$$\Delta=\left\{\left(\prod_{k=1}^{m}\prod_{s=1}^{\rho(k)}(\partial/\partial\alpha_{ks})^{s-1}\right)D\right\}_{\alpha_{ks}=\alpha_k}.$$

Moreover as

$$(\partial/\partial\alpha_{ht})^{t-1}\prod_{l=1}^{t-1}(\alpha_{ht}-\alpha_{hj})=(t-1)!$$

we obtain

$$\left(\prod_{k=1}^{m}\prod_{s=1}^{\rho(k)}(\partial/\partial\alpha_{ks})^{s-1}\right)D=\prod_{h=1}^{m}\prod_{t=1}^{\rho(h)}\left((t-1)!\prod_{k=1}^{h-1}\prod_{s=1}^{\rho(k)}(\alpha_{ht}-\alpha_{ks})\right)+\text{terms}$$

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which vanish as $\alpha_{ks} \to \alpha_k$, all (k, s), and thus

$$\Delta = \prod_{h=1}^{m} \prod_{t=1}^{\rho(h)} \left((t-1)! \prod_{k=1}^{h-1} (\alpha_h - \alpha_k)^{\rho(k)}\right).$$

Both the mode of evaluation and the explicit expression for Δ suggest the terminology employed, namely *generalised Vandermonde determinant.* We see that Δ is precisely a confluent case of the ordinary Vandermonde determinant D; incidentally, confluence is suggested by observing that the exponential polynomial $F(z)$ is the general solution of the differential equation

$$(d/dz - \alpha_1)^{\rho(1)}(d/dz - \alpha_2)^{\rho(2)} \cdots (d/dz - \alpha_m)^{\rho(m)} w = 0$$

For further discussion see van der Poorten (1970).

A similar determinant arises in showing that if $\alpha_1, \alpha_2, \cdots, \alpha_m$ are distinct non-zero complex numbers then a function of the shape

$$F(z) = \sum_{k=1}^{m} \sum_{s=1}^{\rho(k)} a_{ks} z^{s-1} \alpha_k^z$$

cannot vanish at all the points $z = n+1, n+2, \cdots, n+\sigma$ (n a rational integer) without thereby vanishing identically. We obtain

$$\Delta = \left| (n+\lambda)^{s-1} \alpha_k^{n+\lambda} \right|_{(k,s),\lambda},$$

and since

$$(n+\lambda)^{s-1} \alpha_k^{n+\lambda} = \left\{ \left(\alpha_{ks} \frac{\partial}{\partial \alpha_{ks}} \right)^{s-1} \alpha_{ks}^{n+\lambda} \right\}_{\alpha_{ks} = \alpha_k},$$

the above described technique immediately provides the result

$$\Delta = \prod_{h=1}^{m} \prod_{t=1}^{\rho(h)} \left((t-1)! \, \alpha_h^{n+t} \prod_{k=1}^{h-1} (\alpha_h - \alpha_k)^{\rho(k)}\right).$$

Here we observe that the sequence $\{F(n)\}$ is the general solution of the difference equation (where E denotes the operator sending the sequence $\{u_n\}$ to the sequence $\{u_{n+1}\}$, the n-th term of which is u_{n+1}):

$$(E - \alpha_1)^{\rho(1)}(E - \alpha_2)^{\rho(2)} \cdots (E - \alpha_m)^{\rho(m)} \{u_n\} = \{0\}.$$

For further discussion see van der Poorten (1970).

A similar idea is employed in evaluating the determinant which arises in showing that a function of the shape

$$F(z) = \sum_{k=1}^{m} \sum_{s=1}^{\rho(k)} a_{ks} z^{s-1} (1-z)^{\alpha_k}$$

cannot have a multiple zero of order as great as σ at $z = 0$ without vanishing identically.

