Compositio Mathematica **118:** 203–215, 1999.

On Seshadri Constants of Canonical Bundles of Compact Complex Hyperbolic Spaces

JUN-MUK HWANG[1*] and WING-KEUNG TO[2]
[1]*Department of Mathematics, Seoul National University, Seoul 151-742, Korea; e-mail: jmhwang@math.snu.ac.kr*
[2]*Department of Mathematics, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260; e-mail: mattowk@leonis.nus.edu.sg*

(Received: 15 January 1998; accepted in final form: 25 March 1998)

Abstract. Upper and lower bounds for the Seshadri constants of canonical bundles of compact complex hyperbolic spaces are given in terms of metric invariants. The lower bound is obtained by carrying out the symplectic blow-up construction for the Poincaré metric, and the upper bound is obtained by a convexity-type argument.

Mathematics Subject Classification (1991): 32J25, 14E25, 32M15.

Key words: Seshadri constant, complex hyperbolic space, symplectic blow-up, singular Hermitian metric.

0. Introduction

Let X be an n-dimensional projective algebraic manifold, and L be an ample line bundle over X. For a given point $x \in X$, Demailly ([De]) has introduced the Seshadri number $\varepsilon(L, x)$ of L at x as a way to measure the 'local positivity' of L at x. To be precise, let $f: Y \to X$ be the blow-up of X at x. Then $\varepsilon(L, x)$ is defined as the supremum of all positive numbers ε such that the $\mathbb{R}$-divisor class $f^*L - \varepsilon E$ is nef on Y (cf. [De] for other equivalent definitions of $\varepsilon(L, x)$). This number is useful in the investigation of generation of jets at x by sections of $K_X + L$, where K_X denotes the canonical line bundle of X. In fact, Demailly proved the following.

PROPOSITION 0.1 ([De, Prop. 6.8]). *If* $\varepsilon(L, x) > n + s$, *then* $H^0(X, K_X + L)$ *generates all* s*-jets at* x. *If* $\inf_{x \in X} \varepsilon(L, x) > 2n$, *then* $K_X + L$ *is very ample.*

Hence, it is valuable to have lower bounds on Seshadri numbers. Upper bounds of Seshadri numbers are also interesting, as such bounds often give interesting geometric informations. An example is the existence of irreducible curves $C \subset X$ passing through the point x and with bounded $L \cdot C/\mathrm{mult}_x(C)$ (cf. [De, Sect. 6]). Here $L \cdot C$ denotes the degree of L over C, and $\mathrm{mult}_x(C)$ denotes the multiplicity of

---

* Supported by RIBS and by the KOSEF through the GARC at Seoul National University.

PDF VS o/s (editing) INTERPRINT Article COMP4385 (compkap:mathfam) v.1.15
166581.tex; 23/08/1999; 9:11; p.1

C at x. A quite general, but somewhat weak, lower bound is obtained by [EKL] for sufficiently general points of arbitrary projective manifolds and ample line bundles. Better lower bounds are known for surfaces by [EL].

In [La], Lazarsfeld studied the case of principally polarized Abelian varieties and related the Seshadri constants of the theta divisor to metric invariants to get an interesting lower bound. Lazarsfeld used 'symplectic blow-up' ([GS], [MP]), which is a lifting of the flat Kähler metric to the blow-up of one point in $\mathbb{C}^n$. In this paper, we will study the case of compact quotients of the unit ball in $\mathbb{C}^n$, and give upper and lower bounds on the Seshadri numbers in terms of metric invariants. The lower bound is obtained by a variation of Lazarsfeld's method. In our case, we construct 'symplectic blow-up' of the Poincaré metric. Our upper bound of the Seshadri number is obtained by using a convexity-type argument.

It should be mentioned that our methods depend heavily on the radial symmetry of the Poincaré metric, and do not seem to generalize easily to compact quotients of other bounded symmetric domains.

1. Statement of Results

Let $\mathrm{B}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z| < 1\}$ be the unit ball in $\mathbb{C}^n$, where $|z|^2 = z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$. B^n is equipped with the Poincaré metric whose Kähler form is given by

$$\omega = \sqrt{-1}\left(\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{1-|z|^2} + \frac{\bar{z}\,\mathrm{d}z \wedge z\,\mathrm{d}\bar{z}}{(1-|z|^2)^2}\right),$$

where we used the notations

$$\mathrm{d}z \wedge \mathrm{d}\bar{z} = \sum_{i=1}^{n} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i,$$

$$\bar{z}\,\mathrm{d}z = \sum_{j=1}^{n} \bar{z}_j\,\mathrm{d}z_j, \qquad z\,\mathrm{d}\bar{z} = \sum_{k=1}^{n} z_k\,\mathrm{d}\bar{z}_k.$$

The Ricci form of ω is $-(n+1)\omega$. For any two points $z, z' \in \mathrm{B}^n$, we denote by $d(z, z')$ the Poincaré distance between them.

Let $\Gamma \subset \mathrm{PU}(1, n)$ be a discrete torsion-free cocompact subgroup and $X = \mathrm{B}^n/\Gamma$ be the associated smooth compact quotient. It is well-known that ω is invariant under $\mathrm{PU}(1, n)$ and thus descends to a Kähler form on X, which we denote by the same symbol. Such an (X, ω) is called a compact complex hyperbolic space. For a given point $x \in X$, choose an inverse image $x_0 \in \mathrm{B}^n$ of x. The injectivity radius of X at x is defined to be $\rho_x = \frac{1}{2}\min_{\gamma\in\Gamma, \gamma x_0 \neq x_0} d(x_0, \gamma x_0)$.

Denote also by $d(\cdot, \cdot)$ the distance function on X with respect to ω. Then the diameter D_x of X at x is defined to be $D_x = \max_{y\in X} d(x, y)$.

