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# DENSE SUBSPACES OF PRODUCT SPACES

TOSHIJI TERADA

**1. Introduction.** Unless otherwise specified, all spaces considered here are regular $T_1$-spaces. A space $X$ is called *$\sigma$-discrete* if $X$ is the union of a countable family of discrete subspaces. Arhangel'skii [**2**] showed that the class of spaces which contain dense $\sigma$-discrete subspaces is productive. The fact that the class of spaces which contain dense subspaces of countable pseudocharacter is productive is obtained by Amirdžanov [**1**]. On the other hand, the class of spaces which contain metrizable spaces as dense subspaces is obviously not productive. As a generalized concept of metrizable spaces there is the concept of $\sigma$-spaces [**14**]. This class of spaces has many similar properties to the class of metrizable spaces. However we will point out a remarkable difference between the class of metrizable spaces and the class of $\sigma$-spaces by showing that the class of spaces which contain $\sigma$-spaces as dense subspaces is productive. It will be also shown that the class of spaces which contain dense subspaces with $G_\delta$-diagonals and the class of spaces which contain dense subspaces with point-countable separating open covers are productive. These results have applications to the theory of cardinal invariants.

In Section 3 the following result will be proved: For an arbitrary space $X$, if $m$ is a sufficiently large cardinal, then $X^m$ contains a $\sigma$-space as a dense subspace. Section 4 will be devoted to some remarks. Applications to the theory of cardinal invariants will be given in Section 5. In particular, the answer to a question of Ginsburg-Woods [**9**] and Arhangel'skii [**3**] will be obtained. Further a connected left-separated space will be constructed. This is also a counterexample for another problem of Arhangel'skii [**3**].

Basic cardinal functions used in this paper are found in [**11**].

**2. Productive classes.** For a space $X$ let $\Delta_X$ be the diagonal of $X \times X$. If $\Delta_X$ is $G_\delta$ in $X \times X$, then it is said that $X$ has a *$G_\delta$-diagonal*. In [**6**], Ceder proved that a space $X$ has a $G_\delta$-diagonal if and only if there is a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open covers of $X$ such that given any point $x$ in $X$,

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$$\cap \{\text{st}(x, \mathscr{G}_n):n = 1, 2, \ldots\} = \{x\}$$

is satisfied, where st $(x, \mathscr{G}_n)$ is the union of all members of $\mathscr{G}_n$ containing $x$. Such a sequence of open covers is called $G_\delta$-diagonal sequence for $X$. A space $X$ is called a *σ-space* if $X$ has a σ-locally finite net [**14**]. An open cover $\mathscr{U}$ of a space $X$ is called *point-countable* if every point of $X$ is in at most countably many members of $\mathscr{U}$. An open cover $\mathscr{U}$ is called *separating* if given any distinct points $x$ and $y$, there is a member of $U$ of $\mathscr{U}$ such that $x \in U$ and $y \notin U$. These concepts are important in the theory of generalized metric spaces (see [**5**], [**12**], [**14**] and etc.).

It is well known that the class of spaces with $G_\delta$-diagonals, the class of σ-spaces and the class of spaces with point-countable separating open covers are countably productive but not productive. However we can prove the following results.

2.1. THEOREM. *The class of spaces which contain dense subspaces with $G_\delta$-diagonals is productive.*

2.2. THEOREM. *The class of spaces which contain σ-spaces as dense subspaces is productive.*

2.3. THEOREM. *The class of spaces which contain dense subspaces with point-countable separating open covers is productive.*

Let us recall the construction of Amirdžanov (see [**3**] ).

2.4. *Construction.* Let $\{X_\alpha:\alpha \in A\}$ be an infinite family of spaces without isolated points. For each $\alpha$ let $p_\alpha$ and $q_\alpha$ be distinct points in $X_\alpha$ and let

$$X'_\alpha = X_\alpha - \{p_\alpha, q_\alpha\}.$$

Let $\mathscr{F}(A)$ be the family of all nonempty finite subsets of $A$. Then, by a transfinite induction, we can construct a one-to-one map $s:\mathscr{F}(A) \rightarrow A$ such that $s(B) \notin B$ for each member $B$ of $\mathscr{F}(A)$. Now, for each member $B$ of $\mathscr{F}(A)$, we define a map

$$f_B: \prod \{X'_\beta:\beta \in B\} \rightarrow \prod \{X_\alpha:\alpha \in A\}$$

in the following way: For each element $\langle y_\beta \rangle$ of $\prod \{X'_\beta:\beta \in B\}$ and each member $\alpha$ of $A$,

$$\pi_\alpha(f_B(\langle y_\beta \rangle)) = \begin{cases} y_\alpha & \text{if } \alpha \in B \\ q_\alpha & \text{if } \alpha = s(B) \\ p_\alpha & \text{otherwise.} \end{cases}$$

Here $\pi_\alpha:\prod \{X_\alpha:\alpha \in A\} \rightarrow X_\alpha$ is the natural projection. Now let $Y_A$ be the subspace

$$\cup \{f_B(\prod \{X'_\beta:\beta \in B\}):B \in \mathscr{F}(A)\}$$

in $\prod \{X_\alpha:\alpha \in A\}$.

