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Dimension estimates and approximation in non-uniformly hyperbolic systems

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Abstract. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M and μ a hyperbolic ergodic f-invariant probability measure. This paper obtains an upper bound for the stable (unstable) pointwise dimension of μ, which is given by the unique solution of an equation involving the sub-additive measure-theoretic pressure. If μ is a Sinai–Ruelle–Bowen (SRB) measure, then the Kaplan–Yorke conjecture is true under some additional conditions and the Lyapunov dimension of μ can be approximated gradually by the Hausdorff dimension of a sequence of hyperbolic sets $\{\Lambda_n\}_{n\geq 1}$. The limit behaviour of the Carathéodory singular dimension of Λ_n on the unstable manifold with respect to the super-additive singular valued potential is also studied.

Key words: dimension, hyperbolic measure, hyperbolic set
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1. *Introduction*

Hyperbolic approximation plays a fundamental role in the study of smooth dynamical systems. Roughly speaking, for a hyperbolic ergodic measure μ of positive entropy, one can always find a sequence of horseshoes $\{\Lambda_n\}_{n\geq 1}$ so that the dynamical quantities on them

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are close to the corresponding ones of the measure μ. Such results can be traced back to the landmark work by Katok [18] or Katok and Hasselblatt [19]. An earlier related work was obtained by Misiurewicz and Szlenk [25] for piecewise continuous and monotone maps of interval. For more results of this type, we would like to refer the reader to [2, 8, 10, 14, 15, 27, 30, 34, 35] and the references therein.

From the point of dimension theory of dynamical systems, it is natural and non-trivial to use Hausdorff dimension to estimate how large that part of the dynamics described by these horseshoes is. If μ is an ergodic hyperbolic Sinai–Ruelle–Bowen (SRB) measure of a surface diffeomorphism, Mendoza [24] proved that the Hausdorff dimension of the horseshoes on the unstable manifolds approaches to one. For the higher dimensional case, Sánchez-Salas [31] proved that the measure μ can be approximated in the weak topology by ergodic measures supported on the horseshoes $\{\Lambda_n\}_{n\geq 1}$. Moreover, he established some interesting results concerning the Hausdorff dimension of the horseshoes. Using Cao, Pesin and Zhao's ideas [8], Wang, Qu and Cao [34] generalized Mendoza's result [24] for diffeomorphisms on a higher dimensional manifold. In fact, the authors proved that the Hausdorff dimension of the horseshoes $\{\Lambda_n\}_{n\geq 1}$ on the unstable manifold tends to the dimension of the unstable manifold. Furthermore, if the stable direction is one dimension, then the Hausdorff dimension of the measure μ can be approximated by the Hausdorff dimension of $\{\Lambda_n\}_{n\geq 1}$. The first result in this paper shows that the Lyapunov dimension of μ (see equation (1) for the definition) can be approximated gradually by the Hausdorff dimension of a sequence of hyperbolic sets $\{\Lambda_n\}_{n\geq 1}$, provided that the stable direction is one or μ satisfies the Pesin's entropy formula in the stable direction.

The main motivation of our first result is the study of the Kaplan–Yorke conjecture [13]. To be more precise, let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M and let μ be a hyperbolic ergodic f-invariant probability measure. For $x \in M$, *the pointwise dimension of* μ *at* x is defined by

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

provided the limit exists, where $B(x, r)$ denotes the ball of radius r centred at x. A measure μ is called *exact dimensional* if $d_\mu(x)$ is constant almost everywhere and let $\dim_H \mu$ denote the *Hausdorff dimension* of the measure μ (see [29] for the detailed definition). Young [36] proved that almost all the known characteristics of dimension type of a measure μ coincide if μ is exact dimensional. This indicates that it is very important to show the exactness of a measure in dimension theory of dynamical systems.

Let Γ be the set of points which are regular in the sense of Oseledec multiplicative ergodic theorem [26]. For every $x \in \Gamma$, denote the Lyapunov exponents of f at x by

$$\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_u(\mu) > 0 > \lambda_{u+1}(\mu) \geq \cdots \geq \lambda_{m_0}(\mu),$$

where u and $s := m_0 - u$ are the dimension of the unstable and stable subspaces of $T_x M$, respectively.

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The *Lyapunov dimension of* μ is defined as follows:

$$\dim_L \mu = \begin{cases} m_0 & \text{if } \ell = m_0; \\ \ell + \dfrac{\lambda_1(\mu) + \cdots + \lambda_u(\mu) + \cdots + \lambda_\ell(\mu)}{|\lambda_{\ell+1}(\mu)|} & \text{otherwise,} \end{cases} \tag{1}$$

where $\ell = \max\{i : \lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_i(\mu) \geq 0\}$. It is not difficult to show that $\dim_H \mu \leq \dim_L \mu$, e.g., see [32, Proposition 4.2] for details. It was conjectured in [13] that if μ is an SRB measure, which is absolutely continuous along the unstable leaves, then generically,

$$\dim_H \mu = \dim_L \mu. \tag{2}$$

By Young's dimension formula in [36], the conjecture is true if M is a surface. This paper proves the conjecture in the higher dimensional case under the assumption that the stable direction is one or μ satisfies the 'Pesin's entropy formula in the stable direction'. Moreover, the measure μ is exact dimensional in this case (see Theorem A).

To summarize, let $h_\mu(f)$ denote the metric entropy of f with respect to μ (see Walters' book [33] for details of metric entropy), the first result is stated as the following theorem.

THEOREM A. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M and μ a hyperbolic ergodic SRB measure on M. Assume that either one of the following properties holds:*

(i) *μ has a one-dimensional stable manifold;*

(ii) *μ satisfies $h_\mu(f) = -\lambda_{u+1}(\mu) - \lambda_{u+2}(\mu) - \cdots - \lambda_{m_0}(\mu)$,*

then $\dim_H \mu = \dim_L \mu$. *Furthermore, there exists a sequence of hyperbolic sets $\{\Lambda_n\}$ such that*

$$\dim_H \Lambda_n \to \dim_L \mu \ \ (n \to \infty).$$

Example 1.1. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M. Assume that the volume measure ϱ is f-invariant ergodic and hyperbolic. Let

$$\lambda_1(\varrho) \geq \lambda_2(\varrho) \geq \cdots \geq \lambda_u(\varrho) > 0 > \lambda_{u+1}(\varrho) \geq \cdots \geq \lambda_{m_0}(\varrho)$$

denote the Lyapunov exponents of f with respect to ϱ. By Pesin's entropy formula [28] (see also [23] for a simple proof), one has that

$$h_\varrho(f) = \lambda_1(\varrho) + \lambda_2(\varrho) + \cdots + \lambda_u(\varrho) = -\lambda_{u+1}(\varrho) - \cdots - \lambda_{m_0}(\varrho),$$

where the second equality holds since f is volume-preserving. By Theorem A, there exists a sequence of hyperbolic sets $\{\Lambda_n\}$ such that

$$\dim_H \Lambda_n \to m_0 \ \ (n \to \infty),$$

since $\dim_L \mu = m_0$ in this case.

Ledrappier [20] proved the existence of the pointwise dimension of each SRB measure. For a hyperbolic invariant measure μ of a C^2 (or $C^{1+\alpha}$) diffeomorphism f of a smooth compact Riemannian manifold M without boundary, Ledrappier and Young [22] proved the existence of dimension of μ on stable/unstable manifolds, and that the *upper pointwise*

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dimension of μ is upper bounded by the sum of the dimension of μ on stable and unstable manifolds. Later, Barreira, Pesin and Schmeling [4] proved that the *lower pointwise dimension* of μ is also lower bounded by the sum of the dimension of μ on stable and unstable manifolds. This showed that the measure μ is exact dimensional, which finally solves the Eckmann–Ruelle conjecture.

Motivated by the work in [12], where it is proved that the unique solution of the measure-theoretic pressure is exactly the dimension of an invariant measure supported on an average conformal repeller, the second result in this paper shows that the unique solution of measure-theoretic pressure gives an upper bound of the dimension of a hyperbolic ergodic measure μ on stable/unstable manifolds. To be more precise, we introduce some notation first. For each $x \in M$ and $n \geq 1$, consider the differentiable operator $D_x f^n : T_x M \to T_{f^n(x)} M$ and denote the singular values of $D_x f^n$ in the decreasing order by

$$\alpha_1(x, f^n) \geq \alpha_2(x, f^n) \geq \cdots \geq \alpha_u(x, f^n) \geq \cdots \geq \alpha_{m_0}(x, f^n).$$

Recall that u and s are the dimension of the unstable and stable subspace of $T_x M$, respectively. For every $t \in [0, u]$, define

$$\phi^t(x, f^n) := \sum_{i=1}^{[t]} \log \alpha_i(x, f^n) + (t - [t]) \log \alpha_{[t]+1}(x, f^n)$$

and

$$\psi^t(x, f^n) := \sum_{i=u-[t]+1}^{u} \log \alpha_i(x, f^n) + (t - [t]) \log \alpha_{u-[t]}(x, f^n).$$

For every $t \in [0, s]$, define

$$\varphi^t(x, f^n) := \sum_{i=u+1}^{u+[t]} \log \alpha_i(x, f^n) + (t - [t]) \log \alpha_{u+[t]+1}(x, f^n).$$

Since f is smooth, the functions $x \mapsto \alpha_i(x, f^n)$, $x \mapsto \phi^t(x, f^n)$, $x \mapsto \psi^t(x, f^n)$ and $x \mapsto \varphi^t(x, f^n)$ are continuous. It is easy to see that the sequences of functions

$$\Phi_f(t) := \{-\phi^t(\cdot, f^n)\}_{n\geq 1} \tag{3}$$

are super-additive and

$$\Psi_f(t) := \{-\psi^t(\cdot, f^n)\}_{n\geq 1}, \quad \Xi_f(t) := \{\varphi^t(\cdot, f^n)\}_{n\geq 1} \tag{4}$$

are sub-additive. Ledrappier and Young [22] proved the existence of stable and unstable pointwise dimension $d^s_\mu(x)$, $d^u_\mu(x)$ of a hyperbolic ergodic measure μ for μ-almost every (a.e.) x. The following theorem shows that the unique solution of the sub-additive measure-theoretic pressure equation

$$P_\mu(f, \Psi_f(t)) = 0 \quad (P_\mu(f, \Xi_f(t)) = 0)$$

is an upper bound for the unstable (stable) dimension of μ, see §2 for the definitions of measure-theoretic pressure and stable and unstable dimension of an invariant measure.