Then the injectivity radius ρ_X and diameter D_X of (X,ω) are given by $\rho_X := \min_{x\in X}\rho_x$ and $D_X := \max_{x\in X} D_x$ respectively. It is easy to see that one always has $\rho_x \leqslant D_x$ and thus also $\rho_X \leqslant D_X$. For $x \in X$, let $\varepsilon(K_X, x)$ be the Seshadri number of K_X at x as defined in Section 0, and let $\varepsilon(K_X) := \inf_{x\in X}\varepsilon(K_X, x)$. Our main result in this paper is the following

THEOREM 1.1. *Let (X,ω) be an n-dimensional compact complex hyperbolic space.*

(i) *Then for any point $x \in X$, we have*

$$(n+1)\sinh^2(\rho_x) \leqslant \varepsilon(K_X, x) \leqslant (n+1)\sinh^2(D_x).$$

(ii) *In particular, we have*

$$(n+1)\sinh^2(\rho_x) \leqslant \varepsilon(K_X) \leqslant (n+1)\sinh^2(D_X).$$

Combining Theorem 1.1 with Proposition 0.1 of Demailly, we immediately have

COROLLARY 1.2. *Let (X,ω) be as in Theorem* 1.1. *If $\rho_X > \sinh^{-1}\sqrt{2n/(n+1)}$, then $2K_X$ is very ample. In particular, for any given X, there exists a finite etale cover X' so that $2K_{X'}$ is very ample.*

We remark that our upper bound combined with [EKL] gives a uniform lower bound of the diameter D_X. But this is weaker than the one obtained using Gauss–Bonnet.

2. Proof of the Lower Bound

In this section, we are going to prove the lower bound for $\varepsilon(K_X, x)$. The idea is to produce a lift of ω to the blow-up of B^n at the origin with sufficient positivity along the exceptional divisor by carrying out the Guillemin–Sternberg construction using the Poincaré radius.

For a point $z \in \mathrm{B}^n$, let $|z|_P = d(z,0)$ be the Poincaré distance of z from the origin 0. We have the well-known relations

$$|z|_P = \tfrac{1}{2}\log\frac{1+|z|}{1-|z|},$$

$$|z|^2 = 1 - \frac{1}{\cosh^2|z|_P} = \tanh^2|z|_P.$$

For a number $c > 0$, we define $\mathrm{B}^n(c) := \{z \in \mathrm{B}^n \mid |z|_P < c\}$. Using the notations in Section 1, we have

LEMMA 2.1. *Let $\phi(r)$ be a positive real-valued smooth function of a positive real variable r, such that $\phi(r)r$ is monotone increasing. Let $\delta > 0$ be a small positive* *number such that* $|\phi(|z|)z| < 1$ *for* $z \in \mathrm{B}^n(\delta)\backslash\{0\}$. *Define* $\Phi\colon \mathrm{B}^n(\delta)\backslash\{0\} \to \mathrm{B}^n$ *by* $\Phi(z) = \phi(|z|)z$. *Then*

$$\begin{aligned}\Phi^*\omega \;=\; & \sqrt{-1}\cosh^2(|\Phi(z)|_P)\phi^2\,\mathrm{d}z\wedge\mathrm{d}\overline{z} + \\ & +\sqrt{-1}[\cosh^2(|\Phi(z)|_P)\frac{\phi\phi'}{|z|} + \\ & + \cosh^4(|\Phi(z)|_P)(\phi^4+\phi^3\phi'|z|)]\overline{z}\,\mathrm{d}z\wedge z\,\mathrm{d}\overline{z}.\end{aligned}$$

Moreover, it is a Kähler form on $B^n(\delta)\backslash\{0\}$.

Proof. First we remark that it follows from the arguments in [MP, Sect. 5.1] that $\Phi^*\omega$ is a Kähler form on $\mathrm{B}^n(\delta)\backslash\{0\}$. Writing $\Phi_i(z) = \phi(|z|)z_i$, $1 \leqslant i \leqslant n$, and using $d\phi = (\phi'/2|z|)(\overline{z}\,\mathrm{d}z + z\,\mathrm{d}\overline{z})$, we have

$$\begin{aligned}\sum_{i=1}^n d\Phi_i\wedge d\overline{\Phi}_i \;&=\; \sum_{i=1}^n(\phi\,\mathrm{d}z_i + z_i\,\mathrm{d}\phi)\wedge(\phi\,\mathrm{d}\overline{z}_i + \overline{z}_i\,\mathrm{d}\phi)\\ &=\; \phi^2\,\mathrm{d}z\wedge\mathrm{d}\overline{z} + \phi\overline{z}\,\mathrm{d}z\wedge\mathrm{d}\phi + \phi\,\mathrm{d}\phi\wedge z\,\mathrm{d}\overline{z}\\ &=\; \phi^2\,\mathrm{d}z\wedge\mathrm{d}\overline{z} + \frac{\phi\phi'}{|z|}\overline{z}\,\mathrm{d}z\wedge z\,\mathrm{d}\overline{z};\end{aligned}$$

$$\begin{aligned}&\sum_{j,k=1}^n \overline{\Phi}_j\,\mathrm{d}\Phi_j\wedge\Phi_k\,\mathrm{d}\overline{\Phi}_k\\ &\quad= \sum_{j,k=1}^n \phi\overline{z}_j(\phi\,\mathrm{d}z_j + z_j\,\mathrm{d}\phi)\wedge\phi z_k(\phi\,\mathrm{d}\overline{z}_k + \overline{z}_k\,\mathrm{d}\phi)\\ &\quad= (\phi^4+\phi^3\phi'|z|)\overline{z}\,\mathrm{d}z\wedge z\,\mathrm{d}\overline{z}.\end{aligned}$$

Now one can use $(1-|\Phi(z)|^2)^{-1} = \cosh^2(|\Phi(z)|_P)$ to conclude Lemma 2.1.