The following proposition is obvious.

2.5. PROPOSITION. *$Y_A$ is dense in $\prod \{X_\alpha:\alpha \in A\}$.*

2.6. PROPOSITION. *If $X_\alpha$ has a $G_\delta$-diagonal for each $\alpha$ in $A$, then $Y_A$ has also a $G_\delta$-diagonal.*

*Proof.* For each $\alpha$ in $A$ let $\mathscr{G}_{\alpha 1}, \mathscr{G}_{\alpha 2}, \ldots$ be a $G_\delta$-diagonal sequence for $X_\alpha$. Without loss of generality we can assume that $\mathscr{G}_{\alpha i+1}$ is a refinement of $\mathscr{G}_{\alpha i}$ for each $i$. Let

$$\mathscr{G}'_{\alpha i} = \{G - \{p_\alpha, q_\alpha\}:G \in \mathscr{G}_{\alpha i}\}$$

for each $\alpha$ and $i$. Now, for each $B$ in $\mathscr{F}(A)$, let

$$\mathscr{G}_{Bi} = \{\prod \{U_\alpha:\alpha \in A\} \cap Y_A:$$
$$U_\alpha \in \mathscr{G}'_{\alpha i} \text{ for } \alpha \in B,\ U_\alpha = \text{st}(q_\alpha, \mathscr{G}_{\alpha i}) - \{p_\alpha\}$$
$$\text{for } \alpha = s(B) \text{ and } U_\alpha = X_\alpha \text{ for } \alpha \in A - (B \cup \{s(B)\})\}.$$

Further, let

$$\mathscr{G}_i = \cup \{\mathscr{G}_{Bi}:B \in F(A)\}.$$

Then obviously $\mathscr{G}_i$ is an open cover of $Y_A$ for each $i$. Hence we will show that $\mathscr{G}_1, \mathscr{G}_2, \ldots$ is a $G_\delta$-diagonal sequence for $Y_A$.

Let $\langle x_\alpha \rangle$ be an arbitrary point of $Y_A$. We assume that

$$\langle x_\alpha \rangle = f_B(\langle y_\beta \rangle).$$

Assertion 1.

$$\text{st}(\langle x_\alpha \rangle, \mathscr{G}_i) = \cup \{\text{st}(\langle x_\alpha \rangle, \mathscr{G}_{Ci}):C \subset B\}.$$

Assume that $C$ is a member of $\mathscr{F}(A)$ such that $C - B$ is nonempty. Let $\alpha$ be an element in $C - B$. Let $U$ be a member of $\mathscr{G}_{Ci}$. Then $\pi_\alpha(U)$ is a subset of $X_\alpha - \{p_\alpha, q_\alpha\}$ since $\alpha \in C$. On the other hand, $\pi_\alpha(\langle x_\alpha \rangle)$ is $p_\alpha$ or $q_\alpha$ since $\alpha \notin B$. Hence $\langle x_\alpha \rangle$ is not contained in $U$.

Assertion 2. For each proper subset $C$ of $B$ there is a number $n_C$ such that $\langle x_\alpha \rangle$ is not contained in any member of $\mathscr{G}_{Ci}$ for each $i \geqq n_C$.

Since $\pi_{s(C)}(\langle x_\alpha \rangle) \neq q_{s(C)}$, there is a number $n_C$ such that

$$\pi_{s(C)}(\langle x_\alpha \rangle) \notin \text{st}(q_{s(C)}, \mathscr{G}_{s(C)i}) \quad \text{for each } i \geqq n_C.$$

On the other hand, if $U$ is a member of $\mathscr{G}_{Ci}$, then

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$$\pi_{s(C)}(U) \subset \text{st}\,(q_{s(C)}, \mathscr{G}_{s(C)i})$$

by the construction of $\mathscr{G}_{Ci}$. Hence $\langle x_\alpha \rangle$ is not contained in any member of $\mathscr{G}_{Ci}$ such that $i \geqq n_C$.

From Assertion 1 and Assertion 2 it follows that:

Assertion 3.

$$\cap \{\text{st}\,(\langle x_\alpha \rangle, \mathscr{G}_i): i = 1, 2, \dots\}$$

$$= \cap \{\text{st}\,(\langle x_\alpha \rangle, \mathscr{G}_{Bi}): i = 1, 2, \dots\}.$$

Assertion 4.

$$\cap \{\text{st}\,(\langle x_\alpha \rangle, \mathscr{G}_{Bi}): i = 1, 2, \dots\} = \{\langle x_\alpha \rangle\}.$$

In fact, for each $\alpha$ in $B \cup \{s(B)\}$,

$$\pi_\alpha(\cap \{\text{st}\,(\langle x_\alpha \rangle, \mathscr{G}_{Bi}): i = 1, 2, \dots\}) = \{x_\alpha\}.$$

This means that this assertion is true.