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THEOREM B. *Suppose $f : M \to M$ is a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M and μ is a hyperbolic ergodic measure on M. Then one has*

$$d_\mu^u(x) \le t_u^* \quad \text{and} \quad d_\mu^s(x) \le t_s^* \ \ \mu\text{-a.e. } x,$$

where t_u^* and t_s^* are the unique solutions of the equations $P_\mu(f, \Psi_f(t)) = 0$ and $P_\mu(f, \Xi_f(t)) = 0$, respectively.

For each hyperbolic ergodic measure μ of positive entropy, there exists a sequence of hyperbolic sets $\{\Lambda_n\}_{n\ge1}$ such that the dynamical quantities on Λ_n gradually approach to those of the measure μ (see Theorem 2.4). Since the hyperbolic sets $\{\Lambda_n\}_{n\ge1}$ are non-conformal, it is difficult to compute their Hausdorff dimension. Following the approach described in [8], this paper introduces the concept of Carathéodory singular dimension of a hyperbolic set on unstable manifolds (see §2 for the detailed definition). The third result of this paper shows that the zero of the super-additive/sub-additive measure-theoretic pressure $P_\mu(f, \Phi_f(t))/P_\mu(f, \Psi_f(t))$ gives a lower/upper bound of the Carathéodory singular dimension of Λ_n on the unstable manifold. In addition, if μ is an SRB measure, then the Carathéodory singular dimension of Λ_n on the unstable manifold tends to the dimension of the unstable manifold, and the Lyapunov dimension of μ is exactly the sum of t_s^* and the dimension of the unstable manifold, where t_s^* is the unique root of the equation $P_\mu(f, \Xi_f(t)) = 0$.

THEOREM C. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M, and let μ be a hyperbolic ergodic measure on M. Then there exists a sequence of hyperbolic sets $\{\Lambda_\varepsilon\}_{\varepsilon\ge0}$ such that the following properties hold:*

(i) $\liminf_{\varepsilon\to0} \dim_C^{\Phi_f}(\Lambda_\varepsilon \cap W_{\rm loc}^u(x, f)) \ge t_{u*}$ *for every $x \in \Lambda_\varepsilon$, where t_{u*} is the unique root of the equation $P_\mu(f, \Phi_f(t)) = 0$;*

(ii) $\limsup_{\varepsilon\to0} \dim_C^{\Psi_f}(\Lambda_\varepsilon \cap W_{\rm loc}^u(x, f)) \le t_u^*$ *for every $x \in \Lambda_\varepsilon$, where t_u^* is the unique root of the equation $P_\mu(f, \Psi_f(t)) = 0$.*

Furthermore, if μ is an SRB measure, then $\dim_L \mu = u + t_s^*$ and

$$\lim_{\varepsilon\to0} \dim_C^{\Phi_f}(\Lambda_\varepsilon \cap W_{\rm loc}^u(x, f)) = u$$

for every $x \in \Lambda_\varepsilon$, where u is the dimension of the unstable manifold and t_s^* is the unique root of the equation $P_\mu(f, \Xi_f(t)) = 0$.

The paper is organized as follows. Section 2 gives some basic notions and properties, including Hausdorff dimension, hyperbolic set, pressure and singular dimension. All the proofs of the main results will be given in §3.

2. *Preliminaries*

In this section, we will recall some definitions and preliminary results which are used in the proofs of the main results.

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2.1. *Hyperbolic set.* Let f be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M. We say an f-invariant compact subset $\Lambda \subset M$ is a *hyperbolic set* if for any $x \in \Lambda$, the tangent space admits a decomposition $T_xM = E^s(x) \oplus E^u(x)$ such that the following properties hold:

(1) the splitting is Df-invariant, that is, for every $x \in \Lambda$, $D_x f E^\sigma(x) = E^\sigma(f(x))$ for $\sigma = s, u$;

(2) the stable subspace $E^s(x)$ is uniformly contracting and the unstable subspace $E^u(x)$ is uniformly expanding in the sense that there are constants $C \geq 1$ and $0 < \chi < 1$ such that for every $n \geq 0$ and $v^\sigma \in E^\sigma(x)$ ($\sigma = s$ or u), we have

$$\|D_x f^n v^s\| \leq C\chi^n \|v^s\| \quad \text{and} \quad \|D_x f^{-n} v^u\| \leq C\chi^n \|v^u\|.$$

Recall that a hyperbolic set Λ is *locally maximal* if there exists an open neighbourhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$, and a diffeomorphism f is called *topologically transitive* on Λ if for every two non-empty (relative) open subsets $U, V \subset \Lambda$, there exists $n > 0$ such that $f^n(U) \cap V \neq \emptyset$. Given a point $x \in \Lambda$, for each small $\beta > 0$, the *local stable and unstable manifolds* at the point x are defined as follows:

$$W^s_{\mathrm{loc}}(x, f) = \{y \in M : d(f^n(x), f^n(y)) \leq \beta \text{ for all } n \geq 0\},$$

and

$$W^u_{\mathrm{loc}}(x, f) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) \leq \beta \text{ for all } n \geq 0\}.$$

The *global stable and unstable sets* of $x \in \Lambda$ are given as follows:

$$W^s(x, f) = \bigcup_{n\geq 0} f^{-n}(W^s_{\mathrm{loc}}(f^n(x), f)), \quad W^u(x, f) = \bigcup_{n\geq 0} f^n(W^u_{\mathrm{loc}}(f^{-n}(x), f)).$$

Let d^s/d^u be the metric induced by the Riemannian structure on the stable/unstable manifold W^s/W^u.

2.2. *Dimension.* Let X be a compact Riemannian manifold with a Riemannian metric. Given a subset Z of X, for $s \geq 0$ and $\delta > 0$, define

$$\mathcal{H}^s_\delta(Z) := \inf\left\{\sum_i |U_i|^s : Z \subset \bigcup_i U_i, \ |U_i| \leq \delta \text{ for all } i\right\},$$

where $|\cdot|$ denotes the diameter of a subset. The quantity

$$\mathcal{H}^s(Z) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(Z)$$

is called the *s-dimensional Hausdorff measure* of Z. It is easy to show that there is a jump-up value

$$\dim_H Z := \inf\{s : \mathcal{H}^s(Z) = 0\} = \sup\{s : \mathcal{H}^s(Z) = \infty\},$$

which is called the *Hausdorff dimension* of Z.

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Given a Borel probability measure μ on X, *the Hausdorff dimension of the measure* μ is defined as

$$\dim_H \mu = \inf\{\dim_H Y : Y \subset X,\ \mu(Y) = 1\}.$$

The *lower and upper pointwise dimension of* μ at point $x \in X$ are defined respectively by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x, r)$ denotes the ball of radius r centred at x. If $\underline{d}_\mu(x) = \overline{d}_\mu(x)$, then we denote the common value by $d_\mu(x)$. In particular, Barreira and Wolf [5] proved that

$$\dim_H \mu = \text{ess sup}\{\underline{d}_\mu(x) : x \in X\}, \tag{5}$$

where the essential supremum is taken with respect to μ. The following well-known result gives the relation between the Hausdorff dimension and the lower pointwise dimension.

PROPOSITION 2.1. *The following properties hold:*

(1) *if* $\underline{d}_\mu(x) \geq \alpha$ *for* μ*-a.e.* $x \in X$*, then* $\dim_H \mu \geq \alpha$*;*
(2) *if* $\underline{d}_\mu(x) \leq \alpha$ *for every* $x \in Z \subseteq X$*, then* $\dim_H Z \leq \alpha$*.*

Let $f : X \to X$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact Riemannian manifold X, and let μ be a hyperbolic ergodic measure on X. Let Γ be the set of points which are regular in the sense of Oseledets [26]. A measurable partition ξ^u/ξ^s of X is said to be *subordinate to the unstable/stable manifold* if for μ-almost every x, $\xi^u(x) \subset W^u(x, f)/\xi^s(x) \subset W^s(x, f)$ and contains an open neighbourhood of x in $W^u(x, f)/W^s(x, f)$. Let $\{\mu^u_x\}$ and $\{\mu^s_x\}$ be the collections of conditional measures associated with ξ^u and ξ^s, respectively. For every $x \in \Gamma$, Ledrappier and Young [22] proved the existence of the following limits:

$$d^u_\mu(x) := \lim_{r\to 0} \frac{\log \mu^u_x(B^u(x,r))}{\log r} \quad \text{and} \quad d^s_\mu(x) := \lim_{r\to 0} \frac{\log \mu^s_x(B^s(x,r))}{\log r}, \tag{6}$$

which are called the stable and unstable dimension of the measure μ, respectively. Here $B^\sigma(x, r) := \{y \in W^\sigma(x, f) : d^\sigma(x, y) < r\}$ with $\sigma \in \{u, s\}$. Since we consider the limit $r \to 0$ in equation (6), the definition of $d^u_\mu(x)$ will remain unchanged if we consider the global metric d in the dynamical ball $B^\sigma(x, r)$ instead.

2.3. *Pressure.* Let (M, f) be a topological dynamical system (TDS for short), that is, $f : M \to M$ is a continuous map on a compact metric space M equipped with the metric d. Denote by $\mathcal{M}_{\text{inv}}(f|_M)$ and $\mathcal{M}_{\text{erg}}(f|_M)$ the set of all f-invariant and ergodic Borel probability measures on M, respectively. Given $n \in \mathbb{N}$ and $x, y \in M$, let

$$d_n(x, y) = \max\{d(f^k(x), f^k(y)) : 0 \leq k < n\}.$$

Given $\varepsilon > 0$, denote by $B_n(x, \varepsilon) = \{y : d_n(x, y) < \varepsilon\}$ the *Bowen's ball* of radius ε centred at x of length n. A subset $E \subset M$ is called (n, ε)*-separated* if $d_n(x, y) > \varepsilon$ for any two

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distinct points $x, y \in E$. A sequence of continuous functions $\Psi = \{\psi_n\}_{n\geq 1}$ on M is called *sub-additive* if

$$\psi_{m+n} \leq \psi_n + \psi_m \circ f^n \quad \text{for all } m, n \geq 1.$$

Similarly, one calls a sequence of continuous functions $\Phi = \{\phi_n\}_{n\geq 1}$ on M *super-additive* if $-\Phi = \{-\phi_n\}_{n\geq 1}$ is sub-additive.

Let $\Psi = \{\psi_n\}_{n\geq 1}$ be a sub-additive sequence of continuous potentials on M, set

$$P_n(f, \Psi, \varepsilon) = \sup \left\{ \sum_{x\in E} e^{\psi_n(x)} : E \text{ is an } (n, \varepsilon)\text{-separated subset of } M \right\}.$$

The quantity

$$P(f, \Psi) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_n(f, \Psi, \varepsilon)$$

is called the *sub-additive topological pressure* of Ψ.

The sub-additive topological pressure satisfies the following variational principle, see [6] for more details.