Let $V \subset \mathbb{P}^{n-1}\times\mathrm{B}^n$ be the blow-up of B^n at the origin 0, and let $\alpha\colon V \to \mathbb{P}^{n-1}$ and $\beta\colon V \to \mathrm{B}^n$ be the natural maps. Let σ be the Fubini–Study (1,1)-form on $\mathbb{P}^{n-1}$ normalized so that $\int_{\mathbb{P}^1}\sigma = 2\pi$ for a complex line $\mathbb{P}^1 \subset \mathbb{P}^{n-1}$. The pull-back of the Euclidean coordinates $z_1\ldots,z_n$ by β define holomorphic functions on V, which we denote by the same letters. Let $E = \beta^{-1}(0)$ be the exceptional divisor in V. We will skip the proof of the following lemma, which can be checked directly using standard local coordinates on V, say $u_1,\ldots,u_n$ satisfying $z_1 = u_1$, $z_2 = u_1u_2,\ldots,z_n = u_1u_n$, etc. (Also see [GS, Sect. 4].)

LEMMA 2.2. *The expression* $\sqrt{-1}(1/|z|^2)\overline{z}\,\mathrm{d}z\wedge z\,\mathrm{d}\overline{z}$ *defines a smooth* $(1,1)$*-form on* V *whose restriction to a differential form on* E *is zero, and*

$$\alpha^*\sigma = \sqrt{-1}\left(\frac{\mathrm{d}z\wedge\mathrm{d}\overline{z}}{|z|^2} - \frac{\overline{z}\,\mathrm{d}z\wedge z\,\mathrm{d}\overline{z}}{|z|^4}\right).$$

https://doi.org/10.1023/A:1001074801110 Published online by Cambridge University Press

For $0 < c < 1$, let $V(c) = \beta^{-1}(\mathrm{B}^n(c))$.

LEMMA 2.3. *For a fixed number $\lambda > 0$, choose a small number $\delta > 0$ and $\phi(r)$ as in Lemma 2.1 such that the associated map $\Phi(z)\colon \mathrm{B}^n(\delta)\backslash\{0\} \to \mathrm{B}^n$ satisfies $|\Phi(z)|_P^2 = \lambda^2 + |z|_P^2$ for $z \in \mathrm{B}^n(\delta)\backslash\{0\}$ (cf. the proof below for the explicit expression of $\phi(r)$). Then the Kähler form $\beta^*\Phi^*\omega$ on $V(\delta)\backslash E$ extends to a continuous semi-positive $(1,1)$-form on $V(\delta)$ whose restriction to E is equal to $(\sinh^2\lambda)\alpha^*\sigma$.*

Proof. To satisfy $|\Phi(z)|_P^2 = \lambda^2 + |z|_P^2$, one chooses $\phi(r)$ given by, for sufficiently small r,

$$r^2\phi(r)^2 = 1 - \frac{1}{\cosh^2\sqrt{\lambda^2 + \frac{1}{4}\left(\log\frac{1+r}{1-r}\right)^2}}.$$

By taking derivatives of both sides, one can easily see that

$$\lim_{r\to 0}(\phi^2 + r\phi\phi') = \frac{\sinh\lambda}{\lambda\cosh^3\lambda} < +\infty.$$

Putting $r = |z|$, we can thus regard $r^2\phi^2$ and $\phi^2 + r\phi\phi'$ as continuous functions on $V(\delta)$. Then by Lemma 2.1, we have

$$\begin{aligned}\Phi^*\omega \;&=\; \sqrt{-1}\cosh^2(|\Phi|_P)r^2\phi^2\left(\frac{\mathrm{d}z\wedge\mathrm{d}\overline{z}}{r^2} - \frac{\overline{z}\,\mathrm{d}z\wedge z\,\mathrm{d}\overline{z}}{r^4}\right) + \\ &\quad+\sqrt{-1}\cosh^2(|\Phi|_P)(\phi^2 + r\phi\phi')\times \\ &\quad\times(1 + r^2\phi^2\cosh^2(|\Phi|_P))\frac{\overline{z}\,\mathrm{d}z\wedge z\,\mathrm{d}\overline{z}}{r^2}.\end{aligned}$$

The first term is just $\cosh^2(|\Phi|_P)r^2\phi^2\alpha^*\sigma$. Since $|\Phi(z)|_P$ converges to λ as $z \to 0$, the first term converges to $\cosh^2\lambda(1 - (1/\cosh^2\lambda))\alpha^*\sigma = \sinh^2\lambda\alpha^*\sigma$. From Lemma 2.2 and the boundedness of $\phi^2 + r\phi\phi'$, the second term restricted to E is zero. This finishes the proof of Lemma 2.3.

Now we are ready to get the desired lifting of Poincaré metric to V.

PROPOSITION 2.4. *Fix a number $\lambda > 0$. Given any small number $\eta > 0$, there exists a semi-positive closed real $(1,1)$-form τ on V, which is smooth on $V\backslash E$ and is continuous on E, such that $\tau = \beta^*\omega$ on $V\backslash V(\lambda(1+\eta))$ and $\tau = \sinh^2\lambda\alpha^*\sigma$ on E.*

Proof. By choosing a sufficiently small $\delta = \delta(\lambda,\eta) > 0$, it is clear that one can define $\Phi\colon \mathrm{B}^n\backslash\{0\} \to \mathrm{B}^n$ such that $\Phi(z) = z$ for $|z|_P > \lambda(1+\eta)$, $|\Phi(z)|_P$ depends only on the value of $|z|_P$, and it is a monotone increasing smooth function of $|z|_P$; moreover, $\Phi(z)$ takes the form in Lemma 2.1 with $|\Phi(z)|_P^2 = \lambda^2 + |z|_P^2$ for $|z|_P < \delta$. By Lemma 2.3, $\beta^*\Phi^*\omega$ on $V\backslash E$ can be extended to the desired τ on V, which gives Proposition 2.4.

https://doi.org/10.1023/A:1001074801110 Published online by Cambridge University Press

Now that we have the lifting of ω, the proof of the lower bound for $\varepsilon(K_X, x)$ given below is just a direct translation of the proof in [La], replacing 'Abelian varieties' by 'ball quotients'.

