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ON THE NUMBER OF EIGENVALUES IN THE SPECTRAL GAP OF A DIRAC SYSTEM

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1. Introduction

We consider the one-dimensional operator,

$$Ly := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left\{ y' - \begin{pmatrix} p(x) & c_1 + V_1(x) \\ c_2 - V_2(x) & -p(x) \end{pmatrix} y \right\}, \tag{1.1}$$

on $0 < x < \infty$ with $y = \binom{y_1}{y_2}$. The coefficients p, V_1, and V_2 are assumed to be real, locally Lebesgue integrable functions; c_1 and c_2 are positive numbers. The operator L acts in the Hilbert space H of all equivalence classes of complex vector-value functions $f = \binom{f_1}{f_2}$ such that $\int_0^\infty (|f_1|^2 + |f_2|^2)\,dx < \infty$. L has domain $D(L)$ consisting of all $y \in H$ such that y is locally absolutely continuous and $Ly \in H$; thus in the language of differential operators L is a maximal operator. Associated with L is the minimal operator L_0 defined as the closure of L_0' where L_0' is the restriction of L to the functions with compact support in $(0, \infty)$.

The singular structure of the coefficients at 0 determine whether $L = L_0$ (and hence L is selfadjoint—the limit-point case) or L contains L_0 properly (the limit-circle case). In the latter case the selfadjoint operators L_1 generated by L satisfy $L_0 \subset L_1 \subset L$ and are determined by imposing a boundary condition at $x = 0$ on the elements of $D(L)$. The limit-point and limit-circle terminology arises from the geometric method of Weyl and is discussed in **[18]**. We use L_1 to denote a selfadjoint extension of L_0 in either case. In any event the essential spectra of L_0, L_1, and L coincide. Essential selfadjointness criteria and construction of selfadjoint extensions of L_0 have been discussed in many papers, e.g. **[1, 8, 9, 10, 11, 12, 20, 22, 23, 24]**.

Under rather general conditions with p, V_1, V_2 "small" at infinity and sufficiently "regular" at 0, the essential spectrum of L_1 is $(-\infty, -c_1] \cup [c_2, \infty)$. Results on location of essential spectra are given in **[3, 8, 9, 14, 23]**. When the gap $(-c_1, c_2)$ contains no essential spectrum of L_1, it can contain only eigenvalues of L_1. The purpose of this paper is to obtain conditions on the coefficients which determine if the gap contains finitely or infintely many eigenvalues of L_1. Since L_1 is a finite dimensional extension of L_0, the finiteness of the gap spectra is independent of L_1. Another problem, also independent of L_1, is to determine when the gap spectrum is infinite which of the endpoints $-c_1$ and c_2 are cluster points of eigenvalues. While our theorems do not assume the existence of a gap, it is the case of primary interest.

For $V_1 = V_2$, equation (1.1) arises from the three-dimensional Dirac equation with a spherically symmetric potential after a separation of variables. For the choices $p(x) = k/x$, $V_1(x) = V_2(x) = z/x$, and $c_1 = c_2 = c$, (1.1) is the radial wave equation in relativistic quantum mechanics for a particle in a field of potential $V(x) = z/x$. When the anomalous magnetic moment of the particle is considered, $p(x) = k/x$ is replaced by $p(x) = k_1/x + k_2/x^2$.

The discreteness of the spectrum of L_1 in $(-c_1, c_2)$ was studied by Birman [**3**, Section 5]. Using the methods of Birman, Kurbenin [**15**] gave criteria for the spectrum in the gap to be finite or to be infinite. The work of Kurbenin requires $p(x) = k/x$ and $V_1(x) = V_2(x)$ to be uniformly bounded on $(0, \infty)$. The one-dimensional results of Kurbenin will follow from the results given below. Further discussion is given in [**7**, Section 62].

In the three-dimensional Dirac equation with gap $[-1, 1]$ and with a regularly growing potential that decays like $|x|^{-m}$, $0 < m < 2$, as $|x| \to \infty$, Tamura [**21**] has given asymptotic behavior for the number of eigenvalues in $(0, 1-r)$ as $r \to 0$. Also in the three-dimensional case with a parameter Λ multiplying the potential, Klaus [**13**] has given an asymptotic formula for the number of eigenvalues in the gap as $\Lambda \to \infty$.