From Assertions 3 and 4 it follows that $\mathscr{G}_1, \mathscr{G}_2, \dots$ is a $G_\delta$-diagonal sequence for $Y_A$.

2.7. PROPOSITION. *If $X_\alpha$ is a $\sigma$-space for each $\alpha$ in $A$, then $Y_A$ is also a $\sigma$-space.*

*Proof.* For each $\alpha$ in $A$ let $\mathscr{N}_\alpha$ be a $\sigma$-locally finite net of $X_\alpha$. We can assume that

$$\mathscr{N}_\alpha = \cup \{\mathscr{N}_{\alpha i}: i = 1, 2, \dots\}$$

where each $\mathscr{N}_{\alpha i}$ is locally finite in $X_\alpha$ and $\mathscr{N}_{\alpha i} \subset \mathscr{N}_{\alpha i+1}$ for each $i$. We assume also that each member of $\mathscr{N}_\alpha$ is closed in $X_\alpha$. Let

$$\mathscr{N}'_\alpha = \{F \in \mathscr{N}_\alpha: F \cap \{p_\alpha, q_\alpha\} = \emptyset\}$$

and let

$$\mathscr{N}'_{\alpha i} = \mathscr{N}_{\alpha i} \cap \mathscr{N}'_\alpha.$$

Then obviously $\mathscr{N}'_\alpha$ is a net of $X'_\alpha$. For each member $B$ of $\mathscr{F}(A)$ let

$$\mathscr{N}'_{Bi} = \{\prod \{F_\beta: \beta \in B\}: F_\beta \in \mathscr{N}'_{\beta i} \text{ for each } \beta \in B\}.$$

Then $\cup \{\mathscr{N}'_{Bi}: i = 1, 2, \dots\}$ is a net of $\prod \{X'_\beta: \beta \in B\}$. Let

$$\mathscr{N}_{Bi} = \{f_B(F): F \in \mathscr{N}'_{Bi}\}$$

and let

$$\mathscr{N}_i = \cup \{\mathscr{N}_{Bi}: B \in \mathscr{F}(A)\}.$$

Now, we will show that $\mathcal{N} = \cup \{\mathcal{N}_i : i = 1, 2, \ldots\}$ is a $\sigma$-locally finite net of $Y_A$. It is obvious that $\mathcal{N}$ is a net of $Y_A$. Hence we will show that $\mathcal{N}_i$ is locally finite in $Y_A$.

Let $\langle x_\alpha \rangle$ be an arbitrary point of $Y_A$. We assume that

$$\langle x_\alpha \rangle = f_B(\langle y_\beta \rangle) \quad \text{for } B \text{ in } \mathcal{F}(A).$$

Let $\prod \{U_\beta : \beta \in B\}$ be a canonical open neighborhood of $\langle y_\beta \rangle$ in $\prod \{X_\beta : \beta \in B\}$ which intersects with only a finite number of members of $\mathcal{N}'_{Bi}$. Since $\mathcal{N}'_{s(B)i}$ is locally finite in $X_{s(B)}$ and $\cup \mathcal{N}'_{s(B)i}$ does not contain $q_{s(B)}$, there is an open neighborhood $U_{s(B)}$ of $q_{s(B)}$ such that

$$U_{s(B)} \cap (\cup \mathcal{N}'_{s(B)i} \cup \{p_{s(B)}\}) = \emptyset.$$

For each $\alpha$ in $A - (B \cup \{s(B)\})$ let $U_\alpha = X_\alpha$. Then $\prod \{U_\alpha : \alpha \in A\} \cap Y_A$ is an open neighborhood of $\langle x_\alpha \rangle$ in $Y_A$ which intersects with only a finite number of members of $\mathcal{N}_i$. In fact,

Assertion 1. If $C$ is a member of $\mathcal{F}(A)$ which is distinct from $B$, then each member of $\mathcal{N}_{Ci}$ is disjoint from $\prod \{U_\alpha : \alpha \in A\}$.

Let $F$ be an arbitrary member of $\mathcal{N}_{Ci}$. Since $s(C) \neq s(B)$,

$$\pi_{s(B)}(F) \subset (\cup \mathcal{N}'_{s(B)i}) \cup \{p_{s(B)}\}$$

by the construction of $F$. On the other hand,

$$\pi_{s(B)}(\prod \{U_\alpha : \alpha \in A\}) = U_{s(B)}$$

is disjoint from $\cup \mathcal{N}'_{s(B)i} \cup \{p_{s(B)}\}$. Hence $F$ is disjoint from $\prod \{U_\alpha : \alpha \in A\}$.

Assertion 2. The number of members of $\mathcal{N}_{Bi}$ which intersect with $\prod \{U_\alpha : \alpha \in A\}$ is finite.

This is obvious since $\prod \{U_\beta : \beta \in B\}$ intersects with only a finite number of members of $\mathcal{N}'_{Bi}$. This completes the proof.