PROPOSITION 2.5. *For any point $x \in X$, we have $\varepsilon(K_X, x) \geqslant (n+1)\sinh^2(\rho_x)$.*

Proof. Let $f: Y \to X$ be the blow-up of X at x and E be the exceptional divisor. For any number λ satisfying $0 < \lambda < \rho_x$, we want to show that the $\mathbb{R}$-divisor class $f^*K_X - (n+1)(\sinh^2\lambda)E$ is nef. Fix $\eta > 0$ so that $\lambda(1+3\eta) < \rho_x$. Using the covering projection map, we have a Kähler isometric embedding of $\mathrm{B}^n(\lambda(1+3\eta))$ into X sending 0 to x. For $\nu < \lambda(1+3\eta)$, $V(\nu)$ can be viewed as being embedded in Y as a neighborhood of E. From Proposition 2.4, we get a semi-positive closed real (1, 1) -form τ on Y agreeing with ω off $V(\lambda(1+2\eta))$ and with $\sinh^2\lambda\alpha^*\sigma$ on E. Since the class of E restricted to E itself is equal to $-(1/2\pi)\sigma$ and $K_X = ((n+1)/2\pi)\omega$ on X as Kähler classes, $((n+1)/2\pi)\tau$ represents $f^*K_X - (n+1)(\sinh^2\lambda)E$ (cf. [La]) and the required nefness follows easily.

3. Proof of the Upper Bound

Let (X, ω) be as in Theorem 1.1. We are going to prove the upper bound for $\varepsilon(K_X, x)$ stated in Theorem 1.1.

First we recall the following definitions in [De, Sect. 2]. For any point $x \in X$, we denote by $\mathcal{C}(K_X, x)$ the class of singular Hermitian metrics of K_X which are $\mathcal{C}^\infty$ on $X\backslash\{x\}$ and of class L^1_{loc} on X. Also we let

$$\mathcal{C}^+(K_X, x) := \{h \in \mathcal{C}(K_X, x) \mid c_1(K_X, h) \geqslant 0\}. \tag{3.1}$$

Here $2\pi c_1(K_X, h)$ denotes the curvature (1, 1)-current of h on X. In terms of a holomorphic trivialization $K_X|_U \sim U \times \mathbb{C}$ over an open coordinate neighborhood U of x, suppose h is given by a function ψ_h satisfying $h(v, v) = \mathrm{e}^{-2\psi_h}|v|^2$, $v \in K_X|_U$. Then it is well known that the Lelong number

$$\nu(\psi_h, x) := \liminf_{z\to x} \frac{\psi_h(z)}{\log|z-x|} \tag{3.2}$$

is independent of the choice of the trivialization of K_X, i.e. $\nu(\psi_h, x)$ depends only on h and x. By [De, Thm 6.4 and Rem. 6.6], $\varepsilon(K_X, x)$ is also given by

$$\varepsilon(K_X, x) = \sup_{h\in\mathcal{C}^+(K_X,x)} \nu(\psi_h, x). \tag{3.3}$$

We denote by h_0 the smooth Hermitian metric on K_X induced by the Poincaré metric ω (so that $2\pi c_1(K_X, h_0) = (n+1)\omega)$. For any point $x \in X$ and $h \in \mathcal{C}(K_X, x)$, we let ϕ_h be the unique function on X (possibly singular at x) satisfying

$$h = h_0\,\mathrm{e}^{-2\phi_h} \quad \text{on } X. \tag{3.4}$$

Since h_0 is smooth, it follows from (3.3) that we also have

$$\varepsilon(K_X, x) = \sup_{h\in\mathcal{C}^+(K_X,x)} \nu(\phi_h, x). \tag{3.5}$$

Now we fix a point $x \in X$, and we fix an $h \in \mathcal{C}^+(K_X, x)$. Without loss of generality, we will assume that $\phi_h(y) \to -\infty$ as $y \to x$ (otherwise, $\nu(\phi_h, x) \leqslant 0$). Denote by $\pi\colon \mathrm{B}^n \to X$ the covering projection map sending the origin 0 to x. Then we define a function $\mu_h\colon \mathrm{B}^n \to \mathbb{R} \cup \{-\infty\}$ given by

$$\begin{aligned}\mu_h(z) \;&=\; 2\phi_h(\pi(z)) - (n+1)\log(1-|z|^2)\\ &\qquad \text{for } \; z \in \mathrm{B}^n\backslash\pi^{-1}\{x\}\end{aligned} \tag{3.6}$$

and $\mu_h(z) = -\infty$ for $z \in \pi^{-1}\{x\}$. Using (3.4), it is easy to check that $\sqrt{-1}\partial\overline{\partial}\mu_h = 2\pi \cdot \pi^* c_1(K_X, h) \geqslant 0$ on $\mathrm{B}^n\backslash\pi^{-1}\{x\}$, and thus μ_h is a plurisubharmonic function on B^n (here one may regard μ_h as extended from $\mathrm{B}^n\backslash\pi^{-1}\{x\}$). Also, it is easy to see from (3.6) that

$$\nu(\mu_h, 0) = 2\nu(\phi_h, x). \tag{3.7}$$

DEFINITION 3.1. A real-valued function f defined on a subset $S \subset \mathbb{C}^n$ is said to be *radially symmetric about the origin* 0 if $f(z)$ depends only on $|z|$ for all $z \in S$, i.e. $f(z) = F(|z|^2)$ for some function $F\colon \mathbb{R} \to \mathbb{R}$.