In Section 2 below we give conditions for gap spectra to be infinite. By means of oscillation theory we generate a family of second order scalar differential equations for which the oscillation of one of them at 0 or ∞ is sufficient to give infinite gap spectra. Section 3 treats the converse problem and gives a comparison second order vector equation whose non-oscillation implies a finite gap spectra. Additional results are given by treating the operator L directly. An interesting corollary of the latter method is a sufficient condition for the length of the gap to be "infinite", i.e. L_1 has empty essential spectrum. Finally in Section 4, we use a shifting technique which for certain equations permits one to determine which endpoint of a gap is a cluster point of eigenvalues.

We use the following notation: $(\ ,\ )$ represents the inner product in both H and R^n with corresponding norm $\|\ \|$, I is an identity operator or matrix, $\sigma(A)$ is the spectrum of A, and if J is an interval, $C_0(J)$ is the set of all continuously differentiable 2-vector-valued functions with compact support contained in J.

2. Infinite gap spectra

First we consider the square of L given by

$$L^2 y = -(y' + JPy)' - P(Jy' - Py) \tag{2.1}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} V_2 - c_2 & p \\ p & V_1 + c_1 \end{pmatrix}.$$

The gap spectrum of L_1 has a close connection with L^2 which we now describe. For $d > 0$, set $M_d(y) = L^2 y - d^2 y$; we say M_d is *non-oscillatory* at $\infty(0)$ provided that there is a number $b > 0$ such that on (b, ∞) $((0, b))$ no nontrivial solution y of $M_d y = 0$ vanishes twice, i.e., satisfies $y(b_1) = 0 = y(b_2)$ for $b < b_1 < b_2$. Let K_0 be the minimal operator associated with L^2. By the spectral theorem for self-adjoint operators, the spectrum of a self-adjoint extension K of K_0 has infinitely many points in $(-\infty, d^2)$ if, and only if, there is an infinite dimensional subspace $G \subset D(K)$ such that $(Ky, y) < d^2(y, y)$ for all $y \in G$ [7, Section 3]. By standard arguments (cf. [7, Sections 10, 12, 13]) this is equivalent to M_d being oscillatory at either 0 or ∞. Note that L_1^2 is a self-adjoint extension of K_0. By the spectral mapping theorem [5, p. 604], $\sigma(L_1^2) = \{\lambda^2 : \lambda \in \sigma(L_1)\}$; hence $\sigma(L_1) \cap (-d, d)$ is finite if, and only if, $\sigma(L_1^2) \cap (-\infty, d^2)$ is finite. Thus we have the criterion:

Theorem 2.1. *For $d > 0$, $\sigma(L_1) \cap (-d, d)$ is finite if, and only if, M_d is nonoscillatory at both 0 and ∞.*

A further useful object is the quadratic form Q_d of M_d given by

$$Q_d(y) = \int_0^\infty y^* M_d(y)\, dx$$

$$= \int_0^\infty [(y')^* y' + (y')^* JPy - y^*(PJ)y' + y^*(P^2 - d^2 I)y]\, dx \tag{2.2}$$

where the domain of Q_d is all y which have compact support in $(0, \infty)$, are absolutely continuous, and satisfy $\int_0^\infty (y')^* y' dx < \infty$. The basic connection between the oscillatory properties of M_d and Q_d is (cf. [4, 7]):

Theorem 2.2. *M_d is nonoscillatory at $\infty(0)$ if, and only if, there is a number $b > 0$ such that $Q_d(y) > 0$ for all nontrivial y in domain Q_d with support in (b, ∞) $((0, b))$.*

To establish oscillation criteria for M_d we use positive linear functionals. These were first introduced in oscillation theory by Etgen and Pawlowski [6]. A nontrivial linear functional g defined on the real $n \times n$ matrices is said to be *positive* if $g(B) \geqq 0$ whenever B is symmetric and positive semidefinite ($B \geqq 0$). All such g have the representation $g(B) = \sum_{i=1}^k (Bu_i, u_i)$ where u_i are non-zero n-vectors [cf. 17]. Recall a second-order scalar differential operator l is said to be oscillatory at $\infty(0)$ if all solutions of $l(z) = 0$ have infinitely many zeros in a neighborhood of $\infty(0)$.