2.8. PROPOSITION. *If $X_\alpha$ has a point-countable separating open cover, then $Y_A$ has also a point-countable separating open cover.*

*Proof.* For each $\alpha$ in $A$ let $\mathcal{G}_\alpha$ be a point-countable separating open cover of $X_\alpha$. Let

$$\mathcal{G}'_\alpha = \{G - \{p_\alpha, q_\alpha\} : G \in \mathcal{G}_\alpha\}$$

and let

$$\mathcal{H}_\alpha = \{G - \{p_\alpha\} : q_\alpha \in G \in \mathcal{G}_\alpha\}.$$

For each $B$ in $\mathcal{F}(A)$ let

$\mathscr{G}_B = \{\prod \{U_\alpha : \alpha \in A\} \cap Y_A : U_\alpha \in \mathscr{G}'_\alpha$ for $\alpha \in B$, $U_\alpha \in \mathscr{H}_\alpha$ for $\alpha = s(B)$ and $U_\alpha = X_\alpha$ for $\alpha \in A - (B \cup \{s(B)\})\}$. We will show that the family $\mathscr{G} = \cup \{\mathscr{G}_B : B \in F(A)\}$ is a point-countable separating open cover of $Y_A$. It is obvious that $\mathscr{G}$ is an open cover of $Y_A$.

Let $\langle x_\alpha \rangle$ be an arbitrary point of $Y_A$. We can assume that $\langle x_\alpha \rangle = f_B(\langle y_\beta \rangle)$.

Assertion 1. If a member $U$ of $\mathscr{G}_C$ contains $\langle x_\alpha \rangle$, then $C$ is a subset of $B$.

Assume that there is an element $\gamma$ in $C - B$. Then

$$\pi_\gamma(U) \subset X_\gamma - \{p_\gamma, q_\gamma\}.$$

On the other hand,

$$\pi_\gamma(\langle x_\alpha \rangle) \in \{p_\gamma, q_\gamma\}.$$

Hence $\langle x_\alpha \rangle \notin U$.

Assertion 2. For each $C \subset B$ the number of members of $\mathscr{G}_C$ which contain $\langle x_\alpha \rangle$ is countable.

For each $\alpha$ in $C$ the number of members of $\mathscr{G}'_\alpha$ which contain $x_\alpha$ is countable. The number of members of $\mathscr{H}_{s(C)}$ which contain $x_{s(C)}$ is also countable. Hence the assertion is obvious by the construction of $\mathscr{G}_C$.

Next, let $\langle x_\alpha \rangle$ and $\langle z_\alpha \rangle$ be two distinct points in $Y_A$. We assume also that $\langle x_\alpha \rangle = f_B(\langle y_\beta \rangle)$. Then it is not so difficult to see that there is an element $\lambda$ in $B \cup \{s(B)\}$ such that $x_\lambda \neq z_\lambda$. Let $U_\lambda$ be a member of $\mathscr{G}_\lambda$ such that $x_\lambda \in U_\lambda$ and $z_\lambda \notin U_\lambda$. Let

$$U'_\lambda = U_\lambda - \{p_\lambda, q_\lambda\} \quad \text{if } \lambda \in B$$

and let

$$U'_\lambda = U_\lambda - \{p_\lambda\} \quad \text{if } \lambda = s(B).$$

Then there is a member of $U$ of $\mathscr{G}_B$ such that $U$ contains $\langle x_\alpha \rangle$ and that $\pi_\lambda(U) = U'_\lambda$. This shows also that $\langle z_\alpha \rangle$ is not contained in $U$. This completes the proof.

Since the class of spaces with $G_\delta$-diagonals, the class of $\sigma$-spaces and the class of spaces with point-countable separating open covers are countably productive, we can easily prove Theorems, 2.1, 2.2 and 2.3 by Propositions 2.5, 2.6, 2.7 and 2.8.

**3. Other properties of infinite products.** For a space $X$ the smallest cardinality of dense subspaces is called the *density* of $X$ and denoted by $d(X)$. Let us call a space $X$ *$\sigma$-closed-discrete* if $X$ is the union of a countable family of closed, discrete subspaces. We will prove the following.

3.1. THEOREM. *Let $X$ be an arbitrary space. If $m$ is a cardinal such that $m \geqq d(X)$, then $X^m$ contains a σ-closed-discrete space as a dense subspace.*

3.2. COROLLARY. *If $m \geqq d(X)$, then $X^m$ contains a space with a $G_\delta$-diagonal as a dense subspace.*

3.3. COROLLARY. *If $m \geqq d(X)$, then $X^m$ contains a σ-space as a dense subspace.*

The following is also proved.