Now we define another function μ_h^* given by

$$\mu_h^*(z) = \max_{|w|=|z|} \mu_h(w) \quad \text{for } \; z \in \mathrm{B}^n. \tag{3.8}$$

We shall need some simple properties of μ_h^* given in the following lemma.

LEMMA 3.2. (i) μ_h^* *takes values on* $\mathbb{R} \cup \{-\infty\}$, *and* $\mu_h^*(z) = -\infty$ *only at* $z = 0$.

(ii) μ_h^* *is radially symmetric about 0* (*cf. Definition* 3.1).

(iii) μ_h^* *is a plurisubharmonic function on* B^n.

(iv) μ_h^* *is locally Lipschitz on* $\mathrm{B}^n\backslash\{0\}$, *i.e. for any compact subset* $K \subset \mathrm{B}^n\backslash\{0\}$, *there exists a constant* $C_K > 0$ *such that* $|\mu_h^*(z) - \mu_h^*(z')| \leqslant C_K|z - z'|$ *for* $z, z' \in K$. *In particular,* μ_h^* *is bounded on compact subsets of* $\mathrm{B}^n\backslash\{0\}$.

(v) $\nu(\mu_h^*, 0) = \nu(\mu_h, 0)$.

Proof. (i) follows easily from the facts that $\mu_h \in \mathcal{C}^\infty(\mathrm{B}^n\backslash\pi^{-1}\{x\})$ and $\mu_h(z) \to -\infty$ as z approaches the discrete set $\pi^{-1}\{x\}$. (ii) follows trivially from (3.8). Next we observe that $\mu_h^*(z) = \max_{U\in U(n)} \mu_h(Uz)$, where $U(n)$ denotes the set of $n \times n$ unitary matrices. It is well known that if the supremum of an (infinite) set of plurisubharmonic functions is an upper semi-continuous function (and $< \infty$), then it is automatically plurisubharmonic (cf., e.g., [H, pp. 16]). Thus, by (i), in order to prove (iii), it suffices to check that μ_h^* is upper semi-continuous on B^n. For any $c \in \mathbb{R}$, let $z_0 \in \mathrm{B}^n$ be a point such that $\mu_h^*(z_0) < c$. Then $\mu_h^*(z) < c$ for all

z satisfying $|z| = |z_0|$. Write $\pi^{-1}\{x\}$ as $\{x_i\}_{i\in I}$, where I is a (countable) index set. For each $i \in I$, choose an open neighborhood U_i of x_i such that $\mu_h < c$ on U_i. Together with the smoothness of μ_h on $\mathrm{B}^n\backslash\bigcup_{i\in I} U_i$, it follows that there exists $\varepsilon > 0$ such that $\mu_h(z) < c$ (and hence $\mu_h^*(z) < c$) for all z satisfying $|z_0| - \varepsilon < |z| < |z_0| + \varepsilon$. Thus μ_h^* is upper semi-continuous on B^n, which then leads to (iii). Next we proceed to prove (iv). Obviously we just have to consider compact subsets of the form $K_{a,b} := \{z \in \mathrm{B}^n | a \leqslant |z| \leqslant b\}$, where $0 < a < b < 1$. As in the proof of (iii), it is clear that one can choose open neighborhoods V_i of x_i, $i \in I$, such that the values of μ_h^* on $K_{a,b}$ depend only on those of μ_h on $K_{a,b}\backslash\bigcup_{i\in I} V_i$. Then the Lipschitzness of μ_h^* on $K_{a,b}$ follows easily from that of μ_h on $K_{a,b}\backslash\bigcup_{i\in I} V_i$, and (iv) follows. To prove (v), we first observe that (3.8) implies easily that $\nu(\mu_h^*, 0) \leqslant \nu(\mu_h, 0)$. Also the usual convexity properties of plurisubharmonic functions imply that $\mu_h(z) \leqslant \nu(\mu_h, 0)\cdot\log|z| + \mathrm{O}(1)$ near 0 (cf., e.g., [De, Proof of Lemma 2.8]). Together with (3.8), one also sees that $\mu_h^*(z) \leqslant \nu(\mu_h, 0)\cdot\log|z| + O(1)$ near 0, which then implies $\nu(\mu_h^*, 0) \geqslant \nu(\mu_h, 0)$, and (v) follows. Thus we have finished the proof of Lemma 3.2.

Next we construct smooth plurisubharmonic approximations of μ_h^* as follows. Fix a nonnegative function $\eta \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ such that η is radially symmetric about 0 with $\int_{\mathbb{C}^n} \eta(\zeta)\,\mathrm{d}\lambda(\zeta) = 1$. Here λ denotes the standard Lebesque measure on $\mathbb{C}^n$. Then for $\tau > 0$, we define

$$\mu_{h,\tau}^*(z) := \int_{\mathbb{C}^n} \mu_h^*(z - \tau\zeta)\eta(\zeta)\,\mathrm{d}\lambda(\zeta). \tag{3.9}$$

It is clear that for any $0 < r < 1$, there exists $\tau_0 > 0$ (depending on r and $\mathrm{supp}(\eta)$) such that for $0 < \tau < \tau_0$, $\mu_{h,\tau}^*$ is well-defined on $\{z \in \mathrm{B}^n \mid |z|^2 < r\}$. Moreover, we have

LEMMA 3.3. (i) *For each* $\tau > 0$, $\mu_{h,\tau}^*$ *is a smooth and plurisubharmonic function, and for* $z \in \mathrm{B}^n$, $\mu_{h,\tau}^*(z) \to \mu_h^*(z)$ *as* $\tau \to 0$.