Theorem 2.3. *Let $d > 0$, g be a nontrivial positive linear functional and assume P of (2.1) is locally absolutely continuous. Then $\sigma(L_1) \cap (-d, d)$ is infinite if the scalar differential equation,*

$$-g(I)z'' + g(P^2 - d^2 I + [P'J - JP']/2)z = 0 \tag{2.3}$$

is oscillatory either at 0 or at ∞.

Proof. Suppose to the contrary that $\sigma(L_1) \cap (-d, d)$ is finite. Let (2.3) be oscillatory say at ∞. Then by Theorems 2.1 and 2.2 there is a number b such that $Q_d(y) > 0$ for all nontrivial y in domain Q_d with support in (b, ∞). Since (2.3) is oscillatory at ∞ there is a real nontrivial solution z such that $z(b_1) = z(b_2) = 0$ with $b < b_1 < b_2$. Thus multiplying

(2.3) by z and integrating by parts yields

$$\int_{b_1}^{b_2} [g(I)|z'|^2 + g(P^2 - d^2 I + [P'J - JP']/2)z^2]\,dx = 0.$$

Thus by the above representation for g there is a non-zero vector u_i such that

$$\int_{b_1}^{b_2} \{(u_i, u_i)|z'|^2 + ([P^2 - d^2 I + (P'J - JP')/2]u_i, u_i)z^2\}\,dx \leqq 0. \tag{2.4}$$

Set $y(x) = z(x)u_i$ on $[b_1, b_2]$ and zero elsewhere. Then y is in domain Q_d with support in (b, ∞). Further, an integration by parts in (2.2) gives that

$$Q_d(y) = \int_{b_1}^{b_2} \{u_i^* u_i |z'|^2 + u_i^*[(P'J - JP')/2 + P^2 - d^2 I]u_i z^2\}\,dx$$

which is the same as (2.4). Thus $Q_d(y) \leqq 0$ contrary to $Q_d(y) > 0$; hence $\sigma(L_1) \cap (-d, d)$ is infinite.

For $g(B) = (Bu, u)$, $u = \binom{\alpha}{\beta} \neq \binom{0}{0}$, equation (2.3) reduces to

$$-z'' + \Gamma(x)z = 0 \tag{2.5}$$

with

$$\Gamma(x) = (\alpha^2 + \beta^2)^{-1}\{(\alpha^2 - \beta^2)p' - \alpha\beta(V_2' - V_1')$$
$$+ \alpha^2((V_2 - c_2)^2 + p^2 - d^2) + 2\alpha\beta p(V_1 + V_2 + c_1 - c_2) + \beta^2((V_1 + c_1)^2 + p^2 - d^2).$$

For $V_1 = V_2 = V$, we note the following cases of (2.5) by taking (α, β) equal to $(1, 0)$, $(0, 1)$, $(1, \pm 1)$ respectively and also of (2.3) with $g(B) = \text{trace}\, B$.

$$-z'' + \{p' + (V - c_2)^2 + p^2 - d^2\}z = 0 \tag{2.6}$$

$$-z'' + \{-p' + (V + c_1)^2 + p^2 - d^2\}z = 0 \tag{2.7}$$

$$-z'' + (1/2)\{(V - c_2)^2 + (V + c_1)^2 + 2p^2 - 2d^2 \pm 2p(2V + c_1 - c_2)\}z = 0 \tag{2.8}$$

$$-z'' + (1/2)\{(V - c_2)^2 + (V + c_1)^2 + 2p^2 - 2d^2\}z = 0. \tag{2.9}$$

Example 1. Let $V_1(x) = V_2(x) = a/x$, $a \neq 0$, $c_1 = c_2 = d = c$, and $p(x) = k/x + r/x^2$. Then (2.6) is

$$-z'' + \{p'(x) + a^2/x^2 - 2ac/x + p(x)^2\}z = 0$$

which is oscillatory at ∞ if $a > 0$. Similarly (2.7) is oscillatory at ∞ if $a < 0$. Thus $\sigma(L_1) \cap (-c, c)$ is infinite.

For $c_1=c_2=d$, p' and p^2 being $0(x^{-2})$ as $x\to\infty$, and V of constant sign, it follows that (2.6) or (2.7) will be oscillatory at ∞ if $|V(x)|\leqq\varepsilon<2c$ on some $[x_0,\infty)$ and $|V(x)|x^2\to\infty$ as $x\to\infty$. This gives Theorem 3 of [15].

