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A two parameter eigenvalue problem

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An ordinary second order differential equation is considered in which the coefficients are dependent on two parameters ω and F as well as the independent variable μ. The equation arises in the study of free oscillations of incompressible inviscid fluid in global shells. An asymptotic technique is presented which estimates the eigenvalues (that is the values of ω for which the solution is bounded for all $|\mu| \leq 1$) as functions of F, as $F \to \infty$. The agreement of the results with numerical computations is also discussed.

1. Introduction

In recent studies of free oscillations of incompressible inviscid fluid in both spherical and spheroidal shells with rigid or free outer surfaces equations like that below are encountered (see Rickard [2], [3], [4]):

$$(1.1)\quad \omega\left(1-\mu^2\right)\left(\omega^2-4\mu^2\right) \frac{d^2\chi}{d\mu^2} - \mu\omega\left\{4\left(\omega^2-2-2\mu^2\right)+\tfrac{1}{2}F\left(1-\mu^2\right)\left(\omega^2-4\mu^2\right)\right\} \frac{d\chi}{d\mu} + \left\{-2\left(\omega^3+\omega^2+4\mu^2\right)+\tfrac{1}{2}F\left(\omega^2-4\mu^2\right)\left(\omega^3+(2-3\omega)\mu^2\right)\right\}\chi = 0 .$$

In free-surface problems, like that above (see Rickard [2], [4]) F represents a non-dimensionalised Froude number and ω the free periods of oscillation (apart from various constants). In rigid boundary problems a similar parameter K occurs (see Rickard [2], [3]), which is essentially a measure of the deviation of the shell from a spherical shell. The dependent variable χ represents a reduced pressure and the independent variable μ is such that $|\mu| \leq 1$.

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It is required to calculate the eigenvalues ω, that is the values of ω for which χ is bounded in $|\mu| \leq 1$, as functions of F (or K, as the case may be). The analytical solution of (1.1) for small F and numerical calculations for moderate F have been carried out by Rickard ([2], [3], [4]). In most geophysical applications F is not large. For example, F is approximately 12 for the Pacific Ocean (see Rickard [4]). However, the structure of the eigenvalues of (1.1) and similar equations as $F \to \infty$ is of considerable mathematical interest.

In this paper an asymptotic technique is used to estimate the values of ω as $F \to \infty$. Specifically, the paper is made up as follows. In Section 2 the solution of (1.1) away from $\mu = 0$ and $|\mu| = 1$ is examined for $F \to \infty$. Sections 3 and 4 examine the solution in these boundary layers while Section 5 discusses the matching procedure necessary to estimate ω. Finally, in Section 6 the results are compared with numerical calculations and restrictions on the general applicability of the method are discussed.

2. Solutions for large positive F

Since we are primarily concerned with calculating the eigenvalues of (1.1), (the values of ω for which χ is bounded) and not in the detailed structure of the solution, all we require of the boundary conditions is a means of distinguishing between even and odd eigenfunctions. It will be convenient to distinguish between even and odd solutions of (1.1) by imposing the boundary conditions

$$\chi(1) = 1 , \quad \chi(0) = 0 \tag{2.1}$$

for odd solutions, and

$$\chi(1) = 1 , \quad \chi'(0) = 0 \tag{2.2}$$

for even solutions.

Further, we shall restrict our attention to positive values of μ, the extension to negative values being straightforward. In (1.1), as $F \to \infty$, the coefficient of the most highly differentiated term becomes singular and we may expect the presence of boundary layers, enabling us to satisfy (2.1), (2.2).

Away from the neighbourhood of any such boundary layers the solution

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of (1.1) may be written

$$\chi(\mu) = AF(\mu) + BG(\mu) \ , \tag{2.3}$$

where A and B are arbitrary constants and $F(\mu)$, $G(\mu)$ may be written in the form

$$F(\mu) = F_1(\mu) + (2/F)F_2(\mu) + (2/F)^2F_3(\mu) + \ldots \ , \tag{2.4}$$

$$G(\mu) = e^{\mu^2F/4}\left\{G_1(\mu) + (2/F)G_2(\mu) + (2/F)^2G_3(\mu) + \ldots\right\} \ , \tag{2.5}$$

where $F_n(\mu)$, $G_n(\mu)$, $n = 1, 2, \ldots$, are functions of μ only.

Although ω is also a function of F , we shall for the present regard it as constant; a simplification which in no way affects the analysis to follow.