It is convenient to introduce the following notation: write for non-negative integers l:

$$x^{(l)} = x(x-1)(x-2)\cdots(x-l+1), \qquad x^{(0)} = 1,$$

and write

$$\delta/\delta x\, f(x) = f(x+1) - f(x),$$

to create a useful notational analogy with the usual derivative. Then it is easy to see that

$$\frac{\delta}{\delta x} x^{(l)} = l x^{(l-1)},$$

and the Leibniz formula

$$(\delta/\delta x)^p g(x) f(x) = \sum_{\mu=0}^{p} \binom{p}{\mu} ((\delta/\delta x)^{p-\mu} f(x+\mu))(\delta/\delta x)^{\mu} g(x).$$

The determinant involved, is

$$\Delta = \left| \{(d/dz)^{\lambda-1} z^{s-1}(1-z)^{\alpha_k}\}_{z=0} \right|_{(k,s),\lambda} = \left| \frac{(\lambda-1)!}{(\lambda-s)!}(-1)^{\lambda-s} \alpha_k^{(\lambda-s)} \right|_{(k,s),\lambda}$$

But

$$\{(\delta/\delta\alpha_{ks})^{s-1} \alpha_{ks}^{(\lambda-1)}\}_{\alpha_{ks}=\alpha_k} = \frac{(\lambda-1)!}{(\lambda-s)!} \alpha_k^{(\lambda-s)},$$

and the determinant $|\alpha_{ks}^{(\lambda-1)}|_{(k,s),\lambda}$ is in fact a Vandermonde determinant equal to the difference product of the α_{ks}. We have in effect, the first problem we discussed above, except that a differencing operator replaces differentiation. Taking into account the Leibniz formula above we thus similarly obtain

$$(-1)^{\sigma(\sigma-1)/2} \Delta = \prod_{h=1}^{m} \prod_{t=1}^{\rho(h)} (-1)^{t-1}(t-1)! \prod_{k=1}^{h-1} \prod_{s=1}^{\rho(k)} (\alpha_h - \alpha_k + t - s).$$

Δ is thus essentially a disguised Vandermonde determinant in the quantities $\alpha_k + s - 1$.

Determinants involving the Γ–function may be dealt with by the same treatment. Thus if $\alpha_1, \alpha_2, \cdots, \alpha_m$ are complex numbers, not rational integers, one shows that a function of the shape

$$\sum_{k=1}^{m} \sum_{s=1}^{\rho(k)} a_{ks} z^{s-1} \Gamma(\alpha_k + z)$$

cannot vanish at all the points $z = n+1, n+2, \cdots, n+\sigma$ (n a positive integer) without thereby vanishing identically. The determinant involved, is

$$\Delta = \left| (n+\lambda)^{s-1} \Gamma(\alpha_k + n + \lambda) \right|_{(k,s),\lambda}$$

and as

$$\Gamma(\alpha_k + n + \lambda) = \Gamma(\alpha_k)(\alpha_k + n + \lambda - 1)^{(n+\lambda)},$$

and

$$(\alpha_{ks}\,\delta/\delta\alpha_{ks})^{s-1}(\alpha_{ks}+n+\lambda-1)^{(n+\lambda)}=(n+\lambda)^{s-1}(\alpha_{ks}+n+\lambda-1)^{(n+\lambda)},$$

we obtain after some manipulation:

$$\Delta=\prod_{h=1}^{m}\prod_{t=1}^{\rho(h)}((t-1)!\,\Gamma(\alpha_h+n+t)\prod_{k=1}^{h-1}\prod_{s=1}^{\rho(k)}(\alpha_h-\alpha_k+t-s).$$

A further detailed discussion of the above techniques applied to an apparently intractable determinant and its cofactors is basic to the author's paper (1970).

Readers interested in identities involving Vandermonde determinants will wish to be referred to the first chapter of Hua (1963).

5. Generalised Kronecker Products — Periodic Functions

Let $f_1(z),\cdots,f_m(z)$ be m distinct functions defined on some domain G, and let each function be periodic with period τ. For convenience write $\tau=m\theta$, and suppose that the points $\alpha+\theta,\alpha+2\theta,\cdots,\alpha+mn\theta$ lie in G. Then the non-trivial function

$$F(z)=\sum_{h=1}^{m}\sum_{t=1}^{n}a_{ht}z^{t-1}f_h(z)$$

cannot vanish at all of the points $\alpha+\theta,\alpha+2\theta,\cdots,\alpha+mn\theta$ unless the $mn\times mn$ determinant

$$\Delta=\left|(\alpha+u\theta)^{s-1}f_k(\alpha+u\theta)\right|_{(k,s),u}$$

vanishes. We shall show that Δ vanishes if and only if the $m\times m$ determinant