It is possible to relax the differentiability requirements on p, V_1, and V_2. For the functional $g(B)=(B\binom{1}{0},\binom{1}{0})(g(B)=(B\binom{0}{1},\binom{0}{1}))$, equation (2.6) ((2.7)) is obtained without assuming V_1 and V_2 differentiable. This is because the middle terms of (2.2) involving V_1 and V_2 are replaced by zero. For these same two choices of g, the differentiability requirement on p can also be dropped by avoiding an integration by parts. In this case we get in place of (2.6) and (2.7) that if either of

$$-z''-2pz'+\{(V_2-c_2)^2+p^2-d^2\}z=0, \tag{2.10}$$

$$-z''+2pz'+\{(V_1+c_1)^2+p^2-d^2\}z=0, \tag{2.11}$$

is oscillatory at ∞ or 0, then $\sigma(L_1)\cap(-d,d)$ is infinite. For smooth p, these equations are less effective than (2.6) and (2.7) as is shown by the oscillation preserving substitution $w=z\exp\{\int\pm p\}$ for (2.10), (2.11) respectively.

3. Finite gap spectra

The middle two terms of (2.2) may be written

$$\begin{aligned} MID:&=\int_0^\infty[(y')^*JPy-y^*PJy']\,dx \\ &=\int_0^\infty\{p[-\bar{y}_1'y_1+\bar{y}_2'y_2-y_1'\bar{y}_1+y_2'\bar{y}_2]-(V_1+c_1)(\bar{y}_1'y_2+y_1'\bar{y}_2) \\ &\quad+(V_2-c_2)(\bar{y}_2'y_1+y_2'\bar{y}_1)\}\,dx. \end{aligned} \tag{3.1}$$

We need to distinguish two cases: $c_1=\min(c_1,c_2)$ and $c_2=\min(c_1,c_2)$. First suppose $c_1=\min(c_1,c_2)$. Let $c=(c_1+c_2)/2$ and $\varepsilon=c-c_1$ so that $c_2=c+\varepsilon$. Using these values in (3.1) together with

$$\int_0^\infty\{-c[\bar{y}_1'y_2+y_1'\bar{y}_2]-c[\bar{y}_2'y_1+y_2'\bar{y}_1]\}\,dx=\int_0^\infty -c[\bar{y}_1y_2+y_1\bar{y}_2]'\,dx=0$$

yields after integration by parts that

$$MID=\int_0^\infty\{p'[|y_1|^2-|y_2|^2]-V_1[\bar{y}_1'y_2+y_1'\bar{y}_2]+(V_2-2\varepsilon)[\bar{y}_2'y_1+y_2'\bar{y}_1]\}\,dx.$$

Thus if $0<\beta_i\leqq 1$ for $i=1,2$, we have that

$$Q_d(y)=\int_0^\infty\{(y')^*\Delta_1y'+|\beta_1y_1'-\beta_1^{-1}V_1y_2|^2+|\beta_2y_2'-\beta_2^{-1}(V_2-2\varepsilon)y_1|^2+y^*\Delta_2y\}\,dx \tag{3.2}$$

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where

$$\Delta_1=\begin{pmatrix}1-\beta_1^2 & 0\\ 0 & 1-\beta_2^2\end{pmatrix},$$

$$\Delta_2=P^2-d^2I+\begin{pmatrix}p'-\beta_2^{-2}(V_2-2\varepsilon)^2 & 0\\ 0 & -p'-\beta_1^{-2}V_1^2\end{pmatrix}. \tag{3.3}$$

Thus we have

$$Q_d(y)\geqq\int_0^\infty[(y')^*\Delta_1 y_1'+y^*\Delta_2 y]\,dx, \tag{3.4}$$

and we have proved the following by Theorem 2.2.

Theorem 3.1. *Let $d>0$, $0<\beta_i\leqq 1$ for $i=1,2$. Then $\sigma(L_1)\cap(-d,d)$ is finite if one of the following holds.*

(i) *The matrix Δ_2 for $\beta_1=\beta_2=1$ is positive definite in a neighbourhood of 0 and in a neighbourhood of ∞.*

(ii) *$0<\beta_i<1$ for $i=1,2$ and the matrix equation*

$$-\Delta_1 y''+\Delta_2 y=0 \tag{3.5}$$

is nonoscillatory at both 0 and ∞.