It is now a straightforward matter to determine $F_1(\mu)$ and $G_1(\mu)$ by substituting (2.3)-(2.5) in (1.1) and comparing corresponding powers of F . We find that

$$F_1(\mu) = \alpha\mu^{\omega^2}\left(1-\mu^2\right)^{-\lambda} \ , \quad G_1(\mu) = \frac{\beta\left(\omega^2-4\mu^2\right)\left(1-\mu^2\right)^{\lambda-2}}{\mu^{1+\omega^2}} \ , \tag{2.6}$$

where α and β are arbitrary constants and

$$\lambda = \frac{(\omega-1)^2(\omega+2)}{2\omega} \ . \tag{2.7}$$

From (2.3)-(2.6) it follows that

$$\chi(\mu) = A\left[\mu^{\omega^2}\left(1-\mu^2\right)^{-\lambda} + \ldots\right] + \\ + B\left[\left(\omega^2-4\mu^2\right)\left(1-\mu^2\right)^{\lambda-2}\mu^{-1-\omega^2}e^{\mu^2F/4} + \ldots\right] \ , \tag{2.8}$$

where the constants α and β have been absorbed into A and B . Further terms in (2.8) may be calculated in a straightforward way but here we give only the leading contributions to $F(\mu)$ and $G(\mu)$. It is clear that (2.8) breaks down in the neighbourhood of $\mu = 0$ and $\mu = 1$, and that we cannot at present satisfy (2.1), (2.2). We shall now consider the solutions of (1.1) within these boundary layers.

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3. The boundary layer near $\mu = 0$

A consistent match with the solution (2.8) near $\mu = 0$ can be found by taking $F^{-\frac{1}{2}}$ as scaling factor for this layer, and near $\mu = 0$ we write

(3.1) $$\mu = (2/F)^{\frac{1}{2}}y \ , \quad \chi = \bar{\chi}_1(y) + (2/F)\bar{\chi}_2(y) + \ldots \ .$$

From (1.1), (3.1) it follows that

(3.2) $$\frac{d^2\bar{\chi}_1}{dy^2} - y\frac{d\bar{\chi}_1}{dy} + \omega^2\bar{\chi}_1 = 0 \ ,$$

the complete solution to which is given by

(3.3) $$\bar{\chi}_1(y) = A_1M\left(-\tfrac{1}{2}\omega^2, \tfrac{1}{2}, \tfrac{1}{2}y^2\right) + B_1yM\left(\tfrac{1}{2}(1-\omega^2), \tfrac{3}{2}, \tfrac{1}{2}y^2\right) \ ,$$

where A_1, B_1 are arbitrary constants and $M(a, b, z)$ is Kummer's function (see, for example, Slater, [5], p. 504).

We now have to apply the boundary conditions at $\mu = 0$, as given by (2.1), (2.2). Which of these conditions we impose depends on whether the class of solutions we wish to consider are even or odd, and we shall now proceed to discuss each of these cases separately.

(i) Odd solutions

In this case (2.1) is the appropriate condition, and from (3.3) it follows that

(3.4) $$A_1 = 0 \ ,$$

(3.5) $$\bar{\chi}_1(y) = B_1yM\left(\tfrac{1}{2}(1-\omega^2), \tfrac{3}{2}, \tfrac{1}{2}y^2\right) \ .$$

The asymptotic form of (3.5) as $y \to \infty$ is given by

(3.6) $$\bar{\chi}_1(y) = B_1\left[\frac{\sqrt{\pi/2}(\tfrac{1}{2})^{\omega^2/2}y^{\omega^2}}{(\tfrac{1}{2}\omega^2)!}L_1(y) + \frac{\sqrt{\pi}e^{\frac{1}{2}y^2}y^{-1-\omega^2}(2)^{\omega^2/2}}{(-\tfrac{1}{2}-\tfrac{1}{2}\omega^2)!}L_2(y)\right] \ ,$$

where

$$L_1(y) = 1 + \frac{\omega^2(1-\omega^2)}{2y^2} + \ldots \ ,$$

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$$L_2(y) = 1 + \frac{(1+\omega^2)(2+\omega^2)}{2y^2} + \dots \tag{3.7}$$

(ii) Even solutions

We now have (2.2) instead of (2.1) as the appropriate boundary condition, and from (3.3) it follows that

$$B_1 = 0 \,, \tag{3.8}$$

$$\bar{\chi}_1(y) = A_1 M\left(-\tfrac{1}{2}\omega^2, \tfrac{1}{2}, \tfrac{1}{2}y^2\right) . \tag{3.9}$$