3.4. THEOREM. *If $m \geqq d(X)$, then $X^m$ contains a space with a point-countable separating open cover.*

3.5. *Construction.* Let $\{X_\alpha:\alpha \in A\}$ be an infinite family of spaces without isolated points such that $|X_\alpha| \leqq |A|$ for each $\alpha \in A$. In the same way as the construction 2.4, let $p_\alpha$, $q_\alpha$ be two distinct points in $X_\alpha$, let $X'_\alpha = X_\alpha - \{p_\alpha, q_\alpha\}$ and let $\mathscr{F}(A)$ be the family of all nonempty finite subsets of $A$. Since $|A|$ is infinite, there is a disjoint family $\mathscr{E}$ of countably infinite subsets of $A$ such that $|\mathscr{E}| = |A|$. Then, since the cardinality of the set

$$\cup \{\prod \{X'_\beta:\beta \in B\}:B \in \mathscr{F}(A)\}$$

is not more that $|A|$, there is a one-to-one map

$$t:\cup \{\prod \{X'_\beta:\beta \in B\}:B \in \mathscr{F}(A)\} \rightarrow \mathscr{E}.$$

For each $B$ in $\mathscr{F}(A)$ we define a map

$$h_B:\prod \{X'_\beta:\beta \in B\} \rightarrow \prod \{X_\alpha:\alpha \in A\}$$

in the following way: For each element $\langle y_\beta \rangle$ of $\prod \{X'_\beta:\beta \in B\}$ and each $\alpha$ in $A$,

$$\pi_\alpha(h_B(\langle y_\beta \rangle)) = \begin{cases} y_\alpha & \text{if } \alpha \in B \\ q_\alpha & \text{if } \alpha \in t(\langle y_\beta \rangle) - B \\ p_\alpha & \text{otherwise.} \end{cases}$$

Let $Z_A$ be the subspace $\cup \{h_B(\prod\{X'_\beta:\beta \in B\}):B \in \mathscr{F}(A)\}$ of $\prod \{X_\alpha:\alpha \in A\}$.

The following proposition is obvious.

3.6. PROPOSITION. *$Z_A$ is dense in $\prod \{X_\alpha:\alpha \in A\}$.*

Hence in order to prove Theorem 3.1 it suffices to show the following.

3.7. PROPOSITION. $Z_A$ *is* $\sigma$*-closed-discrete.*

*Proof.* Let

$$\mathscr{F}_n(A) = \{B \in \mathscr{F}(A):|B| = n\}$$

for each $n = 1, 2, \cdots$. Let

$$Z_n = \cup \{h_B(\prod \{X'_\beta:\beta \in B\}:B \in \mathscr{F}_n(A)\}.$$

Then obviously $Z_A = \cup \{Z_n:n = 1, 2, \cdots\}$. Hence it suffices to show that each $Z_n$ is a closed, discrete subspace in $Z_A$. Let $\langle x_\alpha \rangle$ be an arbitrary point of $Z_n$. We assume that

$$\langle x_a \rangle = h_B(\langle y_\beta \rangle).$$

For each $\beta$ in $B$ let $U_\beta$ be an open neighborhood of $x_\beta = y_\beta$ in $X_\beta$ such that

$$U_\beta \cap \{p_\beta, q_\beta\} = \emptyset.$$

Since $t(\langle y_\beta \rangle)$ is infinite and $B$ is finite, there is an element $\alpha_0$ in $t(\langle y_\beta \rangle) - B$. Let $U_{\alpha_0}$ be an open neighborhood of $x_{\alpha_0} = q_{\alpha_0}$ in $X_{\alpha_0}$ such that $p_{\alpha_0} \notin U_{\alpha_0}$. For $\alpha \in A - (B \cup \{\alpha_0\})$ let $U_\alpha = X_\alpha$. Now we set

$$U = \prod \{U_\alpha:\alpha \in A\} \cap Z_A.$$

Then $U$ is obviously an open neighborhood of $\langle x_\alpha \rangle$. Further,

Assertion 1. $U \cap Z_n = \{\langle x_\alpha \rangle\}$.

Assume that there is an element $\langle z_\alpha \rangle$ of $Z_n$ in $U$ which is distinct from $\langle x_\alpha \rangle$. Then $z_\alpha$ is neither $p_\alpha$ nor $q_\alpha$ for each $\alpha$ in $B \cup \{\alpha_0\}$. This shows that

$$|\{\alpha \in A:z_\alpha \notin \{p_\alpha, q_\alpha\}\}| > n.$$

This is a contradiction since $\langle z_\alpha \rangle$ is an element of $Z_n$.

Next, let $\langle w_\alpha \rangle$ be an arbitrary point in $Z_A - Z_n$. We can assume that

$$\langle w_\alpha \rangle = h_C(\langle v_\gamma \rangle) \quad \text{for } C \text{ in } \mathscr{F}(A) - \mathscr{F}_n(A).$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ be distinct elements of $t(\langle v_\gamma \rangle)$ and let $V_{\alpha i}$ be an open neighborhood of $w_{\alpha i}$ such that $p_{\alpha i} \notin V_{\alpha i}$ for each $i = 1, 2, \cdots, n+1$. For each $\alpha$ in $A - \{\alpha_1, \alpha_2, \cdots, \alpha_{n+1}\}$ let $V_\alpha = X_\alpha$. Now, let

$$V = \prod \{V_\alpha:\alpha \in A\} \cap Z_A.$$

Then obviously $V$ is an open neighborhood of $\langle w_\alpha \rangle$ in $Z_A$.

Assertion 2. $V \cap Z_n = \emptyset$.