THEOREM 2.1. *Let $\Psi = \{\psi_n\}_{n\geq 1}$ be a sub-additive sequence of continuous potentials on M. Then*

$$P(f, \Psi) = \sup\{h_\mu(f) + \mathcal{F}_*(\Psi, \mu) |\ \mu \in \mathcal{M}_{\text{inv}}(f|_M), \mathcal{F}_*(\Psi, \mu) \neq -\infty\},$$

where $h_\mu(f)$ is the measure theoretic entropy of f with respect to the measure μ and $\mathcal{F}_*(\Psi, \mu) = \lim_{n\to\infty}(1/n) \int \psi_n \, d\mu$.

Remark 2.1. If $\Psi = \{\psi_n\}_{n\geq 1}$ is *additive* in the sense that $\psi_n(x) = \psi(x) + \psi(fx) + \cdots + \psi(f^{n-1}x) := S_n\psi(x)$ for some continuous function $\psi : M \to \mathbb{R}$, we simply denote the topological pressure $P(f, \Psi)$ as $P(f, \psi)$.

Next we recall the super-additive topological pressure introduced in [8] by the variational relation for topological pressure, although it is unknown whether the variational principle holds for super-additive topological pressure defined via separated sets. Given a sequence of super-additive continuous potentials $\Phi = \{\phi_n\}_{n\geq 1}$ on M, the *super-additive topological pressure* of Φ is defined as

$$P(f, \Phi) := \sup\{h_\mu(f) + \mathcal{F}_*(\Phi, \mu) : \mu \in \mathcal{M}_{\text{inv}}(f|_M)\},$$

where

$$\mathcal{F}_*(\Phi, \mu) = \lim_{n\to\infty} \frac{1}{n} \int \phi_n \, d\mu = \sup_{n\in\mathbb{N}} \frac{1}{n} \int \phi_n \, d\mu.$$

The second equality is due to the standard sub-additive argument. The following result gives the relation between the sub-additive (super-additive) topological pressure and the topological pressure for additive potentials.

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PROPOSITION 2.2. *Let $\Phi = \{\phi_n\}_{n\geq 1}$ be a sequence of continuous potentials on M. Then the following properties hold:*

(1) *if Φ is sub-additive and the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, then*

$$P(f, \Phi) = \lim_{n\to\infty} P(f, \phi_n/n) = \lim_{n\to\infty} (1/n) P(f^n, \phi_n);$$

(2) *if Φ is super-additive, then*

$$P(f, \Phi) = \lim_{n\to\infty} P(f, \phi_n/n) = \lim_{n\to\infty} (1/n) P(f^n, \phi_n).$$

The first statement is proved in [3], where the sub-additive topological pressure is defined via separated sets, so one requires that the entropy map be upper semi-continuous. The second statement is proved in [8], and one does not need any additional condition since the super-additive topological pressure is defined via the variational relations.

Following the approach described in [29], we recall the topological pressure on an arbitrary subset of unstable manifolds. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional smooth compact Riemannian manifold M and let $\Lambda \subset M$ be a hyperbolic set. Let $\Psi = \{\psi_n\}_{n\geq 1}$ be a sub-additive sequence of continuous functions on Λ. For every $x \in \Lambda$, denote $Z = \Lambda \cap W^u_{\mathrm{loc}}(x, f)$. Given $s \in \mathbb{R}$, set

$$m(Z, \Psi, s, \delta) := \lim_{N\to\infty} \inf \bigg\{ \sum_i \exp\bigg(-sn_i + \sup_{y\in B^u_{n_i}(x_i,\delta)} \psi_{n_i}(y) \bigg) \bigg\}, \tag{7}$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i, \delta)\}$ with $x_i \in \Lambda, n_i \geq N$ that cover Z, and

$$B^u_{n_i}(x_i, \delta) := \{y \in W^u(x, f) : d^u(f^j(x_i), f^j(y)) < \delta \text{ for } j = 0, 1, \ldots, n_i - 1\}.$$

It is easy to show that there is a jump-up value

$$P_Z(f, \Psi, \delta) := \inf\{s : m(Z, \Psi, s, \delta) = 0\} = \sup\{s : m(Z, \Psi, s, \delta) = +\infty\}.$$

The quantity

$$P_Z(f, \Psi) := \lim_{\delta\to 0} P_Z(f, \Psi, \delta)$$

is called *the topological pressure of* Ψ *on the subset* Z. It is not difficult to show that $P_\Lambda(f, \Psi) = P(f|_\Lambda, \Psi)$ (see [6, Proposition 4.4]).

Let μ be an f-invariant Borel probability measure on M. Given a sub-additive potential $\Phi = \{\phi_n\}_{n\geq 1}$ on M, for $0 < \delta < 1, n \geq 1$ and $\varepsilon > 0$, a subset $F \subset M$ is called *an* (n, ε, δ)*-spanning set* if the union $\bigcup_{x\in F} B_n(x, \varepsilon)$ has μ-measure more than or equal to $1 - \delta$. Put

$$P_\mu(f, \Phi, n, \varepsilon, \delta) := \inf \bigg\{ \sum_{x\in F} \exp\bigg(\sup_{y\in B_n(x,\varepsilon)} \phi_n(y) \bigg) : F \text{ is an } (n, \varepsilon, \delta)\text{-spanning set} \bigg\}$$

and let further that

$$P_\mu(f, \Phi, \varepsilon, \delta) := \limsup_{n\to\infty} \frac{1}{n} \log P_\mu(f, \Phi, n, \varepsilon, \delta),$$

$$P_\mu(f, \Phi, \delta) := \liminf_{\varepsilon\to 0} P_\mu(f, \Phi, \varepsilon, \delta),$$

$$P_\mu(f, \Phi) := \lim_{\delta\to 0} P_\mu(f, \Phi, \delta),$$

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and we call $P_\mu(f, \Phi)$ *the sub-additive measure-theoretic pressure* of (f, Φ) with respect to μ. If one considers a super-additive potential $\Phi = \{\phi_n\}_{n\geq 1}$ on M, replacing $\sup_{y\in B_n(x,\varepsilon)} \phi_n(y)$ by $\phi_n(x)$ in $P_\mu(f, \Phi, n, \varepsilon, \delta)$, then the corresponding quantity $P_\mu(f, \Phi)$ is called *the super-additive measure theoretic pressure* of (f, Φ) with respect to μ.

Remark 2.2.

(i) It is easy to see that $P_\mu(f, \Phi, \delta)$ increases with δ decreasing to zero. So the limit in the last formula exists. Moreover, it is proved in [7] that $P_\mu(f, \Phi, \delta)$ is independent of δ. Hence, the limit of $\delta \to 0$ is redundant in the definition.

(ii) If $\Phi = \{\phi_n\}_{n\geq 1}$ is an additive potential on M, that is, $\phi_n(x) = \sum_{i=0}^{n-1} \phi_1(f^i x)$ for some continuous function ϕ_1, then we simply write $P_\mu(f, \Phi)$ as $P_\mu(f, \phi_1)$.

In the following, we recall some properties of sub-additive/super-additive measure-theoretic pressure which are proved in [7].

THEOREM 2.2. [7, Theorem A] *Let (M, f) be a TDS and $\Phi = \{\phi_n\}_{n\geq 1}$ a sub-additive potential on M. For every $\mu \in \mathcal{M}_{\mathrm{erg}}(f|_M)$ with $\mathcal{F}_*(\Phi, \mu) \neq -\infty$, we have that*

$$P_\mu(f, \Phi) = h_\mu(f) + \mathcal{F}_*(\Phi, \mu).$$

THEOREM 2.3. [7, Proposition 3.2] *Let (M, f) be a TDS and $\Phi = \{\phi_n\}_{n\geq 1}$ a super-additive potential on M. For every $\mu \in \mathcal{M}_{\mathrm{erg}}(f|_M)$, we have that*

$$P_\mu(f, \Phi) = h_\mu(f) + \mathcal{F}_*(\Phi, \mu).$$

Remark 2.3. In Theorem 2.2, to avoid the indeterminate form $\infty - \infty$, the condition $\mathcal{F}_*(\Phi, \mu) \neq -\infty$ is necessary. However, we do not need this condition in Theorem 2.3. If $\Phi = \{\phi_n\}_{n\geq 1}$ is an additive potential on M, that is, $\phi_n(x) = S_n\phi(x)$ for some continuous function ϕ, then we have

$$P_\mu(f, \phi) = h_\mu(f) + \int \phi \, d\mu \quad \text{for all } \mu \in \mathcal{M}_{\mathrm{erg}}(f|_M).$$

The above formula is also proven in [16].

2.4. *Singular dimension.* Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M and $\Lambda \subset M$ a hyperbolic set. Consider the sub-additive singular valued potential $\Psi_f(t) = \{-\psi^t(\cdot, f^n)\}_{n\geq 1}$ given by equation (4). Fix $x \in \Lambda$ and let $Z = \Lambda \cap W^u_{\mathrm{loc}}(x, f)$. Following the approach described in [8], we introduce the Carathéodory singular dimension of Z. Put

$$m(Z, \Psi_f(t), \delta) := \lim_{N\to\infty} \inf \left\{ \sum_i \exp\left[\sup_{y\in B^u_{n_i}(x_i,\delta)} -\psi^t(y, f^{n_i}) \right] \right\}, \tag{8}$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i, \delta)\}$ with $x_i \in \Lambda, n_i \geq N$ that cover Z. It is easy to see that there is a jump-up value

$$\begin{aligned} \dim^{\Psi_f}_{C,\delta} Z &:= \inf\{t : m(Z, \Psi_f(t), \delta) = 0\} \\ &= \sup\{t : m(Z, \Psi_f(t), \delta) = +\infty\}. \end{aligned} \tag{9}$$

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The quantity

$$\dim_C^{\Psi_f} Z := \lim_{\delta\to 0} \dim_{C,\delta}^{\Psi_f} Z \tag{10}$$

is called *the Carathéodory singular dimension of Z with respect to the sub-additive singular valued potential* Ψ_f.

Consider the super-additive singular valued potential $\Phi_f(t) = \{-\phi^t(\cdot, f^n)\}_{n\geq 1}$ given by equation (3), replacing $\sup_{y\in B^u_{n_i}(x_i,\delta)} -\psi^t(y, f^{n_i})$ by $-\phi^t(x_i, f^{n_i})$ in equation (8), one can define $m(Z, \Phi_f(t), \delta)$ and $\dim_{C,\delta}^{\Phi_f} Z$ in a similar fashion as equations (8) and (9). The corresponding quantity $\dim_C^{\Phi_f} Z$ as in equation (10) is called *the Carathéodory singular dimension of Z with respect to the super-additive singular valued potential* Φ_f.