$$D=\left|f_k(\alpha+y\theta)\right|_{k,y}$$

vanishes. Indeed if for $y=1,2,\cdots,m$, Δ_y denotes the $n\times n$ determinant (in each case trivially a non-vanishing Vandermonde determinant)

$$\Delta_y=\left|(\alpha+(l-1)\tau+y\theta)^{s-1}\right|_{l,s},$$

then an appropriate partitioning of Δ followed by the usual row and column manipulation necessary to evaluate a Vandermonde determinant shows that, in view of the periodicity of the $f_k(z)$,

$$\Delta=D^n\,\Delta_1\Delta_2\cdots\Delta_m.$$

Indeed we show somewhat more finely that *with the notation as above*

$$\text{(A)}\qquad a_{ks}=\sum_{u=1}^{mn}F(\alpha+u\theta)\frac{D_{y,k}}{D}\cdot\frac{\Delta_{s,l,y}}{\Delta_y}$$

$$(k=1,2,\cdots,m;\ s=1,2,\cdots,n)$$

where $u = m(l-1) + y$ $(l = 1, 2, \cdots, n;\ y = 1, 2, \cdots, n)$ *and* $D_{y,k}$ *is the cofactor of* $f_k(\alpha + y\theta)$ *in the determinant* D *(we assume* $D \neq 0$*) and* $\Delta_{s,l,y}$ *is the cofactor of* $(\alpha + (l-1)\tau + y\theta)^{s-1}$ *in the determinant* Δy.

PROOF. By periodicity we have $f_h(\alpha + u\theta) = f_h(\alpha + y\theta)$. Hence substituting for $F(\alpha + u\theta)$ in (A) we see that the coefficient of a_{ht} is

$$\begin{aligned}\gamma &= \sum_{u=1}^{mn} (\alpha + u\theta)^{t-1} f_h(\alpha + y\theta) \frac{D_{y,k}}{D} \cdot \frac{\Delta_{s,l,y}}{\Delta_y} \\ &= \sum_{y=1}^{m} f_h(\alpha + y\theta) \frac{D_{y,k}}{D} \sum_{l=1}^{n} (\alpha + (l-1)\tau + y\theta)^{t-1} \frac{\Delta_{s,l,y}}{\Delta_y} \\ &= \sum_{y=1}^{m} f_h(\alpha + y\theta) \frac{D_{y,k}}{D} \cdot \delta_{s,t} = \delta_{h,k} \cdot \delta_{s,t}\end{aligned}$$

Hence $\gamma = 1$ if $(h,t) = (k,s)$ and $\gamma = 0$ otherwise, proving the assertion (A).

The quoted result slightly generalises a lemma of Fel'man [4; lemma 3].

If, further, the functions $f_h(z)$ are at least $m-1$ times differentiable at α then the $mn \times mn$ determinant

$$\Delta = \left|\{(d/dz)^{h-1} z^{l-1} f_j(z)\}_{z=\alpha+t\tau}\right|_{(j,l),(h,t)}$$

$$\left| \sum_{u=1}^{h} \binom{h-1}{u-1} \frac{(l-1)!}{(l-u)!} (\alpha + t\tau)^{l-u} f_j^{(h-u)}(\alpha + t\tau) \right|_{(j,l),(h,t)}$$

vanishes if and only if the $m \times m$ determinant

$$D = \left| f_k^{(h-1)}(\alpha) \right|_{h,k}$$

vanishes. Indeed if d denotes the $n \times n$ Vandermonde determinant

$$d = \left|(\alpha + t\tau)^{s-1}\right|_{s,t}$$

then appropriate row and column manipulation of Δ shows that Δ is in effect the Kronecker product of D and d whence, neglecting a factorial coefficient Δ is given by

$$d^m D^n.$$

Indeed we show more finely that *with the notation as above*

$$a_{ks} = \sum_{h=1}^{m} \sum_{t=1}^{n} F^{(h-1)}(\alpha + t\tau) \tag{B}$$

$$\cdot \sum_{r=1}^{n-s+1} (-1)^{r-1} \binom{s+r-2}{r-1} \frac{(h+r-2)!}{(h-1)!} \frac{D_{k,h+r-1}}{D} \cdot \frac{d_{t,s+r-1}}{d}.$$

where $D_{k,h+r-1}$ *is the cofactor of* $f_k^{(h+r-2)}(\alpha)$ *in* D, *and* $d_{t,s+r-1}$ *is the cofactor of* $(\alpha+t\tau)^{s+r-2}$ *in* d (*and we assume* $D\neq 0$, *and conventionally* $D_{k,h+r-1=0}$ *if* $h+r-1>m$; $d_{t,s+r-1}=0$ *if* $s+r-1>n$).