Since

$$t(\langle y_\beta \rangle) \cap t(\langle v_\gamma \rangle) = \emptyset$$

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for each $\langle y_\beta \rangle \in \cup \{\prod \{X'_\beta : \beta \in B\} : B \in \mathscr{F}_n(A)\}$, for every point $\langle z_\alpha \rangle$ in $Z_n \cap V$ there must be a number $i$ such that $z_{\alpha_i} = p_{\alpha_i}$. This shows $V \cap Z_n = \emptyset$ by the construction of $V$.

In order to prove Theorem 3.4 it suffices to show the following.

3.8. PROPOSITION. *$Z_A$ has a point-countable separating open cover.*

*Proof.* For each $\alpha$ in $A$ let $U_\alpha$ be an open neighborhood of $q_\alpha$ which does not contain $p_\alpha$. Let

$$W_\alpha = \prod \{V_\alpha : \alpha \in A\} \cap Z_A$$

where $V_\alpha = U_\alpha$ and $V_\beta = X_\beta$ for $\beta \neq \alpha$. Let

$$\mathscr{W} = \{W_\alpha : \alpha \in A\}.$$

Then $\mathscr{W}$ is obviously an open cover of $Z_A$. Further,

Assertion 1. $\mathscr{W}$ is point-countable.

For each point $\langle x_\alpha \rangle$ in $Z_A$,

$$|\{\alpha \in A : \pi_\alpha(\langle x_\alpha \rangle) \neq p_\alpha\}| = \aleph_0.$$

Hence the number of members of $\mathscr{W}$ which contain $\langle x_\alpha \rangle$ is countable since $p_\alpha \notin \pi_\alpha(W_\alpha)$ for each $\alpha$.

Assertion 2. $\mathscr{W}$ is separating.

Let $\langle x_\alpha \rangle$ and $\langle y_\alpha \rangle$ be two distinct points of $Z_A$. We assume that

$$\langle x_\alpha \rangle = h_B(\langle u_\beta \rangle) \quad \text{and} \quad \langle y_\alpha \rangle = h_C(\langle v_\gamma \rangle).$$

Then there is an element $\alpha_0$ in $t(\langle u_\beta \rangle) - (B \cup C)$. Then $\langle x_\alpha \rangle$ is contained in $W_{\alpha_0}$. But $\langle y_\alpha \rangle$ is not an element of $W_{\alpha_0}$. This completes the proof.

3.9. COROLLARY. *Every space is an open perfect image of a space which contains a σ-space with a point-countable separating open cover as a dense subspace.*

*Proof.* Let $X$ be an arbitrary space. Let $Y$ be the product $X \times I^{|X|}$ where $I$ is the closed unit interval. Then $Y$ contains a $\sigma$-space with a point-countable separating open cover as a dense subspace by 3.7 and 3.8. Obviously, the projection

$$p : X \times I^{|X|} \to X$$

is open and perfect.

**4. Remarks.** A space $X$ is called *submetrizable* if there is a one-to-one continuous map from $X$ onto a metrizable space. This concept is closely related to the concept of $G_\delta$-diagonal (see [**5**]). However we have the following.

4.1. PROPOSITION. *The class of spaces which contain dense submetrizable subspaces is not productive.*

*Proof.* It is obvious that each submetrizable space $X$ satisfies the inequality $|X| \leqq \exp(c(X))$. Now, let $Z$ be a countably infinite discrete space. Let

$$m = (\exp^3 \aleph_0)^+.$$

Then for each dense subspace $Y$ of $Z^m$, it is satisfied that

$$|Y| > \exp(|Z|) \quad \text{and} \quad c(Y) \leqq |Z|.$$

This shows that $Y$ is not submetrizable. Hence $Z^m$ contains no dense submetrizable subspace.

By a result of [**8**] the following proposition is obvious.

4.2. PROPOSITION. *For a space $X$ the following are equivalent.*

(1) *$X$ contains a $\sigma$-closed-discrete space as a dense subspace.*

(2) *$X$ contains a $\sigma$-space as a dense subspace.*

(3) *$X$ contains a semi-stratifiable space as a dense subspace.*

From this fact and results in Sections 2 and 3 a question is raised naturally: Is it true that the following are equivalent?

(a) $X$ contains a $\sigma$-space as a dense subspace.

(b) $X$ contains a dense subspace with a $G_\delta$-diagonal.

(c) $X$ contains a dense subspace with a point-countable separating open cover.

The author does not know whether (b) and (c) are equivalent or not. However we can show that (a) and (b), (a) and (c) are not equivalent. More precisely there is a space which contains a dense submetrizable subspace but which does not contain a $\sigma$-space as a dense subspace.

The following lemma is essentially shown by Amirdžanov [**1**].