2.5. *Approximation of hyperbolic measures by hyperbolic sets with dominated splitting.* First we recall the definition of the dominated splitting. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M. Suppose $\Lambda \subset M$ is a compact f-invariant set. We say Λ admits a *dominated splitting* if there is a continuous invariant splitting $T_\Lambda M = E \oplus F$ and constants $C > 0$, $\lambda \in (0, 1)$ such that for each $x \in \Lambda$, $n \in \mathbb{N}$, $0 \neq u \in E(x)$ and $0 \neq v \in F(x)$, it holds that

$$\frac{\|D_x f^n(u)\|}{\|u\|} \leq C\lambda^n \frac{\|D_x f^n(v)\|}{\|v\|}.$$

We say F dominates E and write it as $E \preceq F$. Furthermore, given $0 < \ell \leq m_0$, we say a continuous invariant splitting $T_\Lambda M = E_1 \oplus \cdots \oplus E_\ell$ dominates if there are numbers $\chi_1 < \chi_2 < \cdots < \chi_\ell$, constants $C > 0$ and $0 < \varepsilon < \min_{1\leq i\leq \ell-1}\{(\chi_{i+1} - \chi_i)/100\}$ such that for every $x \in \Lambda$, $n \in \mathbb{N}$ and $1 \leq j \leq \ell$ and each unit vector $u \in E_j(x)$, it holds that

$$C^{-1} \exp[n(\chi_j - \varepsilon)] \leq \|D_x f^n(u)\| \leq C \exp[n(\chi_j + \varepsilon)].$$

In particular, it is clear that $E_1 \preceq \cdots \preceq E_\ell$. We shall use the notion $\{\chi_j\}$-dominated when we want to stress the dependence on the numbers $\{\chi_j\}$.

Refining Katok's approximation theory in non-uniformly hyperbolic dynamical systems [**18**], Avila, Crovisier and Wilkinson [**2**] obtained the following approximation result.

THEOREM 2.4. [**2**] *Let f be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M, and let μ be an ergodic hyperbolic measure with $h_\mu(f) > 0$. Then for every $\varepsilon > 0$ and weak* neighbourhood $\mathcal{V}$ of μ in the space of f-invariant probability measures on M, there exists an f-invariant compact subset $\Lambda_\varepsilon \subset M$ such that:*

(a) *Λ_ε is ε-close to the support set of μ in the Hausdorff distance;*
(b) *$|h_{top}(f|_{\Lambda_\varepsilon}) - h_\mu(f)| \leq \varepsilon$;*
(c) *all the invariant probability measures supported on Λ_ε lie in $\mathcal{V}$;*
(d) *there is a $\{\chi_j(\mu)\}$-dominated splitting $TM = E_1 \oplus E_2 \oplus \cdots \oplus E_\ell$ over Λ_ε, where $\chi_1(\mu) < \cdots < \chi_\ell(\mu)$ are distinct Lyapunov exponents of f with respect to the measure μ.*

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In the second statement, the original result does not show that $h_{\text{top}}(f|_{\Lambda_\varepsilon}) \le h_\mu(f) + \varepsilon$. However, only a slight modification can give the upper bound of the topological entropy of f on the horseshoe.

3. *Proofs*

This section provides the detailed proofs of the main results presented in the previous section.

3.1. *Proof of Theorem A.*

(i) Since μ is a hyperbolic ergodic SRB measure for a $C^{1+\alpha}$ diffeomorphism f and has a one-dimensional stable manifold, by [34, Lemma 15 and 25], one has

$$d_\mu^u(x) := \lim_{r\to 0} \frac{\log \mu_x^u(B^u(x,r))}{\log r} = u \quad \text{and} \quad d_\mu^s(x) := \lim_{r\to 0} \frac{\log \mu_x^s(B^s(x,r))}{\log r} = \frac{h_\mu(f)}{-\lambda_{m_0}(\mu)}$$

for μ-a.e. x. Barreira, Pesin and Schmeling [4] proved that $d_\mu(x) = d_\mu^u(x) + d_\mu^s(x)$ for μ-a.e. x. As a consequence, one has that

$$d_\mu(x) = u + \frac{h_\mu(f)}{-\lambda_{m_0}(\mu)}$$

for μ-a.e. x. Hence, one has

$$\dim_H \mu = u + \frac{h_\mu(f)}{-\lambda_{m_0}(\mu)}.$$

If $\lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_{m_0}(\mu) < 0$, then one can show that

$$\dim_L \mu = u + \frac{h_\mu(f)}{-\lambda_{m_0}(\mu)},$$

since μ is an SRB measure and has a one-dimensional stable manifold. Therefore, we have that

$$\dim_H \mu = \dim_L \mu.$$

If $\lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_{m_0}(\mu) \ge 0$, it follows from the definition of Lyapunov dimension that $\dim_L \mu = m_0$. Since μ is an SRB measure for f and has a one-dimensional stable manifold, one has that

$$h_\mu(f) = \lambda_1(\mu) + \cdots + \lambda_{m_0-1}(\mu) \ge -\lambda_{m_0}(\mu).$$

This together with the fact that

$$1 \ge d_\mu^s(x) = \frac{h_\mu(f)}{-\lambda_{m_0}(\mu)} \quad \mu\text{-a.e. } x$$

implies that $(h_\mu(f)/-\lambda_{m_0}(\mu)) = 1$. This yields that $\dim_H \mu = \dim_L \mu$.

By [34, Theorem B], there exists a sequence of hyperbolic sets Λ_n such that

$$\dim_H \Lambda_n \to \dim_L \mu \ (n \to \infty).$$

(ii) Since μ is a hyperbolic ergodic SRB measure for a $C^{1+\alpha}$ diffeomorphism f, by [34, Lemma 15], one has that

$$d_\mu^u(x) = u \quad \mu\text{-a.e. } x.$$

Considering f^{-1} instead of f, since $h_\mu(f) = -\lambda_{u+1}(\mu) - \lambda_{u+2}(\mu) - \cdots - \lambda_{m_0}(\mu)$, by [21, Theorem A], we have that the measure μ has absolutely continuous conditional measures on stable manifolds of f. Using the same arguments as the proof of [34, Lemma 15], we have that

$$d_\mu^s(x) = s \quad \mu\text{-a.e. } x.$$

Hence, $d_\mu(x) = u + s = m_0$ for μ-a.e. x, which implies that $\dim_H \mu = m_0$. Since μ is an SRB measure for f and $h_\mu(f) = -\lambda_{u+1}(\mu) - \lambda_{u+2}(\mu) - \cdots - \lambda_{m_0}(\mu)$, one can conclude that

$$\lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_{m_0}(\mu) = 0,$$

then $\dim_L \mu = m_0$ by the definition of Lyapunov dimension. This proves that

$$\dim_L \mu = \dim_H \mu.$$

Finally, for each $\varepsilon > 0$, there exists a hyperbolic set Λ_ε satisfying properties (a)–(d) in Theorem 2.4. Fix a positive integer $n \geq 1$. Let t_n be the unique root of Bowen's equation $P(f^{2^n}|\Lambda_\varepsilon, -\psi^t(\cdot, f^{2^n})) = 0$ and let μ_n^u be the unique equilibrium state for the topological pressure $P(f^{2^n}|\Lambda_\varepsilon, -\psi^{t_n}(\cdot, f^{2^n}))$. Similarly, let t_n' be the root of Bowen's equation $P(f^{2^n}|\Lambda_\varepsilon, \phi^t(\cdot, f^{2^n})) = 0$ and let μ_n^s be the unique equilibrium state for the topological pressure $P(f^{2^n}|\Lambda_\varepsilon, \phi^{t_n}(\cdot, f^{2^n}))$. As in the proof of [34, Theorem B], the following properties hold:

(e) $\lim_{\varepsilon\to 0} \lim_{n\to\infty} t_n = u$ and $\lim_{\varepsilon\to 0} \lim_{n\to\infty} t_n' = s$;

(f) there is a Markov partition $\mathcal{P} = \{P_1, P_2, \ldots, P_\ell\}$ of Λ_ε. For every $i \in \{1, 2, \ldots, \ell\}$, there is a family of conditional measures $\{\mu_{n,x}^u\}_{x\in P_i}$ ($\{\mu_{n,x}^s\}_{x\in P_i}$) of μ_n^u (μ_n^s) on the local unstable (stable) sets $W_{P_i}^u$ ($W_{P_i}^s$) such that for every $x \in P_i$, there is small $r_0 > 0$ such that for every $r \in (0, r_0)$,

$$r^{u+\varepsilon} \leq \mu_{n,x}^u(B^u(x, r)) \leq r^{t_n-\varepsilon}$$

and

$$r^{s+\varepsilon} \leq \mu_{n,x}^s(B^s(x, r)) \leq r^{t_n'-\varepsilon},$$

where $W_{P_i}^u(x, f) := W_{loc}^u(x, f) \cap P_i$ and $W_{P_i}^s(x, f) := W_{loc}^s(x, f) \cap P_i$ for every $x \in P_i$.

Define a measure $\hat{\mu}_n$ on P_i as follows:

$$\hat{\mu}_n(B(x, r)) = \mu_{n,x}^u(B^u(x, r)) \cdot \mu_{n,x}^s(B^s(x, r))$$

for every $x \in P_i$ and each sufficiently small $r > 0$. This yields that

$$t_n + t_n' - 2\varepsilon \leq \underline{d}_{\hat{\mu}_n}(x) \leq \overline{d}_{\hat{\mu}_n}(x) \leq m_0 + 2\varepsilon$$

for every $x \in P_i$. By Proposition 2.1 and the fact that $\Lambda_\varepsilon = \bigcup_{i=1}^{\ell} P_i$, we have that

$$\lim_{\varepsilon \to 0} \dim_H \Lambda_\varepsilon = m_0 = \dim_L \mu.$$

This completes the proof of Theorem A.

3.2. *Proof of Theorem B.* Let Γ be the set of points which are regular in the sense of Oseledets [**26**] with respect to the measure μ. For every $x \in \Gamma$, denote its Lyapunov exponents by

$$\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_u(\mu) > 0 > \lambda_{u+1}(\mu) \geq \cdots \geq \lambda_{m_0}(\mu).$$

To prove Theorem B, we need a coarse upper bound for the unstable and stable pointwise dimension $d_\mu^u(x)$, $d_\mu^s(x)$ of an ergodic f-invariant hyperbolic probability measure μ for almost every x. We now provide the following useful lemma, which estimates the Hausdorff measure of the image of a small ball along unstable/stable direction under f.