As $y \to \infty$ the asymptotic form of (3.9) is given by

$$\bar{\chi}_1(y) = A_1\left[\frac{\sqrt{\pi}\,(\frac{1}{2})^{\omega^2/2} y^{\omega^2}}{\left(-\frac{1}{2}+\frac{1}{2}\omega^2\right)!} L_1(y) + \frac{\sqrt{2\pi}\, e^{\frac{1}{2}y^2} y^{-1-\omega^2} (2)^{\omega^2/2}}{\left(-1-\frac{1}{2}\omega^2\right)!} L_2(y)\right] \tag{3.10}$$

Due to the presence of the exponential term in (3.6) we see that every term of the second series dominates even the leading term of the first series. Hence, in the case of odd solutions, it follows from (2.8), (3.6) that B is correctly related to B_1, but that the relationship of A to B_1 may be in error by a term of order B_1.

Analogous remarks apply to the relationships between A, B, A_1 for even solutions.

4. The boundary layer near $\mu = 1$

In this boundary layer, near $\mu = 1$, a consistent match with the solution (2.8) can be found with scaling factor F^{-1}, and on writing

$$\mu = 1 - (2/F)x \,, \quad \chi = \hat{\chi}_1(x) + (2/F)\hat{\chi}_2(x) + \dots \,, \tag{4.1}$$

it follows from (1.1) that

$$x\frac{d^2\hat{\chi}_1}{dx^2} + (x+2)\frac{d\hat{\chi}_1}{dx} + \lambda\hat{\chi}_1 = 0 . \tag{4.2}$$

If we now make the transformation

$$x = -z \,, \tag{4.3}$$

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then (4.2) becomes

$$z \frac{d^2\hat{\chi}_1}{dz^2} + (2-z)\frac{d\hat{\chi}_1}{dz} - \lambda\hat{\chi}_1 = 0 . \tag{4.4}$$

This is the Confluent Hypergeometric or Kummer's Equation, and has solution

$$\hat{\chi}_1(z) = A_2 M(\lambda, 2, z) + B_2 U(\lambda, 2, z) , \tag{4.5}$$

where A_2, B_2 are at present arbitrary constants. For an account of Kummer's equation, together with the definition of $M(a, b, z)$ and $U(a, b, z)$, see, for example, Slater, [5], p. 504.

From (2.1), (2.2), we see that for both even and odd eigensolutions, we require

$$B_2 = 0 , \quad A_2 = 1 , \tag{4.6}$$

so that (4.5) reduces to

$$\hat{\chi}_1(z) = M(\lambda, 2, z) . \tag{4.7}$$

It follows that the asymptotic behaviour of $\hat{\chi}_1(x)$ as $x \to \infty$ $(z \to -\infty)$ is given by

$$\hat{\chi}_1(x) = \frac{x^{\lambda-2}e^{-x}}{(\lambda-1)!}\left[1 - \frac{(\lambda-1)(\lambda-2)}{x} + \frac{(\lambda-1)(\lambda-2)^2(\lambda-3)}{2x^2} + \dots\right] + \frac{x^{-\lambda}}{(1-\lambda)!}\left[1 + \frac{\lambda(\lambda-1)}{x} + \frac{\lambda^2(\lambda-1)(\lambda+1)}{2x^2} + \dots\right] . \tag{4.8}$$

5. Matching of solutions

It is clear from (2.8), on putting $\mu = 1 - \frac{2x}{F}$, that the algebraic terms in (4.8) must have a small coefficient. This can be achieved only if

$$\lambda = n + \delta , \quad n = 2, 3, 4, \dots , \tag{5.1}$$

where $|\delta| << 1$; or if $\lambda \to -\infty$. If $\lambda \to -\infty$, it follows from (2.7) that ω is small and negative, contrary to the known result that $\omega > 0$ for all F (see Rickard, [2]). Hence (5.1) is the only possibility.

It is clear that δ will be different for even and odd solutions, and we shall write

$$(5.2)\qquad \begin{aligned} \delta &= \delta_o \quad \text{for odd solutions,} \\ \delta &= \delta_e \quad \text{for even solutions.} \end{aligned}$$

In an attempt to estimate the values of δ_o, δ_e we shall proceed to match the solutions in the boundary layers near $\mu = 0$ and $\mu = 1$ with (2.8), taking only the leading terms in each series in the asymptotic expansions (3.6), (3.10), (4.8). As remarked previously, this matching procedure may involve errors, due to the dominance of all the terms in one series in (3.6), (3.10) over terms in the other series in these asymptotic expansions. We must therefore anticipate our results to be correct only to within an order of magnitude.