(ii) *Each* $\mu_{h,\tau}^*$ *is radially symmetric about* 0.

(iii) $\mu_{h,\tau}^*$ *converges uniformly to* μ_h^* *on compact subsets of* $\mathrm{B}^n\backslash\{0\}$ *as* $\tau \to 0$.

Proof. (i) is well known and can be found, for example, in [H, pp. 45]. (ii) follows easily from the radial symmetry of μ_h^* and η about 0. Finally, (iii) can be checked easily using (3.9) and Lemma 3.2(iv).

Let $F_h(r)\colon (0, 1) \to \mathbb{R}$ be the function such that $\mu_h^*(z) = F_h(|z|^2)$ for $\mathrm{B}^n\backslash\{0\}$ (cf. Definition 3.1 and Lemma 3.2(i), (ii)). Similarly, for $\tau > 0$, we let $F_{h,\tau}(r)$ be the function such that $\mu_{h,\tau}^*(z) = F_{h,\tau}(|z|^2)$ (cf. Lemma 3.3(ii)). Let D_x be as in Theorem 1.1. We have

PROPOSITION 3.4. (i) *For each* $\tau > 0$, $F_{h,\tau}$ *is a smooth function. Moreover,* $F_{h,\tau}$ *converges uniformly to* F_h *on compact subsets of* $(0, 1)$ *as* $\tau \to 0$.

(ii) *There exists* $r_0 > 0$ *such that for all* ε *satisfying* $0 < \varepsilon < 1$, *there exists* $\tau_\varepsilon > 0$ (*depending on* ε) *such that for* $0 < \tau < \tau_\varepsilon$, *there exists* $r_{\tau,\varepsilon}$ *satisfying* $r_0 < r_{\tau,\varepsilon} < \tanh^2(D_x)$ *such that*

$$|F'_{h,\tau}(r_{\tau,\varepsilon})| \leqslant \frac{1}{1-\varepsilon}((n+1)\cosh^2(D_x)+\varepsilon). \tag{3.10}$$

Proof. (i) Follows easily from the corresponding statements for $\mu^*_{h,\tau}$, μ^*_h in Lemma 3.3 (i), (iii). Next we proceed to prove (ii). Recall that $\phi_h \in \mathcal{C}^\infty(X\backslash\{x\})$ and $\phi_h(y) \to -\infty$ as $y \to x$. It follows that ϕ_h attains maximum at some point $x^* \in X\backslash\{x\}$. By definition of D_x, we necessarily have $\pi(\mathrm{B}^n(D_x)) = X$. Thus there exists $z^* \in \mathrm{B}^n(D_x)\backslash\{0\}$ such that $\pi(z^*) = x^*$. Write $r^* = |z^*|^2$. Then we have

$$0 < r^* = \tanh^2|z^*|_P \leqslant \tanh^2(D_x) \tag{3.11}$$

(cf. Section 2). Since $d\phi_h(x^*) = 0$, it follows that, in terms of Euclidean coordinates of B^n (with z^* identified with x^*), there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that for $0 < \delta < \delta_1$, $|d\phi_h(z)| \leqslant \varepsilon|z^*|/8$ if $|z - z^*| \leqslant \delta/|z^*|$. Then we have

$$0 \leqslant \phi_h(x^*) - \phi_h(\pi(z)) \leqslant \frac{\varepsilon\delta}{8} \quad \text{if } |z - z^*| \leqslant \frac{\delta}{|z^*|}. \tag{3.12}$$

Now define $\phi^*_h(z) := \max_{|w|=|z|} \phi_h(\pi(w))$ for $z \in \mathrm{B}^n$ as in (3.8), so that by (3.6) and (3.8), we have

$$\mu^*_h(z) = 2\phi^*_h(z) - (n+1)\log(1-|z|^2) \quad \text{for } z \in \mathrm{B}^n. \tag{3.13}$$

By definition, ϕ^*_h is radially symmetric about 0. Thus by (3.12), we also have, for $0 < \delta < \delta_1$

$$0 \leqslant \phi_h(x^*) - \phi^*_h(z) \leqslant \frac{\varepsilon\delta}{8} \quad \text{if } |\,|z| - |z^*|\,| \leqslant \frac{\delta}{|z^*|}. \tag{3.14}$$

This implies that, for $0 < \delta < \delta_1$ and $z_1, z_2 \in \mathrm{B}^n$

$$|\phi^*_h(z_1) - \phi^*_h(z_2)| \leqslant \frac{\varepsilon\delta}{4} \quad \text{if } |\,|z_i| - |z^*|\,| \leqslant \frac{\delta}{|z^*|}, \quad i = 1, 2 \tag{3.15}$$

(and in particular, if $|\,|z_i|^2 - |z^*|^2\,| \leqslant \delta$, $i = 1, 2$). Next, using $\mathrm{d}/\mathrm{d}r \log(1-r) = -(1/(1-r))$ and the mean-value theorem, one sees that there exists $\delta_2 = \delta_2(\varepsilon) > 0$ such that

$$\begin{aligned} &|\log(1-|z_1|^2) - \log(1-|z_2|^2)| \\ &\quad \leqslant \left(\frac{1}{1-|z^*|^2} + \frac{\varepsilon}{2(n+1)}\right) |\,|z_1|^2 - |z_2|^2\,| \end{aligned} \tag{3.16}$$

https://doi.org/10.1023/A:1001074801110 Published online by Cambridge University Press

if $|\,|z_i|^2-|z^*|^2\,|\leqslant\delta_2$, $i=1,2$. Now choose

$$r_0=\frac{r^*}{2}>0\quad\text{and}\quad\delta=\min\left(\delta_1,\delta_2,\frac{r^*}{2}\right)>0 \tag{3.17}$$