LEMMA 3.1. *Fix $t \in [0, u]$, then for any $b_0 > 2\sqrt{u}$ and $C_0 > 2^t u^{t/2}$, there is $\rho_0 > 0$ such that for all $x \in \Gamma$, if $B^u(x, \rho) \subset B(x, \rho_0) \cap W^u(x, f)$ for some $0 < \rho < \rho_0$, then we have*

$$\mathcal{H}_{b\rho}^t(B^u(x, \rho)) \leq C\mathcal{H}_\rho^t(f(B^u(x, \rho))),$$

where $b = b_0 \exp\{-\log \alpha_{u-[t]}(x, f)\}$ and $C = C_0 \exp\{-\psi^t(x, f)\}$.

Proof. For simplicity, we just prove the lemma on the assumption that M is the Euclid space $\mathbb{R}^{m_0}$. For the general case, one can use local charts to prove it.

Given a small positive number ε with $e^\varepsilon/(1-\varepsilon) < 2$, since $f : M \to M$ is a $C^{1+\alpha}$ diffeomorphism on M, there exists $\rho_0 > 0$ such that for every $y, z \in B(x, \rho_0) \cap W^u(x, f)$, the following properties hold:

(a) $\|y - z - (D_y f)^{-1}(f(y) - f(z))\| \leq \varepsilon \|y - z\|$;

(b) $|\log \alpha_i(y, f) - \log \alpha_i(z, f)| \leq \varepsilon$ for $i = 1, 2, \ldots, u$.

See [**17**, Lemma 4] for the detailed proof of the above properties. Fix $0 < \rho < \rho_0$. Let $A := B^u(x, \rho)$ and $a = \mathcal{H}_\rho^t(f(A))$. Assume that a is finite, otherwise the conclusion is clear. For every $\eta > 0$, there are points $\{z_j\} \subset f(B(x, \rho_0) \cap W^u(x, f))$ such that

$$f(A) \subset \bigcup_j B^u(z_j, r_j)$$

with $r_j \leq \rho$ for each j and

$$\sum_j r_j^t < a + \eta.$$

Let $B'_j = \{y \in A : f(y) \in B^u(z_j, r_j)\}$, then $A \subset \bigcup_j B'_j$. By property (a), we conclude that B'_j is contained in an ellipse with principal axes

$$\frac{1}{1-\varepsilon} r_j \cdot \alpha_1(y_j, f)^{-1}, \frac{1}{1-\varepsilon} r_j \cdot \alpha_2(y_j, f)^{-1}, \ldots, \frac{1}{1-\varepsilon} r_j \cdot \alpha_u(y_j, f)^{-1},$$

where $y_j \in B^u(x, \rho)$ and $f(y_j) = z_j$. This together with property (b) yield that B'_j is contained in an ellipse with principal axes

$$\frac{e^\varepsilon}{1-\varepsilon} r_j \cdot \alpha_1(x, f)^{-1}, \frac{e^\varepsilon}{1-\varepsilon} r_j \cdot \alpha_2(x, f)^{-1}, \ldots, \frac{e^\varepsilon}{1-\varepsilon} r_j \cdot \alpha_u(x, f)^{-1}.$$

Hence, B'_j is covered by

$$\frac{\exp\{-\sum_{j=u-[t]+1}^{u} \log \alpha_j(x, f)\}}{\exp\{-[t] \log \alpha_{u-[t]}(x, f)\}}$$

balls with radius $(e^\varepsilon/(1-\varepsilon))\sqrt{u} r_j \cdot \exp\{-\log \alpha_{u-[t]}(x, f)\}$. In fact, the radius

$$\begin{aligned} &\frac{e^\varepsilon}{1-\varepsilon}\sqrt{u} r_j \cdot \exp\{-\log \alpha_{u-[t]}(x, f)\} \\ &\quad \leq 2\sqrt{u} \exp\{-\log \alpha_{u-[t]}(x, f)\} \cdot \rho \\ &\quad \leq b\rho. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{H}^t_{b\rho}(B'_j) &\leq \exp\left\{-\sum_{j=u-[t]+1}^{u} \log \alpha_j(x, f) + [t] \log \alpha_{u-[t]}(x, f)\right\} \\ &\quad \cdot \left(\frac{e^\varepsilon}{1-\varepsilon}\sqrt{u}\right)^t r_j^t \cdot \exp\{-t \log \alpha_{u-[t]}(x, f)\} \\ &\leq (2\sqrt{u})^t \cdot \exp\{-\psi^t(x, f)\} \cdot r_j^t. \end{aligned}$$

Summing up over all j, we have that

$$\begin{aligned} \mathcal{H}^t_{b\rho}(A) &\leq \sum_j \mathcal{H}^t_{b\rho}(B'_j) \\ &\leq 2^t(\sqrt{u})^t \exp\{-\psi^t(x, f)\} \cdot \sum_j r_j^t \\ &\leq 2^t(\sqrt{u})^t \exp\{-\psi^t(x, f)\} \cdot (a+\eta). \end{aligned}$$

The choice of C_0 and the arbitrariness of $\eta > 0$ implies the desired result. □

The following result relates the zero of measure-theoretic pressure with the upper bound of the unstable pointwise dimension of μ.

LEMMA 3.2. *For μ-a.e. x, $d^u_\mu(x) \leq t^*_{u,1}$, where $t^*_{u,1}$ is the unique solution of the equation $P_\mu(f, -\psi^t(\cdot, f)) = 0$.*

Proof. Fix a small number $\varepsilon > 0$ such that $-\lambda_u(\mu) + 2\varepsilon < 0$ and choose $t > t^*_{u,1}$ such that

$$h_\mu(f) - \int \psi^t(x, f)\, d\mu = -3\varepsilon.$$

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Claim. *There exists an integer N_1 (depending only on ε) such that, for μ-a.e. x and every $N \geq N_1$, the Birkhoff averages*

$$\frac{1}{kN}\sum_{j=0}^{k-1}\log \alpha_u(f^{jN}x, f^N)$$

converge towards a number bigger than $\lambda_u(\mu) - \varepsilon$, as k goes to $+\infty$.

Proof of the Claim. We give the proof of the Claim by modifying slightly the arguments in the proof of [1, Lemma 8.4].

Since $\lim_{n\to\infty}(1/n)\log\alpha_u(x, f^n) = \lim_{n\to\infty}(1/n)\int\log\alpha_u(x, f^n)\,d\mu = \lambda_u(\mu)$ for μ-a.e. x, there exists a positive integer L such that

$$\int \log\alpha_u(x, f^L)\,d\mu \geq (\lambda_u(\mu) - \varepsilon/2)L. \tag{11}$$

The measure μ may be not ergodic for f^L, one can decompose it as

$$\mu = \frac{1}{m}(\mu_1 + \mu_2 + \cdots + \mu_m),$$

where $m \in \mathbb{N}^+$ divides L and each μ_i is an ergodic f^L-invariant measure such that $f_*\mu_i = \mu_{i+1}$ for each $i\,(\mathrm{mod}\ m)$. Let $A_1 \cup A_2 \cup \cdots \cup A_m$ be a measurable partition of (M, μ) such that $f(A_i) = A_{i+1}$ for each $i\,(\mathrm{mod}\ m)$ and $\mu_i(A_i) = 1$. By equation (11), there exists $j_0 \in \{1, 2, \ldots, m\}$ such that

$$\int \log\alpha_u(x, f^L)\,d\mu_{j_0} \geq (\lambda_u(\mu) - \varepsilon/2)L.$$

For every $N \geq 1$ and μ-a.e. x, one decomposes the orbit $\{f^i(x)\}_{i=0}^{N-1}$ as $(x, \ldots, f^{j-1}(x))$, $(f^j(x), \ldots, f^{j+(r-1)L-1}(x))$ and $(f^{j+(r-1)L}(x), \ldots, f^{N-1}(x))$, where $j < L$, $j + rL \geq N$ and the points $\{f^{j+sL}(x)\}_{s=0}^{r}$ belong to A_{j_0}. Using the super-additivity of $\{\log\alpha_u(x, f^n)\}_{n\geq 1}$, we have that

$$\log\alpha_u(x, f^N) \geq \log\alpha_u(x, f^j) + \sum_{s=0}^{r-2}\log\alpha_u(f^{j+sL}x, f^L) + \log\alpha_u(f^{j+(r-1)L}x, f^{N-j-L(r-1)}).$$

Hence, one has

$$\log\alpha_u(x, f^N) \geq 2C_f + \sum_{s=0}^{r-2}\log\alpha_u(f^{j+sL}x, f^L),$$

where $C_f = \max_{0\leq i<L}\max_{x\in M}|\log\alpha_u(x, f^i)|$ with the convention that $|\log\alpha_u(x, f^0)| = 0$. Since

$$\lim_{k\to+\infty}\frac{1}{kL}\sum_{\ell=0}^{k-1}\log\alpha_u(f^{j+\ell L}x, f^L) = \frac{1}{L}\int\log\alpha_u(x, f^L)\,d\mu_{j_0} \geq \lambda_u(\mu) - \varepsilon/2$$

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and

$$\lim_{k\to+\infty}\frac{1}{kN}\sum_{j=0}^{k-1}\log\alpha_u(f^{jN}x, f^N)\geq\frac{2C_f}{N}+\lim_{k\to+\infty}\frac{1}{kL}\sum_{\ell=0}^{k-1}\log\alpha_u(f^{j+\ell L}x, f^L),$$

and there exists an integer N_1 (depending on ε) so that $|2C_f/N|<\varepsilon/2$ for every $N>N_1$, for μ-a.e. x and every $N>N_1$, we have that

$$\lim_{k\to+\infty}\frac{1}{kN}\sum_{j=0}^{k-1}\log\alpha_u(f^{jN}x, f^N)>\lambda_u(\mu)-\varepsilon. \qquad \square$$

Take $b_0>2\sqrt{u}$ and $C_0>2^t u^{t/2}$, choose $N>N_1$ large enough such that

$$C_0e^{-N\varepsilon}<1 \quad\text{and}\quad e^{[\lambda_u(\mu)-2\varepsilon]N}>b_0. \tag{12}$$

By the above Claim and Birkhoff ergodic theorem, for μ-a.e. $x\in M$, we have that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\alpha_{u-[t]}(f^{jN}x, f^N)&\geq\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log\alpha_u(f^{jN}x, f^N)\\&\geq(\lambda_u(\mu)-\varepsilon)N\end{aligned}$$

and

$$\lim_{n\to\infty}\frac{1}{nN}\sum_{j=0}^{nN-1}\psi^t(f^jx, f)=\int\psi^t(x, f)\,d\mu.$$

Let ρ_0 be as in Lemma 3.1. Fix $\delta\in(0,\rho_0)$. Ledrappier and Young [22] proved that

$$\begin{aligned}\limsup_{n\to\infty}\frac{-\log\mu_x^u(B^u(x,n,\delta/2))}{n}&\leq\lim_{\delta\to0}\limsup_{n\to\infty}\frac{-\log\mu_x^u(B^u(x,n,\delta/2))}{n}\\&=h_\mu(f)\ \mu\text{-a.e. } x,\end{aligned}$$

where $B^u(x,n,\delta/2):=\{y\in W^u(x,f):d^u(f^jx,f^jy)<\delta/2 \text{ for } 0\leq j<n\}$. Hence, one can find sets $A_n\subset\Gamma$ with $\mu(A_n)\to1$ ($n\to\infty$), for every $x\in A_n$ where the following properties hold:

(a) $\exp[-nN(h_\mu(f)+\varepsilon)]\leq\mu_x^u(B^u(x,nN,\delta/2))$;

(b) $nN(-\int\psi^t(x,f)-\varepsilon)\leq-\sum_{j=0}^{nN-1}\psi^t(f^jx,f)\leq nN(-\int\psi^t(x,f)+\varepsilon)$;

(c) $\sum_{j=0}^{n-1}\log\alpha_{u-[t]}(f^{jN}x, f^N)\geq nN(\lambda_u(\mu)-2\varepsilon)$.