When $|\delta| << 1$, we may make the following approximation

$$(5.3)\qquad \frac{(\lambda-1)!}{(1-\lambda)!} \approx (-1)^{n-1}(n-1)!(n-2)!\delta \ .$$

Further, for values of λ given by (5.1), ω^2 is small compared with unity, and we may approximate certain functions occuring in (3.6), (3.10), for example,

$$(5.4)\qquad \left(-1-\tfrac{1}{2}\omega^2\right)! \approx -2/\omega^2 \ , \quad \left(-\tfrac{1}{2}-\tfrac{1}{2}\omega^2\right)! \approx \sqrt{\pi} \ , \quad \left(\tfrac{1}{2}\right)^{\omega^2} \approx 1 \ , \text{ and so on.}$$

From (2.8), (3.6), (3.10), (4.8), with the aid of (5.3) and approximations like those in (5.4), we find

$$(5.5)\qquad \delta_o = \frac{(-1)^{n-1}\omega^2\sqrt{\pi/2}(4/F)^{2-2n}(2/F)^{-\frac{1}{2}-\omega^2}e^{-F/4}}{\left(\omega^2-4\right)(n-1)!(n-2)!} \ ,$$

$$(5.6)\qquad \delta_e = -\frac{2}{\omega^2\pi}\delta_o \ .$$

From (2.7), (5.1) it follows that

$$(5.7)\qquad \omega = \bar{\omega} + \omega^* \ ,$$

where $\bar{\omega}$ is the root of the cubic equation

$$(5.8)\qquad \omega^3 - (3+2n)\omega + 2 = 0 \ , \quad n = 2, 3, 4, \ldots \ ,$$

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which lies between zero and unity, and ω^* is given by

$$\omega^* = \gamma - \bar{\omega} \,, \tag{5.9}$$

where γ is the root of (5.8), (lying between zero and unity), with n replaced by $n + \delta$ $(\delta_o$ for odd solutions and δ_e for even solutions$)$.

Further we see that $\bar{\omega}$ is the same for even and odd solutions associated with a given n and is independent of F, whereas ω^* is different for even and odd solutions and furthermore is $O(\delta)$ and dependent on F.

We shall now infer that ω is of the form

$$\omega = \omega_1 + (2/F)\omega_2 + (2/F)^2\omega_3 + \dots \,, \tag{5.10}$$

where

$$\omega_n = \bar{\omega}_n + \omega_n^* \,, \quad n = 1, 2, \dots \,, \tag{5.11}$$

$\bar{\omega}_n$ being independent of F and the same for both even and odd solutions, while ω_n^* is dependent on F and different for even and odd solutions. Furthermore ω_n^* tends to zero exponentially as $F \to \infty$.

The determination of ω_2, while straightforward, is probably not worthwhile in view of the possible errors in ω_2^*. However, calculation of $\bar{\omega}_2$ presents no difficulty and below we shall outline the method used to calculate such in the simplest case $n = 2$. The extension to arbitrary n is straightforward.

If we substitute (4.1) into (1.1) we find that $\hat{\chi}_2(x)$ satisfies the following differential equation:

$$\begin{aligned} x\frac{d^2\hat{\chi}_2}{dx^2} + (x+2)\frac{d\hat{\chi}_2}{dx} + \lambda\hat{\chi}_2 = \frac{-1}{2\omega_1\left(\omega_1^2-4\right)} \Bigg\{ & \left[2\omega_2\left(3\omega_1^2-4\right)+\omega_1\left(20-\omega_1^2\right)x\right]x\frac{d^2\hat{\chi}_1}{dx^2} \\ & + \left[2\omega_2(x+2)\left(3\omega_1^2-4\right)+\omega_1\left(32-4\omega_1+28x-3x\omega_1^2\right)x\right]\frac{d\hat{\chi}_1}{dx} \\ & + \left[\omega_2\left(5\omega_1^4-21\omega_1^2+4\omega_1+12\right)+2\left\{\left(7\omega_1^3-2\omega_1^2-24\omega_1+16\right)x-\left(\omega_1^3+\omega_1^2+4\right)\right\}\right]\hat{\chi}_1 \Bigg\} \,. \end{aligned} \tag{5.12}$$