4.3. LEMMA. *Let $Y$ be a submetrizable space such that*

$$\min\{|V| : V \text{ is a non-empty open subset of } Y\} \geqq \pi w(X) \cdot \pi w(Y).$$

*Then $X \times Y$ contains a submetrizable space as a dense subspace.*

*Proof.* Let

$$m = \min \{|V|: V \text{ is a non-empty open subset of } Y\}.$$

Let $\mathscr{F}$ and $\mathscr{G}$ be $\pi$-bases of $X$ and $Y$ such that $|\mathscr{F}| = m$, $|\mathscr{G}| = m$. Let

$$\mathscr{F} \times \mathscr{G} = \{U_\alpha \times V_\alpha : \alpha < m\}.$$

We can assume that every member of $\mathscr{F} \times \mathscr{G}$ is non-empty. Let $(x_0, y_0)$ be an arbitrary point in $U_0 \times V_0$. Assume that, for every $\alpha < \beta$, a point $(x_\alpha, y_\alpha)$ is taken in $U_\alpha \times V_\alpha$. Then there is a point $y_\beta$ in $V_\beta - \{y_\alpha : \alpha < \beta\}$. Let $x_\beta$ be an arbitrary point in $U_\beta$. By this transfinite induction we can get a subspace

$$Z = \{(x_\alpha, y_\alpha) : \alpha < m\}$$

of $X \times Y$. It is obvious that $Z$ is a submetrizable dense subspace of $X \times Y$.

Amirdžanov [**1**] also showed the following result: Let $X$ be a space which contains no $\sigma$-discrete space as a dense subspace. Let $Y$ be a separable metrizable space. Then $X \times Y$ contains no $\sigma$-discrete space as a dense subspace. Hence, by this result and the above lemma, it follows that $(\beta N - N) \times D^{\aleph_0}$ contains a submetrizable space as a dense subspace, but it contains no $\sigma$-space as a dense subspace.

**5. Applications.** In this section spaces are completely regular. The smallest infinite cardinal $\kappa$ such that every closed, discrete subset of a space $X$ has cardinality at most $\kappa$ is called the *extent* of $X$ and denoted by $e(X)$. The *diagonal number* of a $T_1$-space $X$, denoted by $\nu(X)$, is the smallest infinite cardinal $\kappa$ such that $\Delta_X$ is written as the intersection of $\kappa$ open subsets of $X \times X$. For a $T_1$-space $X$, the *point separating weight* of $X$, denoted by psw $(X)$, is the smallest infinite cardinal $\kappa$ such that $X$ has a separating open cover $\mathscr{U}$ with the property that every point of $X$ is in at most $\kappa$ members of $\mathscr{U}$. Obviously a $T_1$-space $X$ has a $G_\delta$-diagonal if and only if $\nu(X) = \aleph_0$. Similarly, a $T_1$-space $X$ has a point-countable separating open cover if and only if psw $(X) = \aleph_0$. These cardinal functions are found in [**4**], [**9**], [**10**] and etc.

Ginsburg and Woods [**9**] proved the following: If $X$ is a $T_1$-space, then

$$|X| \leqq \exp(e(X)\nu(X)).$$

On the other hand, Burke and Hodel [**4**] proved the following: If $X$ is a $T_1$-space, then

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$$|X| \leqq \exp (e(X) \operatorname{psw} (X)).$$

Ginsburg and Woods showed also that there is a Hausdorff space $X$ such that $|X| \leqq \exp (c(X)\psi(X))$ is not true. And they raised the following question: Is there a regular space $X$ such that the inequality $|X| \leqq \exp (c(X)\psi(X))$ is not satisfied? Arhangel'skii also raised this question in [**3**]. Now, we can show the solution of this question in the following manner.

5.1. THEOREM. *For each infinite cardinal $\kappa$, there is a completely regular space $T_\kappa$ with the following properties*;

(1) $|T_\kappa| = \kappa$,

(2) $c(T_\kappa) = \aleph_0$,

(3) $\psi(T_\kappa) = \aleph_0$,

(4) $\operatorname{psw} (T_\kappa) = \aleph_0$,

(5) $\partial(T_\kappa) = \aleph_0$.

*Proof.* Let $X$ be a countable metrizable space without an isolated point. In Construction 2.4, assume that $X_\alpha = X$ for each $\alpha$ in $A$ and that $|A| = \kappa$. Let $T_\kappa$ be the space $Y_A$ constructed in 2.4. Then

$$c(T_\kappa) = c(X^\kappa) = \aleph_0$$

[**11**]. By 2.6 and 2.8,

$$\psi(T_\kappa) = \operatorname{psw} (T_\kappa) = \aleph_0.$$

The construction of $T_\kappa$ shows also that the cardinality of $T_\kappa$ is just $\kappa$. Since $T_\kappa$ is a subspace of the $\Sigma$-product of a family of first countable spaces, $\partial(T_\kappa) = \aleph_0$ by a result of [**13**].

Let us recall that a space $X$ is *left separated* if it has a well-ordering $\prec$ such that every initial segment of $X$ under $\prec$ is closed [**11**]. It is obvious that every space contains a left separated space as a dense subspace. Hence if every left separated space is zero-dimensional in the sense of ind, then it follows that every space contains a zero-dimensional dense subspace. In fact, Arhangel'skii [**3**] raised the problem of whether every left separated space is zero-dimensional. Here, using Construction 3.5, we show that there is a connected left separated space.