In a similar manner, we may prove the following when $c_2=\min(c_1,c_2)$. Set $c=(c_1+c_2)/2$, $\varepsilon=c-c_2$ (so that $c_1=c+\varepsilon$) and

$$\Delta_3=P^2-d^2I+\begin{pmatrix}p'-\beta_2^{-2}V_2^2 & 0\\ 0 & -p'-\beta_1^{-2}(V_1+2\varepsilon)^2\end{pmatrix}. \tag{3.6}$$

Theorem 3.2. *Let $d>0$, $0<\beta_i\leqq 1$ for $i=1,2$. Then $\sigma(L_1)\cap(-d,d)$ is finite if one of the following holds.*

(i) *The matrix Δ_3 for $\beta_1=\beta_2=1$ is positive definite in a neighbourhood of 0 and in a neighbourhood of ∞.*

(ii) *$0<\beta_i<1$ for $i=1,2$ and the matrix equation*

$$-\Delta_1 y''+\Delta_3 y=0 \tag{3.7}$$

is nonoscillatory at both 0 and ∞.

Because of the relation (3.4), the nonoscillation of (3.5) at either 0 or ∞ implies the nonoscillation of M_d at the same endpoint. Similar remarks apply to (3.7).

For the next example recall that the vector equation $-y''+\Gamma(x)y=0$ is nonoscillatory at ∞ if $\int_x^\infty\Gamma(s)ds$ exists and $\|\int_x^\infty\Gamma(s)ds\|_0\leqq 1/4x$ for all x sufficiently large (cf. **[16]**). By

$\| \ \|_0$ we mean the operator norm on matrices where the Euclidean norm is used for vectors. We also use that if $-y''+\Gamma(x)y=0$ is nonoscillatory and $\Gamma_1(x)\geqq\Gamma(x)$, then $-y''+\Gamma_1(x)y=0$ is nonoscillatory.

Example 2. Let $p, V=V_1=V_2$ be bounded on some $(0,\varepsilon)$ (hence (3.5) is nonoscillatory at 0) and for $x\geqq\varepsilon$, let $V(x)=a\sin x/x^\delta$ with $\delta>1$ and p satisfy $\pm p'(x)+p(x)^2\geqq a/x^2$ for some $a<1/4$ and $|\int_x^\infty pV|=o(x^{-1})$ as $x\to\infty$. For $c_1=c_2=c=d$ and $\beta_1=\beta_2=\beta$, Δ_2 of (3.5) satisfies

$$\Delta_2=\begin{pmatrix}(p'+(1-\beta^{-2})V^2-2cV+p^2 & 2pV\\ 2pV & -p'+(1-\beta^{-2})V^2+2cV+p^2\end{pmatrix}$$

$$\geqq\tilde{\Delta}_2:=\begin{pmatrix}(1-\beta^{-2})V^2-2cV+a/x^2 & 2pv\\ 2pV & (1-\beta^{-2})V^2+2cV+a/x^2\end{pmatrix}.$$

Now each of $\int_x^\infty V^2$, $\int_x^\infty V$, $\int_x^\infty pV$ is $o(x^{-1})$ as $x\to\infty$; hence for x sufficiently large, $\|\int_x^\infty\tilde{\Delta}_2(s)\,ds\|_0\leqq a_1/x$ for some $a_1<1/4$. Thus β can be chosen so that (3.5) is non-oscillatory; hence $\sigma(L_1)\cap(-c,c)$ is finite.

For $p(x)=k/x$, similar considerations will yield Theorem 1 of [**15**] except for the case $k=1/2$.

We now consider some discrete spectrum results that do not use the operator L^2 directly.

Theorem 3.3. *Let J be a subinterval of $(0,\infty)$, U be an orthogonal matrix, and d a positive number such that:*

(i) *either $UP+P^*U^*\geqq 2dI$ on J or $UP+P^*U^*\leqq -2dI$ on J.*

(ii) *$\mathrm{Re}\int_0^\infty[y^*UJy']\,dx=0$ for all $y\in C_0(J)$.*

Then for all $y\in C_0(J)$, $\|Ly\|\geqq d\|y\|$.