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When $n = 2$, in order to calculate $\bar{\omega}_2$, it is sufficient to take $\lambda = 2$, $\delta = 0$, $\hat{\chi}_1(x) = e^{-x}$, when (5.12) reduces to

$$x\frac{d^2\hat{\chi}_2}{dx^2} + (x+2)\frac{d\hat{\chi}_2}{dx} + 2\hat{\chi}_2 = K_1e^{-x}\left[K_2\bar{\omega}_2+K_3x^2+K_4x+K_5\right], \tag{5.13}$$

where K_n, $n = 1, 2, \ldots, 5$ are constants depending only on $\bar{\omega}_1$.

The structure of $G_2(\mu)$ (see (2.5)) in this case requires that

$$\hat{\chi}_2(x) = \left(a_1x^2+a_2x+a_3\right)e^{-x}, \tag{5.14}$$

where a_n $(n = 1, 2, 3)$ are constants to be determined.

In order that (5.14) shall satisfy (5.13) we find that

$$\bar{\omega}_2 = -\left[\frac{46\bar{\omega}_1^3-10\bar{\omega}_1^2-208\bar{\omega}_1+56}{5\bar{\omega}_1^4-33\bar{\omega}_1^2+4\bar{\omega}_1+28}\right]. \tag{5.15}$$

In this case $(n = 2)$, $\bar{\omega}_1 = 0.2892$, so from (5.15), $\bar{\omega}_2 = 0.1467$.

The values of δ_o, δ_e together with associated values of ω_1^* are tabulated in Table 1 for the case $n = 2$ when $F = 10, 20, 25, 30$.

Odd solutions				Even solutions		
δ_o	ω_1^*	ω from (5.16)	F	δ_e	ω_1^*	ω from (5.16)
.044	-.0038	.314	10	-.27	.025	.343
.019	-.0017	.302	20	-.13	.012	.315
.0096	-.0008	.300	25	-.068	.0059	.307
.00044	-.0004	.299	30	-.0031	.0027	.302

Table 1. Values of δ_o, δ_e, ω_1^* and ω from (5.16) for case $n = 2$

Further, the values of ω, as calculated from the approximation

$$\omega = \bar{\omega}_1 + \omega_1^* + (2/F)\bar{\omega}_2, \tag{5.16}$$

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are also given in Table 1.

6. Discussion

For small values of F, χ and ω can be expressed as power series in F. The procedure is straightforward, see, for example, Rickard ([2], [3]). It is found that

$$\omega = \tilde{\omega}_1 + \frac{\tilde{\omega}_1(\tilde{\omega}_1-1)\left(-3\tilde{\omega}_1^3-19\tilde{\omega}_1^2-12\tilde{\omega}_1+8\right)}{4(3\tilde{\omega}_1-8)} F + O(F^2) , \tag{6.1}$$

where

$$\tilde{\omega}_1 = \frac{2}{n(n+1)} , \quad n = 1, 2, \ldots . \tag{6.2}$$

Numerical computation of eigenvalues of (1.1) is also straightforward, using a method devised by Picken ([1]) for solving ordinary differential equations in terms of Chebyshev series. The reader is referred to Rickard ([2], [3]) for full details of the method. In Table 2 those eigenvalues which are 1/3, 1/6 when $F = 0$ are tabulated for values of F from 10 to 30. The results are illustrated graphically in Figure 1.

F	ω	
0	1/3	1/6
1	.344	.189
5	.356	.244
10	.346	.274
15	.332	.288
20	.318	.293
25	.309	.296
30	.302	.295

Table 2. Numerical calculation of eigenvalues

The values of ω predicted by (5.16) and given in Table 1 are also illustrated in Figure 1. The even solution (for which $\omega = 1/3$ when $F = 0$) is in good agreement with the numerical results even for F as low