PROOF. By periodicity we have $f_j^{(h-1)}(\alpha+t\tau)=f_j^{(h-1)}(\alpha)$ for all h, j. Thus, differentiating, we obtain

$$F^{(h-1)}(\alpha+t\tau)=\sum_{j=1}^{m}\sum_{l=1}^{n}a_{jl}\sum_{u=1}^{h}\binom{h-1}{u-1}\frac{(l-1)!}{(l-u)!}(\alpha+t\tau)^{l-u}f_j^{(h-u)}(\alpha).$$

Substituting into (B) the coefficient of a_{jl} is seen to be

$$\gamma=\sum_{h=1}^{m}\sum_{u=1}^{h}\sum_{r=1}^{n-s+1}(-1)^{r-1}\binom{s+r-2}{r-1}\frac{(h+r-2)!}{(h-1)!}\frac{D_{k,h+r-1}}{D}$$
$$\cdot\binom{h-1}{u-1}\frac{(l-1)!}{(l-u)!}f_j^{(h-u)}(\alpha)\sum_{t=1}^{n}(\alpha+t\tau)^{l-u}\frac{d_{t,s+r-1}}{d}.$$

Considering the inner sum, we note the sum over t is 1 when $l-u=s+r-2$ and zero otherwise; thus only terms with $u\leqq l-s+1$ remain, and

$$\gamma=\sum_{u=1}^{l-s+1}(-1)^{l-u-s+1}\binom{l-u}{s-1}\frac{(l-1)!}{(l-u)!}$$
$$\cdot\sum_{h=1}^{m}\binom{h-1}{u-1}\frac{(h+l-u-s)!}{(h-1)!}\frac{D_{k,h+l-u-s+1}}{D}f_j^{(h-u)}(\alpha).$$

In the inner sum, the factor $\binom{h-1}{u-1}$ causes terms with $h<u$ to vanish, and the factor $D_{k,h+l-u-s+1}$ makes terms with $h>m-l+u+s-1$ vanish. Thus, replacing h by $h-u+1$ we obtain

$$\gamma=\sum_{u=1}^{l-s+1}(-1)^{l-u-s+1}\binom{l-u}{s-1}\frac{(l-1)!}{(l-u)!}$$
$$\cdot\sum_{h=1}^{m-l+s}\binom{h+u-2}{u-1}\frac{(h+l-s-1)!}{(h+u-2)!}\frac{D_{k,h+l-s}}{D}f_j^{(h-1)}(\alpha)$$
$$=\sum_{u=1}^{l-s+1}(-1)^{l-u-s+1}\frac{(l-1)!}{(s-1)!(l-u-s+1)!}$$
$$\cdot\sum_{h=1}^{m-l+s}\frac{(h+l-s-1)!}{(u-1)!(h-1)!}\frac{D_{k,h+l-s}}{D}f_j^{(h-1)}(\alpha)$$
$$=(-1)^{l-s}\binom{l-1}{s-1}\sum_{h=1}^{m-l+s}\frac{(h+l-s-1)!}{(h-1)!}f_j^{(h-1)}(\alpha)\frac{D_{k,h+l-s}}{D}$$
$$\cdot\sum_{u=1}^{l-s+1}(-1)^{u-1}\binom{l-s}{u-1}.$$

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The inner sum over u is zero or 1 according as $l \neq s$ or $l = s$. In the latter case

$$\gamma = \sum_{h=1}^{m} f_j^{(h-1)}(\alpha) \frac{D_{k,h}}{D} = \delta_{j,k}.$$

Hence $\gamma = 1$ if $(j, l) = (k, s)$ and $\gamma = 0$ otherwise, proving the assertion (B).

The quoted result slightly generalises a lemma of Fel'dman (1951).

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