Take a point $x\in A_n$. Let E be a maximal (nN,δ)-separated subset of $A_n\cap\xi^u(x)$, then

$$A_n\cap\xi^u(x)\subset\bigcup_{x_j\in E}B^u(x_j,nN,\delta).$$

Furthermore, by property (a), the number of balls $B^u(x_j,nN,\delta/2)$ is less than or equal to $\exp\{nN[h_\mu(f)+\varepsilon]\}$. Let

$$b_k(x)=(b_0)^k\exp\bigg[-\sum_{j=n-k}^{n-1}\log\alpha_{u-[t]}(f^{jN}x, f^N)\bigg]$$

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for $k = 1, 2, \ldots, n$ and $\beta_n = \{b_0 \exp[(-\lambda_u(\mu) + 2\varepsilon)N]\}^n \cdot \rho$, where $0 < \rho < \rho_0$. By property (c), we have

$$\begin{aligned} b_n(x)\rho &= (b_0)^n \exp\bigg[-\sum_{j=0}^{n-1} \log \alpha_{u-[t]}(f^{jN}x, f^N)\bigg] \cdot \rho \\ &\le (b_0)^n \exp[nN(-\lambda_u(\mu) + 2\varepsilon)] \cdot \rho \\ &= [b_0 e^{(-\lambda_u(\mu)+2\varepsilon)N}]^n \cdot \rho \\ &= \beta_n. \end{aligned}$$

For each $x_j \in E$, using Lemma 3.1 n times, we conclude that

$$\begin{aligned} \mathcal{H}^t_{\beta_n}(B^u(x_j, nN, \delta)) &\le \mathcal{H}^t_{b_n(x_j)\rho}(B^u(x_j, nN, \delta)) \\ &\le C_0 \exp\{-\psi^t(x_j, f^N)\} \cdot \mathcal{H}^t_{b_{n-1}(x_j)\rho}(f^N(B^u(x_j, nN, \delta))) \\ &\le C_0 \exp\{-\psi^t(x_j, f^N)\} \cdot \mathcal{H}^t_{b_{n-1}(x_j)\rho}(B^u(f^N(x_j), (n-1)N, \delta)) \\ &\le (C_0)^2 \exp\{-\psi^t(x_j, f^N)\} \cdot \exp\{-\psi^t(f^N(x_j), f^N)\} \\ &\quad \cdot \mathcal{H}^t_{b_{n-2}(x_j)\rho}(B^u(f^{2N}(x_j), (n-2)N, \delta)) \\ &\le \cdots \\ &\le (C_0)^n \exp\bigg\{ -\sum_{j=0}^{n-1} \psi^t(f^{jN}x_j, f^N)\bigg\} \cdot \mathcal{H}^t_\rho(B^u(f^{nN}x_j, \delta)) \\ &\le (C_0)^n C_1 \cdot \exp\bigg\{ -\sum_{j=0}^{n-1} \psi^t(f^{jN}x_j, f^N)\bigg\}, \end{aligned}$$

where $C_1 = \sup_{y\in M} \mathcal{H}^t_\rho(B(y, \delta))$. By property (b) and the sub-additivity of $\{-\psi^t(\cdot, f^n)\}_{n\ge1}$, we have that

$$\begin{aligned} \mathcal{H}^t_{\beta_n}(A_n \cap \xi^u(x)) &\le \sum_{x_j\in E} \mathcal{H}^t_{\beta_n}(B^u(x_j, nN, \delta)) \\ &\le \sum_{x_j\in E} (C_0)^n C_1 \cdot \exp\bigg\{ -\sum_{i=0}^{n-1} \psi^t(f^{jN}x_j, f^N)\bigg\} \\ &\le \sum_{x_j\in E} (C_0)^n C_1 \cdot \exp\bigg\{ -\sum_{i=0}^{nN-1} \psi^t(f^i x_j, f)\bigg\} \\ &\le (C_0)^n C_1 \cdot \exp[nN(h_\mu(f) + \varepsilon)] \cdot \exp\bigg[nN\bigg(-\int \psi^t(x, f)\, d\mu + \varepsilon\bigg)\bigg] \\ &= (C_0)^n C_1 \cdot \exp\bigg[nN\bigg(h_\mu(f) - \int \psi^t(x, f)\, d\mu + 2\varepsilon\bigg)\bigg] \\ &= (C_0)^n C_1 \cdot e^{-nN\varepsilon} \\ &= (C_0 e^{-N\varepsilon})^n C_1. \end{aligned}$$

Since N satisfies $C_0 e^{-N\varepsilon} < 1$, we have that

$$\lim_{n\to\infty} \mathcal{H}^t_{\beta_n}(A_n \cap \xi^u(x)) = 0.$$

Since $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} \mu^u_x(A_n \cap \xi^u(x)) = 1$ for μ-a.e. x, by [17, Lemma 6], we obtain that

$$\dim_H \mu^u_x \le t$$

for μ-a.e. x. Combining with equation (5) and the choice of t yield that $d^u_\mu(x) \le t^*_{u,1}$ for μ-a.e. x. □

Now we are ready to present the proof of Theorem B.

Proof of Theorem B. For each $n > 1$, the measure μ is f-invariant ergodic, but it may be not ergodic for f^n although μ is still f^n-invariant. In either case, one can find an f^n-invariant ergodic probability measure ν such that

$$\mu = \frac{1}{m}[\nu + f_*\nu + \cdots + f_*^{m-1}\nu],$$

where $m \in \mathbb{N} \setminus \{0\}$ divides n. Let

$$\widetilde{P}_\mu(f^n, -\psi^n(\cdot, f^n)) := h_\mu(f^n) - \int \psi^n(x, f^n)\, d\mu,$$

then one can show that

$$\begin{aligned}\widetilde{P}_\mu(f^n, -\psi^n(\cdot, f^n)) &= \frac{1}{m}\sum_{i=0}^{m-1}\left(h_{f_*^i\nu}(f^n) - \int \psi^n(x, f^n)\, df_*^i\nu\right)\\ &= \frac{1}{m}\sum_{i=0}^{m-1} P_{f_*^i\nu}(f^n, -\psi^n(\cdot, f^n)).\end{aligned}$$

Hence, there exists $j_0 \in \{0, 1, \ldots, m-1\}$ such that

$$\widetilde{P}_\mu(f^n, -\psi^t(\cdot, f^n)) \ge P_{f_*^{j_0}\nu}(f^n, -\psi^t(\cdot, f^n)).$$

Since $f_*^{j_0}\nu$ is hyperbolic and f^n-invariant ergodic, by Lemma 3.2, there is a set $\tilde{A}$ with $\nu \circ f^{-j_0}(\tilde{A}) = 1$ such that for each $x \in \tilde{A}$,

$$d^u_{f_*^{j_0}\nu}(x) \le t^*_{u,n},$$

where $t^*_{u,n}$ is the unique root of the equation $P_\mu(f^n, -\psi^t(\cdot, f^n)) = 0$. Note that $d^u_\mu(x), d^u_{f_*^{j_0}\nu}(x)$ are constants almost everywhere (see [22]) and $d^u_\mu(x) \le d^u_{f_*^{j_0}\nu}(x) \le t^*_{u,n}$ for each $x \in \tilde{A}$ with $\mu(\tilde{A}) \ge 1/m$. Consequently, we have that

$$d^u_\mu(x) \le t^*_{u,n}$$

for μ-a.e. x.

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By the sub-additive of $\{-\psi^t(\cdot, f^n)\}_{n\geq 1}$, we obtain

$$\frac{1}{2^{k+1}}\bigg[h_\mu(f^{2^{k+1}}) - \int \psi^t(x, f^{2^{k+1}})\,d\mu\bigg] \leq \frac{1}{2^k}\bigg[h_\mu(f^{2^k}) - \int \psi^t(x, f^{2^k})\,d\mu\bigg].$$

Hence,

$$\frac{1}{2^{k+1}}\widetilde{P}_\mu(f^{2^{k+1}}, -\psi^t(\cdot, f^{2^{k+1}})) \leq \frac{1}{2^k}\widetilde{P}_\mu(f^{2^k}, -\psi^t(\cdot, f^{2^k})).$$

This yields that $t^*_{u,2^{k+1}} \leq t^*_{u,2^k}$ for every $k \geq 1$. Let $t^*_u := \lim_{k\to\infty} t^*_{u,2^k}$, then one has that

$$d^u_\mu(x) \leq t^*_u \quad \mu\text{-a.e. } x.$$

Since $P_\mu(f, \{-\psi^t(\cdot, f^n)\})$ is continuous and strictly decreasing with respect to t, there exists at most one solution of the equation. To complete the proof of Theorem B, it suffices to show that $P_\mu(f, \{-\psi^{t^*}(\cdot, f^n)\}) = 0$.