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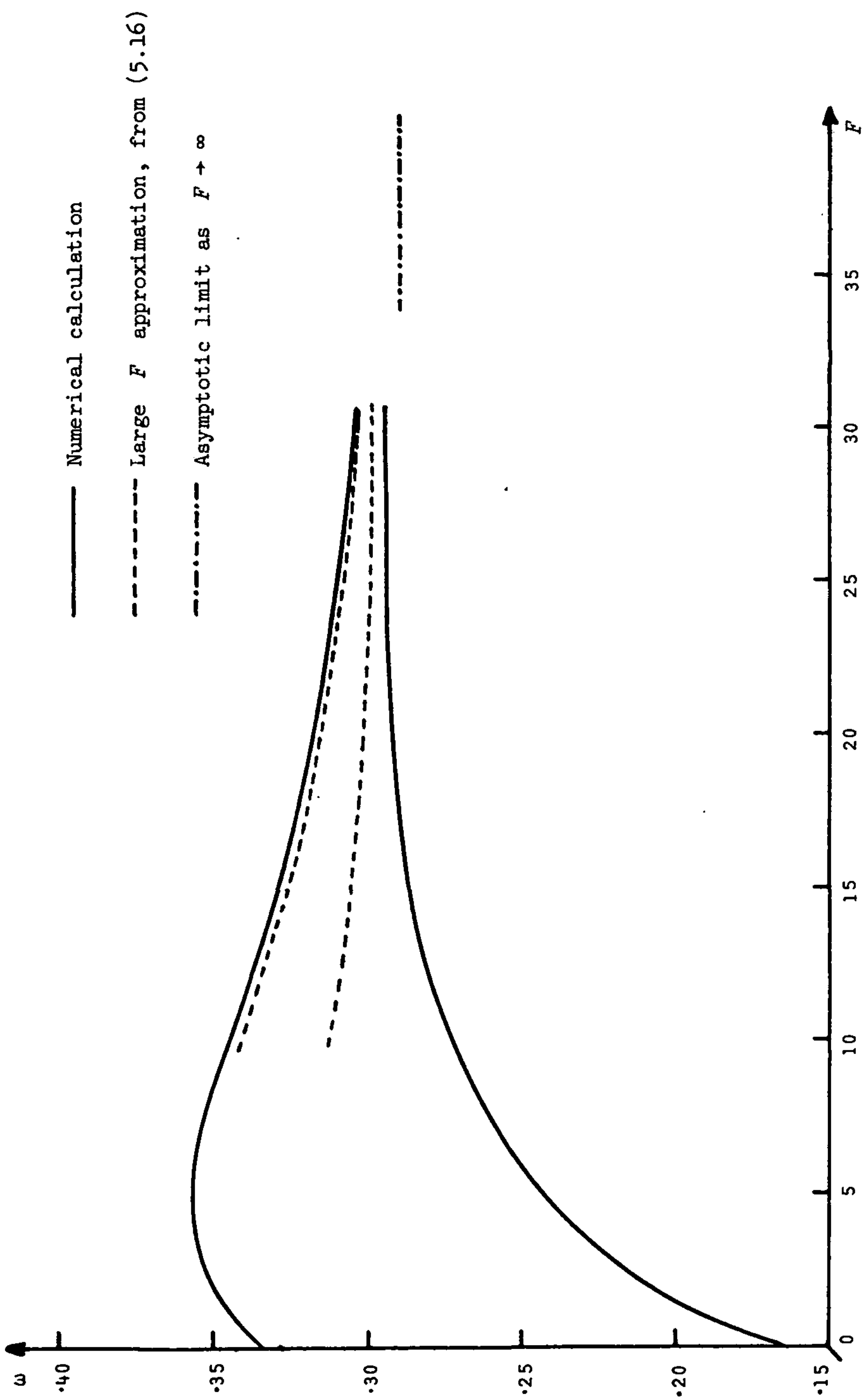

Figure 1. Eigenvalues for $n = 2$

as 10 . However, the odd solution ($\omega = 1/6$, $F = 0$) is not predicted to good approximation by (5.16) until F is greater than about 25 . This is due to the fact that (5.16) will be a good approximation only after the actual solution curve has passed its maximum ($F = 5$ for even solution and $F = 25$ for odd solution, approximately); that is, when exponential decay dominates. In fact, for large F , ω may be taken as $\bar{\omega}_1 + (2/F)\bar{\omega}_2$ to good approximation, but ω_1^* must be considered if it is required to distinguish between the even and odd eigenvalues.

For $F < 0$ similar arguments may be used and it is found that all eigenvalues tend to zero as $F \to -\infty$ (except the exceptional eigenvalue for which $\omega = 1$ for all F).

The method given in this paper can be used to study other equations similar to (1.1) provided that it is possible to find the solution away from the neighbourhood of any boundary layers; see (2.3), (2.6). The solution in the neighbourhood of $\mu = 0$ and $\mu = 1$ is unlikely to cause any difficulty.

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