(thus only δ depends on ε). Then it is clear that there exists $\tau_\varepsilon>0$ (depending on ε and $\mathrm{supp}(\eta)$) such that for any $0<\tau<\tau_\varepsilon$, $\zeta\in\mathrm{supp}(\eta)$ and $z\in\mathrm{B}^n$ satisfying $|\,|z|^2-|z^*|^2\,|\leqslant\delta(1-\varepsilon)$, one has

$$|\,|z-\tau\zeta|^2-|z^*|^2\,|\leqslant\delta,\qquad|\,|z^*-\tau\zeta|^2-|z^*|^2\,|\leqslant\delta$$

and

$$|\,|z-\tau\zeta|^2-|z^*-\tau\zeta|^2\,|\leqslant\delta$$

which, together with (3.9), (3.13), (3.15) and (3.16), imply that

$$\begin{aligned}
&|\mu^*_{h,\tau}(z)-\mu^*_{h,\tau}(z^*)|\\
&\quad\leqslant\int_{\mathbb{C}^n}|\mu^*_h(z-\tau\zeta)-\mu^*_h(z^*-\tau\zeta)|\eta(\zeta)\,\mathrm{d}\lambda(\zeta)\\
&\quad\leqslant\left(2\cdot\frac{\varepsilon\delta}{4}+(n+1)\left(\frac{1}{1-|z^*|^2}+\frac{\varepsilon}{2(n+1)}\right)\delta\right)\int_{\mathbb{C}^n}\eta(\zeta)\,\mathrm{d}\lambda(\zeta)\\
&\quad=\left(\frac{n+1}{1-|z^*|^2}+\varepsilon\right)\delta.
\end{aligned}\tag{3.18}$$

Then (3.18) implies that, for $0<\tau<\tau_\varepsilon$ and $0\leqslant r^*-r\leqslant\delta(1-\varepsilon)$, one has

$$|F_{h,\tau}(r)-F_{h,\tau}(r^*)|\leqslant\left(\frac{n+1}{1-r^*}+\varepsilon\right)\delta. \tag{3.19}$$

By the mean-value theorem, one sees from (3.19) that for $0<\tau<\tau_\varepsilon$, there exists $r_{\tau,\varepsilon}$ satisfying $0<r^*-r_{\tau,\varepsilon}<\delta(1-\varepsilon)$ (and thus $r_0<r_{\tau,\varepsilon}<\tanh^2(D_x)$ by (3.11) and (3.17)) and such that

$$\begin{aligned}
|F'_{h,\tau}(r_{\tau,\varepsilon})|&\leqslant\frac{1}{1-\varepsilon}\left(\frac{n+1}{1-r^*}+\varepsilon\right)\\
&\leqslant\frac{1}{1-\varepsilon}\left(\frac{n+1}{1-\tanh^2(D_x)}+\varepsilon\right),
\end{aligned}\tag{3.20}$$

which leads to (3.10). Thus we have finished the proof of Proposition 3.4.

We shall also need the following.

PROPOSITION 3.5. *For* $c>0$, *let* $f\colon\{z\in\mathbb{C}^n\mid 0<|z|^2<c\}\to\mathbb{R}$ *be a smooth plurisubharmonic function which is radially symmetric about the origin* 0, *and let*

https://doi.org/10.1023/A:1001074801110 Published online by Cambridge University Press

$F(r)\colon (0,c)\to\mathbb{R}$ be the associated smooth function such that $f(z) = F(|z|^2)$ for $0 < |z|^2 < c$ (cf. Definition 3.1). Then for any number r_* satisfying $0 < r_* < c$, we have

$$f(z) \geqslant A\log|z|^2 + B \quad \text{for } 0 < |z|^2 \leqslant r_*, \tag{3.21}$$

where

$$A = r_* F'(r_*) \quad \text{and} \quad B = F(r_*) - r_* F'(r_*)\log r_*. \tag{3.22}$$

Proof. Let $f(z)$ and $F(z)$ be as above. A simple calculation shows that for $0 < |z|^2 < c$

$$\left(\frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}\right)_{1\leqslant i,j\leqslant n} = (F'(|z|^2)\cdot\delta_{ij} + F''(|z|^2)\overline{z}_i z_j)_{1\leqslant i,j\leqslant n}. \tag{3.23}$$

Moreover, the matrix $(\overline{z}_i z_j)_{1\leqslant i,j\leqslant n}$ is of rank one with eigenvalues 0, $|z|^2$ of multiplicity $n-1$ and 1 respectively. Since f is plurisubharmonic, the matrix in (3.23) is positive semi-definite, which implies

$$F'(r) + rF''(r) \geqslant 0, \quad 0 < r < c. \tag{3.24}$$

Next we make another change of variable given by $s = \log r$, and let $G(s)$ be the function such that $F(r) = G(\log r)$ for $0 < r < c$. Using the chain rule, (3.24) implies that

$$G''(s) \geqslant 0, \quad -\infty < s < \log c. \tag{3.25}$$

Consider the function

$$H(s) := G(s) - (As + B), \quad -\infty < s < \log c, \tag{3.26}$$

where A, B are the constants in (3.22). Then one can easily check that (3.22) and (3.25) are equivalent to the following conditions

$$H''(s) \geqslant 0, \quad -\infty < s < \log c$$

and

$$H(\log r_*) = H'(\log r_*) = 0. \tag{3.27}$$

Since $H''(s) \geqslant 0$, $H'(s)$ is increasing on $-\infty < s \leqslant \log r_*$. But $H'(\log r_*) = 0$. Thus, $H'(s) \leqslant 0$ for $-\infty < s \leqslant \log r_*$. Together with the condition $H(\log r_*) = 0$, it follows that $H(s) \geqslant 0$ for $-\infty < s \leqslant \log r_*$, which leads to (3.21). Thus, we have finished the proof of Proposition 3.5.