Since $t^*_{u,2^k} \geq t^*_u$ for every $k \geq 1$, by Theorem 2.2, one has that

$$0 \leq \lim_{k\to\infty}\frac{1}{2^k}\widetilde{P}_\mu(f^{2^k}, -\psi^{t^*_u}(\cdot, f^{2^k})) = P_\mu(f, \{-\psi^{t^*_u}(\cdot, f^n)\}).$$

However, for each small number $\varepsilon > 0$, there exists K so that $t^*_{u,2^k} \leq t^*_u + \varepsilon$ for every $k \geq K$. Hence, we have that

$$\begin{aligned} P_\mu(f, \{-\psi^{t^*_u+\varepsilon}(\cdot, f^n)\}) &= h_\mu(f) - \lim_{n\to\infty}\frac{1}{n}\int \psi^{t^*_u+\varepsilon}(x, f^n)\,d\mu \\ &= \lim_{k\to\infty}\frac{1}{2^k}\widetilde{P}_\mu(f^{2^k}, -\psi^{t^*_u+\varepsilon}(\cdot, f^{2^k})) \leq 0. \end{aligned}$$

The previous arguments imply that $P_\mu(f, \{-\psi^{t^*}(\cdot, f^n)\}) = 0$. One can prove in a similar fashion that $d^s_\mu(x) \leq t^*_s$ for μ-a.e. x. This completes the proof of Theorem B. □

3.3. *Proof of Theorem C.* For each $\varepsilon > 0$, there exists a hyperbolic set Λ_ε satisfying properties (a)–(d) in Theorem 2.4. The following lemma shows that the zero of the super-additive topological pressure of $\Phi_f(t)$ provides a lower bound of the Carathéodory singular dimension of the hyperbolic set on the local unstable leaf with respect to the super-additive singular valued potential $\Phi_f(t)$.

LEMMA 3.3. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M and let $\Lambda \subset M$ be a hyperbolic set. Assume that $f|_\Lambda$ is topologically transitive, then for every $x \in \Lambda$,*

$$\dim_C^{\Phi_f}(\Lambda \cap W^u_{\text{loc}}(x, f)) \geq t_*,$$

where t_* is the unique root of the equation $P(f|_\Lambda, \Phi_f(t)) = 0$.

Proof. For every $x \in \Lambda$, we denote $Z = \Lambda \cap W^u_{\text{loc}}(x, f)$ and $P(t) = P(f|_\Lambda, \Phi_f(t))$. Since the function $P(t)$ is strictly decreasing in t, then for each $t < t_*$, we have that

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$P(t) > 0$. Fix such a number t and take $\varepsilon > 0$ with $P(t) - \varepsilon > 0$. By Proposition 2.2, one has that

$$P(t) = \lim_{n\to\infty} \frac{1}{n} P(f^n|_\Lambda, -\phi^t(\cdot, f^n)),$$

then there exists $N_1 \in \mathbb{N}$ such that for every $n \geq N_1$, we obtain

$$P(f^n|_\Lambda, -\phi^t(\cdot, f^n)) > n(P(t) - \varepsilon) > 0.$$

Fix an integer $L \geq N_1$, by [**11**, Proposition 5.4], one has that

$$P(f^L|_\Lambda, -\phi^t(\cdot, f^L)) = P_Z(f^L|_\Lambda, -\phi^t(\cdot, f^L)).$$

Hence, there is $\delta_1 > 0$ such that

$$P_Z(f^L|_\Lambda, -\phi^t(\cdot, f^L), \delta) > (P(t) - \varepsilon)L$$

for every $0 < \delta < \delta_1$. Consequently, fixing such a $\delta > 0$, one has that

$$m(Z, -\phi^t(\cdot, f^L), (P(t) - \varepsilon)L, \delta) = +\infty.$$

Hence, for each $K > 0$, there exists $S \in \mathbb{N}$ such that for each $N \geq S$, we have that

$$\begin{aligned} K &\leq \inf \sum_i \exp[-(P(t) - \varepsilon)Lm_i - S_{m_i}\phi^t(x_i, f^L)] \\ &\leq e^{-NL(P(t)-\varepsilon)} \inf \sum_i \exp[-S_{m_i}\phi^t(x_i, f^L)], \end{aligned} \tag{13}$$

where the infimum is taken over all collections $\{B^u_{m_i}(x_i, \delta, f^L)\}$ with $x_i \in \Lambda$, $m_i \geq N$ which cover Z, $-S_{m_i}\phi^t(x_i, f^L) = -\phi^t(x_i, f^L) - \phi^t(f^L x_i, f^L) - \cdots - \phi^t(f^{(m_i-1)L}x_i, f^L)$ and

$$B^u_{m_i}(x_i, \delta, f^L) := \left\{ y \in W^u(x_i, f) : \max_{0\leq j<m_i} d^u(f^{jL}(y), f^{jL}(x_i)) < \delta \right\}.$$

Fixing such an N and taking an integer $R \geq NL$, let the collection of balls $\{B^u_{n_i}(x_i, \delta)\}$ with $x_i \in \Lambda$, $n_i \geq R$ be a cover of Z. One can write $n_i = m_i L + s_i$ with $0 \leq s_i < L$ and $m_i \geq N$ for each i. Since $B^u_{n_i}(x_i, \delta) \subset B^u_{m_i}(x_i, \delta, f^L)$ for each i, the collection of balls $B^u_{m_i}(x_i, \delta, f^L)$ is also a cover of Z with $x_i \in \Lambda$, $m_i \geq N$. By the super-additivity of $\{-\phi^t(\cdot, f^n)\}_{n\geq 1}$, one has

$$\begin{aligned} \sum_i \exp[-\phi^t(x_i, f^{n_i})] &\geq \sum_i \exp[-S_{m_i}\phi^t(x_i, f^L) - \phi^t(f^{m_i L}y, f^{s_i})] \\ &\geq C \sum_i \exp[-S_{m_i}\phi^t(x_i, f^L)], \end{aligned}$$

where $C = \min_{0\leq s<L} \min_{x\in M} \exp[-\phi^t(x, f^s)]$. This together with equation (13) yield that

$$\sum_i \exp[-\phi^t(x_i, f^{n_i})] \geq CKe^{NL(P(t)-\varepsilon)}.$$

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Since the cover of Z is taken arbitrarily, one can conclude that

$$\inf \sum_i \exp[-\phi^t(x_i, f^{n_i})] \geq CKe^{NL(P(t)-\varepsilon)},$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i, \delta)\}$ with $x_i \in \Lambda$, $n_i \geq NL$ which cover Z. Letting $N \to \infty$, we obtain

$$m(Z, \Phi_f(t), \delta) = +\infty$$

for every $t < t_*$. This implies that

$$\dim_C^{\Phi_f} Z \geq t_*. \qquad \square$$

Proof of Theorem C(i). By Lemma 3.3, for every $x \in \Lambda_\varepsilon$, we obtain

$$\dim_C^{\Phi_f}(\Lambda_\varepsilon \cap W^u_{\mathrm{loc}}(x, f)) \geq t_{\varepsilon *},$$

where $t_{\varepsilon *}$ is the unique root of the equation $P(f|_{\Lambda_\varepsilon}, \Phi_f(t)) = 0$. By the variational principle of topological entropy, take $\nu \in \mathcal{M}_{\mathrm{inv}}(f|_{\Lambda_\varepsilon})$ such that $h_{\mathrm{top}}(f|_{\Lambda_\varepsilon}) = h_\nu(f|_{\Lambda_\varepsilon})$. By properties (b) and (d) in Theorem 2.4, it holds that

$$\begin{aligned} 0 &= P(f|_{\Lambda_\varepsilon}, \Phi_f(t_{\varepsilon *})) \\ &= \sup\left\{h_\nu(f|_{\Lambda_\varepsilon}) - \sum_{i=1}^{[t_{\varepsilon *}]}\lambda_i(\nu) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\nu) : \ \nu \in \mathcal{M}_{\mathrm{inv}}(f|_{\Lambda_\varepsilon})\right\} \\ &\geq h_{\mathrm{top}}(f|_{\Lambda_\varepsilon}) - \sum_{i=1}^{[t_{\varepsilon *}]}\lambda_i(\nu) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\nu) \\ &\geq h_\mu(f) - \sum_{i=1}^{[t_{\varepsilon *}]}\lambda_i(\mu) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\mu) - (u+1)\varepsilon, \end{aligned} \tag{14}$$

where $\lambda_1(\nu) \geq \lambda_2(\nu) \geq \cdots \geq \lambda_{m_0}(\nu)$ are the Lyapunov exponents of ν. However, let $\tau \in \mathcal{M}_{\mathrm{inv}}(f|_{\Lambda_\varepsilon})$ be an equilibrium state of $P(f|_{\Lambda_\varepsilon}, \Phi_f(t_{\varepsilon *}))$, then one has that

$$\begin{aligned} 0 &= P(f|_{\Lambda_\varepsilon}, \Phi_f(t_{\varepsilon *})) \\ &= h_\tau(f|_{\Lambda_\varepsilon}) - \sum_{i=1}^{[t_{\varepsilon *}]}\lambda_i(\tau) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\tau) \\ &\leq h_{\mathrm{top}}(f|_{\Lambda_\varepsilon}) - \sum_{i=1}^{[t_{\varepsilon *}]}\lambda_i(\tau) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\tau) \\ &\leq h_\mu(f) - \sum_{i=1}^{[t_{\varepsilon *}]}\lambda_i(\mu) - (t_{\varepsilon *} - [t_{\varepsilon *}])\lambda_{[t_{\varepsilon *}]+1}(\mu) + (u+1)\varepsilon. \end{aligned}$$

This together with equation (14) yield that

$$-(u+1)\varepsilon \leq P_\mu(f, \Phi_f(t_{\varepsilon *})) \leq (u+1)\varepsilon.$$

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Hence, we have that

$$\lim_{\varepsilon\to 0} P_\mu(f, \Phi_f(t_{\varepsilon *})) = 0.$$

This implies that $\lim_{\varepsilon\to 0} t_{\varepsilon *} = t_{u*}$, where t_{u*} is the unique root of $P_\mu(f, \Phi_f(t)) = 0$. Consequently, we have that

$$\liminf_{\varepsilon\to 0} \dim_C^{\Phi_f}(\Lambda_\varepsilon \cap W^u_{\text{loc}}(x, f)) \ge t_{u*}. \tag{15}$$

□

As a counterpart of Lemma 3.3, we have the following result.

LEMMA 3.4. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact smooth Riemannian manifold M and let $\Lambda \subset M$ be a hyperbolic set. Assume that $f|_\Lambda$ is topologically transitive. Then for every $x \in \Lambda$,*

$$\dim_C^{\Psi_f}(\Lambda \cap W^u_{\text{loc}}(x, f)) \le t^*,$$

where t^* is the unique root of the equation $P(f|_\Lambda, \Psi_f(t)) = 0$.