Since $[0, 1)^n$-(countable subset) is obviously connected for every natural number $n \geqq 2$, where $[0, 1)$ is the usual interval in the real line. The following lemma is obvious.

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5.2. LEMMA. *Let $A$ be an infinite set and let $X_\alpha = [0, 1)$ for each $\alpha \in A$. Let $Y$ be the $\sigma$-product of $\{X_\alpha : \alpha \in A\}$ with the base point $\langle 0 \rangle$. Then $Y - K$ is connected for every countable subset $K$ of $Y$.*

Since every $\sigma$-closed-discrete space is left separated, it suffices to show the following.

5.3. THEOREM. *There is a connected, $\sigma$-closed-discrete space.*

*Proof.* In Construction 3.5, let $|A| \geqq 2^{\aleph_0}$, let each $X_\alpha$ be the closed unit interval $[0, 1]$ and let $p_\alpha = 0$, $q_\alpha = 1$. Then $Z_A$ is $\sigma$-closed-discrete by 3.7. Assume that $Z_A$ is not connected. Then there are disjoint nonempty open subsets $U_1$, $U_2$, of $Z_A$ such that $U_1 \cup U_2 = Z_A$. Let $V_1$, $V_2$ be open subsets of $\prod \{X_\alpha : \alpha \in A\}$ such that $V_1 \cap Z_A = U_1$, $V_2 \cap Z_A = U_2$. Then $V_1$ and $V_2$ are disjoint. Hence, there are a countably infinite subset $C$ of $A$ and disjoint open subsets $W_1$, $W_2$ of $\prod \{X_\gamma : \gamma \in C\}$ such that

$$\pi_C(V_1) \subset W_1, \quad \pi_C(V_2) \subset W_2,$$

where

$$\pi_C : \prod \{X_\alpha : \alpha \in A\} \rightarrow \prod \{X_\gamma : \gamma \in C\}$$

is the natural projection (see [7, 2.7.12]). Let $\mathscr{F}(C)$ be the family of all nonempty finite subsets of $C$. Let

$$H = \{ \langle y_\beta \rangle \in \cup \{\prod \{X'_\beta : \beta \in B\} : B \in \mathscr{F}(C)\} : t(\langle y_\beta \rangle) \cap C \neq \emptyset\}.$$

Since $C$ is a countable subset of $A$, $H$ is also countable. Now, let $Y$ be the $\sigma$-product of $\{X_\gamma - \{q_\gamma\} : \gamma \in C\}$ with the base point $\langle p_\gamma \rangle$. For each $B \in \mathscr{F}(C)$, let

$$k_B : \prod \{X'_\beta : \beta \in B\} \rightarrow Y$$

be the map defined in the following way: For each $\langle y_\beta \rangle \in \prod \{X'_\beta : \beta \in B\}$ and $\gamma \in C$,

$$\pi_\gamma(k_B(\langle y_\beta \rangle)) = \begin{cases} y_\gamma & \text{if } \gamma \in B \\ p_\gamma & \text{otherwise.} \end{cases}$$

Let

$$K_0 = \{k_B(\langle y_\beta \rangle) : \langle y_\beta \rangle \in H, B \in \mathscr{F}(C)\}.$$

Then $K_0$ is a countable subset of $Y$. We will show that

$$\pi_C(Z_A) \supset Y - K_0.$$

Let $\langle z_\gamma \rangle$ be an arbitrary point of $Y - K_0$ and assume that

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$$\langle z_\gamma \rangle = k_B(\langle y_\beta \rangle)$$

for some $\langle y_\beta \rangle \in \prod \{X'_\beta; \beta \in B\}$. Since $\langle y_\beta \rangle \notin H$,

$$t(\langle y_\beta \rangle) \cap C = \emptyset.$$

Therefore

$$\pi_\gamma(h_B(\langle y_\beta \rangle)) = \begin{cases} y_\gamma \text{ if } \gamma \in B \\ p_\gamma \text{ if } \gamma \in C - B. \end{cases}$$

This shows that

$$\pi_C(h_B(\langle y_\beta \rangle)) = \langle z_\gamma \rangle.$$

$\langle p_\gamma \rangle \in \pi_C(Z_A)$ is obvious. Since $Y - K_0$ is connected by 5.2, $\pi_B(Z_A)$ is connected. Hence there is a point $\langle x_\alpha \rangle$ in $Z_A$ such that

$$\pi_C(\langle x_\alpha \rangle) \notin W_1 \cup W_2.$$

This is a contradiction since $\pi_C(Z_A) \subset W_1 \cup W_2$.

5.4. *Remark.* Since every $\sigma$-closed-discrete normal space is zero-dimensional by the countable sum theorem of Ind, the space considered in the above theorem is not normal. E. Pol **[15]** constructed $\sigma$-closed-discrete spaces which are not zero-dimensional. However her examples are not connected.

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*Yokohama National University,*
*Hodogaya, Yokohama, Japan*