Finally we are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. First we remark that Theorem 1.1(ii) is a direct consequence of Theorem 1.1(i). In view of Proposition 2.5, it remains to prove the upper bound for $\varepsilon(K_X, x)$ in Theorem 1.1(i). Fix a point $x \in X$, and let $h \in \mathcal{C}^+(K_X, x)$. Let ϕ_h, μ_h, μ_h^*, $\mu_{h,\tau}^*$, F_h, $F_{h,\tau}$ be the associated functions as constructed earlier. By Proposition 3.4(ii), there exists $r_0 > 0$ such that for $0 < \varepsilon < 1$, there exists $\tau_\varepsilon > 0$ such that for $0 < \tau < \tau_\varepsilon$, there exists $r_{\tau,\varepsilon}$ satisfying $r_0 < r_{\tau,\varepsilon} < \tanh^2(D_x)$ and such that (3.10) holds. In view of Lemma 3.3, we can apply Proposition 3.5 to the functions $\mu_{h,\tau}^*$, $F_{h,\tau}$ to conclude that for $0 < \varepsilon < 1$ and $0 < \tau < \tau_\varepsilon$,

$$\mu_{h,\tau}^*(z) \geqslant A_{\tau,\varepsilon} \log|z|^2 + B_{\tau,\varepsilon} \quad \text{if } 0 < |z|^2 < r_0, \tag{3.28}$$

where

$$\begin{aligned} A_{\tau,\varepsilon} &:= r_{\tau,\varepsilon} F'_{h,\tau}(r_{\tau,\varepsilon}) \quad \text{and} \\ B_{\tau,\varepsilon} &:= F_{h,\tau}(r_{\tau,\varepsilon}) - r_{\tau,\varepsilon} F'_{h,\tau}(r_{\tau,\varepsilon}) \log r_{\tau,\varepsilon}. \end{aligned} \tag{3.29}$$

By Lemma 3.2(iv), F_h is bounded on the interval $r_0 \leqslant r \leqslant \tanh^2(D_x)$. Hence, by Proposition 3.4(i), there exists $\tau^* > 0$ such that the functions $\{F_{h,\tau}\}_{0<\tau<\tau^*}$ are uniformly bounded on the interval $r_0 \leqslant r \leqslant \tanh^2(D_x)$. Together with (3.10), it follows that there exists a constant $B^* > 0$ such that

$$|B_{\tau,\varepsilon}| \leqslant B^* \quad \text{for } 0 < \varepsilon < \tfrac{1}{2}, \quad 0 < \tau < \min\{\tau_\varepsilon, \tau^*\}. \tag{3.30}$$

Combining (3.10), (3.28), (3.29) and (3.30), one has, for $0 < \varepsilon < \frac{1}{2}$, $0 < \tau < \min\{\tau_\varepsilon, \tau^*\}$ and $0 < |z|^2 < r_0$

$$\begin{aligned} \frac{\mu_{h,\tau}^*(z)}{\log|z|^2} &\leqslant A_{\tau,\varepsilon} + \frac{B^*}{|\log|z|^2|} \\ &\leqslant \frac{\tanh^2(D_x)}{1-\varepsilon}((n+1)\cosh^2(D_x) + \varepsilon) + \frac{B^*}{|\log|z|^2|}. \end{aligned} \tag{3.31}$$

By first letting $\tau \to 0$ (and using Lemma 3.3(i)) and then letting $\varepsilon \to 0$, it follows that for $0 < |z|^2 < r_0$

$$\frac{\mu_h^*(z)}{\log|z|^2} \leqslant (n+1)\sinh^2(D_x) + \frac{B^*}{|\log|z|^2|}. \tag{3.32}$$

By (3.7), Lemma 3.2(v) and taking $\liminf$ on both sides of (3.32) as $z \to 0$, one easily gets $\nu(\phi_h, x) \leqslant (n+1)\sinh^2(D_x)$. Finally by varying $h \in \mathcal{C}^+(K_X, x)$ and using (3.5), one gets $\varepsilon(K_X, x) \leqslant (n+1)\sinh^2(D_x)$, and this finishes the proof of Theorem 1.1.

References

[De] Demailly, J.-P.: *Singular Hermitian Metrics on Positive Line Bundles*, Lecture Notes in Math. 1507, Springer, New York, 1992, pp. 84–104.

[EKL] Ein, L., Küchle, O. and Lazarsfeld, R.: Local positivity of ample line bundles, *J. Differential Geom.* **42** (1995), 193–219.

[EL] Ein, L. and Lazarsfeld, R.: Seshadri constants on smooth surfaces, *Journées de Géométrie Algébriques d'Orsay, Astérisque* **218** (1993), 177–186.

[GS] Guillemin, V. and Sternberg, S.: Birational equivalence in the symplectic category, *Invent. Math.* **97** (1989), 485–522.

[H] Hörmander, L.: *An Introduction to Complex Analysis in Several Variables*, North-Holland, Amsterdam, 1973.

[La] Lazarsfeld, R.: Lengths of periods and Seshadri constants of Abelian varieties, *Math. Res. Lett.* **3** (1996), 439–447.

[MP] McDuff, D. and Polterovich, L.: Symplectic packings and algebraic geometry, *Invent. Math.* **115** (1994), 405–429.