Proof. Denote $P(t) = P(f|_\Lambda, \Psi_f(t))$. For each $t > t_*$,

$$0 > P(t) = \lim_{n\to\infty} \frac{1}{n} P(f^n, -\psi^t(\cdot, f^n)).$$

Fix such a number t and take $\varepsilon > 0$ with $P(t) + \varepsilon < 0$. Then there exists $N_1 \in \mathbb{N}$ such that for every $n \ge N_1$, we obtain

$$P(f^n, -\psi^t(\cdot, f^n)) < n(P(t) + \varepsilon) < 0.$$

Fix an integer $L \ge N_1$ such that

$$P(f^L, -\psi^t(\cdot, f^L)) < L(P(t) + \varepsilon) < 0.$$

For each $x \in \Lambda$, set $Z = \Lambda \cap W^u_{\text{loc}}(x, f)$. By [11, Proposition 5.4], one has that

$$P(f^L, -\psi^t(\cdot, f^L)) = P_Z(f^L, -\psi^t(\cdot, f^L)).$$

Thus, there is $\delta_1 > 0$ such that for every $0 < \delta < \delta_1$, one has

$$P_Z(f^L, -\psi^t(\cdot, f^L), \delta) < (P(t) + \varepsilon)L.$$

Hence, one has that

$$m(Z, -\psi^t(\cdot, f^L), (P(t) + \varepsilon)L, \delta) = 0.$$

For each $\xi > 0$, there exists $N \in \mathbb{N}$ and a cover $\{B^u_{n_i}(x_i, \delta, f^L)\}$ of Z with $x_i \in \Lambda$, $n_i \ge N$ such that

$$\begin{aligned}\xi &\ge \sum_i \exp\Big[-(P(t)+\varepsilon)Ln_i + \sup_{y\in B^u_{n_i}(x_i,\delta,f^L)} -S_{n_i}\psi^t(y, f^L)\Big].\\ &\ge e^{-NL(P(t)+\varepsilon)} \sum_i \exp\Big[\sup_{y\in B^u_{n_i}(x_i,\delta,f^L)} -S_{n_i}\psi^t(y, f^L)\Big].\end{aligned}$$

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Note that $d^u(f^L x, f^L y) < \delta$ implies $d^u(f^i x, f^i y) < \delta$ for $i = 0, 1, \ldots, L-1$, since f is expanding along the unstable manifold. This implies that $B^u_{(n_i-1)L+1}(x_i, \delta) = B^u_{n_i}(x_i, \delta, f^L)$ for every i. Since

$$\begin{aligned}
-S_{n_i}\psi^t(y, f^L) &= -\psi^t(y, f^L) - \psi^t(f^L y, f^L) - \cdots - \psi^t(f^{(n_i-1)L} y, f^L) \\
&\geq -\psi^t(y, f^{(n_i-1)L}) + C_1 \\
&= -\psi^t(y, f^{(n_i-1)L}) - \psi^t(f^{(n_i-1)L} y, f) + \psi^t(f^{(n_i-1)L} y, f) + C_1 \\
&\geq -\psi^t(y, f^{(n_i-1)L+1}) + C_1 + C_2,
\end{aligned}$$

where $C_1 = \min_{x\in M}\{-\psi^t(x, f^L)\}$ and $C_2 = \min_{x\in M} \psi^t(x, f)$, we have that

$$\begin{aligned}
\xi &\geq e^{-NL(P(t)+\varepsilon)} e^{C_1+C_2} \sum_i \exp\Big[\sup_{y\in B^u_{(n_i-1)L+1}(x_i,\delta)} -\psi^t(y, f^{(n_i-1)L+1})\Big] \\
&\geq e^{-NL(P(t)+\varepsilon)} e^{C_1+C_2} \inf \sum_i \exp\Big[\sup_{y\in B^u_{m_i}(x_i,\delta)} -\psi^t(y, f^{m_i})\Big]
\end{aligned}$$

and

$$\inf \sum_i \exp\Big[\sup_{y\in B^u_{m_i}(x_i,\delta)} -\psi^t(y, f^{m_i})\Big] \leq \xi e^{NL(P(t)+\varepsilon)} e^{-C_1-C_2},$$

where the infimum is taken over all collections $\{B^u_{m_i}(x_i, \delta)\}$ with $x_i \in \Lambda$, $m_i \geq (N-1)L$ which cover Z. Letting $N \to \infty$, we obtain

$$m(Z, \Psi_f(t), \delta) = 0$$

for every $t > t_*$. This yields that

$$\dim_C^{\Psi_f} Z \leq t_*. \qquad \square$$

Remark 3.1. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact Riemannian manifold M and $\Lambda \subset M$ be a hyperbolic set. Assume that $f|_\Lambda$ is topologically transitive. Then for every $x \in \Lambda$,

$$\dim_C^{\Phi_f}(\Lambda \cap W^u_{\mathrm{loc}}(x, f)) = t_u^{\Phi_f}, \quad \dim_C^{\Psi_f}(\Lambda \cap W^u_{\mathrm{loc}}(x, f)) = t_u^{\Psi_f},$$

where $t_u^{\Phi_f}$, $t_u^{\Psi_f}$ are the unique roots of the equations

$$P_{\Lambda\cap W^u(x,f)}(f, \Phi_f(t)) = 0, \quad P_{\Lambda\cap W^u(x,f)}(f, \Psi_f(t)) = 0,$$

respectively. The proof is a slight modification of Lemmas 3.3 and 3.4. See [9] for more details about the Carathéodory singular dimension of each subset of a repeller. However, we do not know whether $P_{\Lambda\cap W^u(x,f)}(f, \Phi_f(t)) = P_\Lambda(f, \Phi_f(t))$ and $P_{\Lambda\cap W^u(x,f)}(f, \Psi_f(t)) = P_\Lambda(f, \Psi_f(t))$ hold.

Proof of Theorem C(ii). By Lemma 3.4, we have that

$$\dim_C^{\Psi_f}(\Lambda_\varepsilon \cap W^u_{\mathrm{loc}}(x, f)) \leq t^*_\varepsilon,$$

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where t_ε^* is the unique root of the equation $P(f|_{\Lambda_\varepsilon}, \Psi_f(t)) = 0$. Take $\nu \in \mathcal{M}_{\text{inv}}(f|_{\Lambda_\varepsilon})$ such that $h_\nu(f|_{\Lambda_\varepsilon}) = h_{\text{top}}(f|_{\Lambda_\varepsilon})$, by properties (b) and (d) in Theorem 2.4, it holds that

$$\begin{aligned}
0 &= P(f|_{\Lambda_\varepsilon}, \Psi_f(t_\varepsilon^*)) \\
&= \sup\left\{h_\nu(f|_{\Lambda_\varepsilon}) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\nu) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{u-[t_\varepsilon^*]}(\nu) : \quad \nu \in \mathcal{M}_{\text{inv}}(f|_{\Lambda_\varepsilon})\right\} \\
&\geq h_{\text{top}}(f|_{\Lambda_\varepsilon}) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\nu) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{u-[t_\varepsilon^*]}(\nu) \\
&\geq h_\mu(f) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\mu) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{u-[t_\varepsilon^*]}(\mu) - (u+1)\varepsilon, \qquad (16)
\end{aligned}$$

where $\lambda_1(\nu) \geq \lambda_2(\nu) \geq \cdots \geq \lambda_{m_0}(\nu)$ are the Lyapunov exponents of ν. Similarly, let $\tau \in \mathcal{M}_{\text{inv}}(f|_{\Lambda_\varepsilon})$ be an equilibrium state of $P(f|_{\Lambda_\varepsilon}, \Psi_f(t_\varepsilon^*))$, then one has that

$$\begin{aligned}
0 &= P(f|_{\Lambda_\varepsilon}, \Psi_f(t_\varepsilon^*)) \\
&= h_\tau(f|_{\Lambda_\varepsilon}) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\tau) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{[t_\varepsilon^*]+1}(\tau) \\
&\leq h_{\text{top}}(f|_{\Lambda_\varepsilon}) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\tau) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{[t_\varepsilon^*]+1}(\tau) \\
&\leq h_\mu(f) - \sum_{i=u-[t_\varepsilon^*]+1}^{u} \lambda_i(\mu) - (t_\varepsilon^* - [t_\varepsilon^*])\lambda_{[t_\varepsilon^*]+1}(\mu) + (u+1)\varepsilon.
\end{aligned}$$

This together with equation (16) yield that

$$-(u+1)\varepsilon \leq P_\mu(f, \Psi_f(t_\varepsilon^*)) \leq (u+1)\varepsilon.$$

Hence, we have that

$$\lim_{\varepsilon\to 0} P_\mu(f, \Psi_f(t_\varepsilon^*)) = 0.$$

This implies that $\lim_{\varepsilon\to 0} t_\varepsilon^* = t_u^*$, where t_u^* is the unique root of $P_\mu(f, \Psi_f(t)) = 0$. Consequently, we have that

$$\limsup_{\varepsilon\to 0} \dim_C^{\Psi_f}(\Lambda_\varepsilon \cap W_{\text{loc}}^u(x, f)) \leq t_u^*. \qquad \square$$

Finally, to complete the proof of Theorem C, assume that μ is an SRB measure from now on, then $h_\mu(f) = \lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_u(\mu)$. Thus, $P_\mu(f, \Phi_f(u)) = 0$. Since the Carathéodory singular dimension with respect to Ψ_f is always less than u, by property (i) of Theorem C, we have that

$$\lim_{\varepsilon\to 0} \dim_C^{\Phi_f}(\Lambda_\varepsilon \cap W_{\text{loc}}^u(x, f)) = u.$$

If $\lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_{m_0}(\mu) \geq 0$, then $P_\mu(f, \Xi_f(m_0 - u)) \geq 0$ since μ is an SRB measure for *f*. Consider f^{-1}, by Margulis–Ruelle inequality, we have that

$$h_\mu(f) = h_\mu(f^{-1}) \leq -\lambda_{u+1}(\mu) - \cdots - \lambda_{m_0}(\mu),$$

which implies that $P_\mu(f, \Xi_f(m_0 - u)) \leq 0$. Hence, we have that

$$P_\mu(f, \Xi_f(m_0 - u)) = 0.$$

Thus, we have that $t_s^* = m_0 - u$. By the definition of Lyapunov dimension, we have that $\dim_L \mu = m_0 = u + t_s^*$.

Now, we assume that $\lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_{m_0}(\mu) < 0$ and let ℓ be the largest integer such that $\lambda_1(\mu) + \lambda_2(\mu) + \cdots + \lambda_\ell(\mu) \geq 0$. By a standard computation, one can show that

$$t_s^* = \ell - u - \frac{h_\mu(f) + \lambda_{u+1}(\mu) + \cdots + \lambda_\ell(\mu)}{\lambda_{\ell+1}(\mu)}.$$

Combining with

$$\dim_L \mu = \ell + \frac{h_\mu(f) + \lambda_{u+1}(\mu) + \cdots + \lambda_\ell(\mu)}{|\lambda_{\ell+1}(\mu)|},$$

one has

$$\dim_L \mu = u + t_s^*.$$

This completes the proof of Theorem C.

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