Proof. For $y\in C_0(J)$, we have

$$\|Ly\|^2\|y\|^2=\|Ly\|^2\|Uy\|^2\geqq|(Ly,U^*y)|^2$$

$$=\left|\int_0^\infty[y^*U(Jy'-Py)]\,dx\right|^2$$

$$\geqq\left|\mathrm{Re}\int_0^\infty[y^*U(Jy'-Py)]\,dx\right|^2$$

$$=\left|(1/2)\int_0^\infty y^*(UP+P^*U^*)y\,dx\right|^2$$

$$\geqq\left|d\int_0^\infty y^*y\,dx\right|^2=d^2\|y\|^4.$$

Corollary 3.1. *Suppose the conditions of Theorem 3.3 hold for $J=[b_2,\infty)$ $(J=(0,b_1])$. Then L^2-d^2I is nonoscillatory at $\infty(0)$. If the conditions hold for $J=(0,\infty)$, then $\sigma(L_1)\cap(-d,d)=\emptyset$ if $L_0=L$ and is either ϕ or degenerate otherwise.*

Proof. For $y\in C_0([b_2,\infty))$ with $y\in D(L^2)$,

$$\|L^2y\|\,\|y\|\geqq|(L^2y,y)|=\|Ly\|^2\geqq d^2\|y\|^2$$

or $\|L^2y\|\geqq d^2\|y\|$. Thus the minimal operator associated with L^2 on $[b_2,\infty)$ is bounded below by d^2. Thus a self-adjoint operator associated with L^2 on $[b_2,\infty)$ has a finite spectrum on $(-\infty,d^2)$ since it is a finite dimensional extension of the minimal operator. Thus L^2-d^2I is nonoscillatory at ∞. Similar remarks apply to 0. Note that L_1 is a 0-dimensional (1-dimensional) extension of L_0 if $L_0=L$ $(L_0\neq L)$ so that $\sigma(L_1)\cap(-d,d)=\emptyset$ (contains at most one element).

Two special cases of Theorem 3.3 are of interest.

(A) Suppose $U=\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$. Then a calculation shows (ii) of Theorem 3.3 holds. Also

$$UP+P^*U=2\begin{pmatrix}c_2-V_2 & 0\\ 0 & c_1+V_1\end{pmatrix}. \tag{3.8}$$

This yields the following corollaries.

Corollary 3.2. *If V_1 and V_2 in (1.1) have compact support, then $\sigma(L_1)\cap(-c_1,c_2)$ is finite for all p.*

Corollary 3.3. *If $|V_i(x)|\to\infty$ as $x\to 0$ and as $x\to\infty$ for $i=1,2$ and $V_1(x)V_2(x)<0$ in both a neighbourhood of 0 and of infinity, then $\sigma(L_1)\cap(-d,d)$ is finite for all $d>0$, i.e., L_1 has a purely discrete spectrum.*

(B) Suppose $U=\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. Again (ii) of Theorem 3.3 holds and

$$(1/2)(UP+P^*U^*)=\left(\begin{smallmatrix}p & v\\ v & p\end{smallmatrix}\right),\quad v:=(1/2)(V_1+V_2+c_1-c_2). \tag{3.9}$$

The eigenvalues of (3.9) are $p\pm v$. This gives

Corollary 3.4. *Suppose $d>0$ and in a neighbourhood of $0(\infty)$, $p\pm v\geqq d$ or $p\pm v\leqq -d$. Then L^2-d^2I is nonoscillatory at $0(\infty)$.*

4. The shifting method

Consider (1.1) with $c_1=c_2=c$ say,

$$S(y)=Jy'-\begin{pmatrix}V_2-c & p\\ p & V_1+c\end{pmatrix}y, 0<x<\infty, \tag{4.1}$$

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and let S_1 be a self-adjoint operator determined by S. We assume the essential spectrum of S_1 is $(-\infty, -c]\cup[c,\infty)$ and that S_1 has infinitely many eigenvalues in the gap $[-c,c]$. We wish to determine which of the points $\pm c$ is a cluster point of the eigenvalues. For $0<\varepsilon<c$, set $T_1=S_1+\varepsilon I$, $T_2=S_1-\varepsilon I$. Then spectrum of T_1 (T_2) is the spectrum of S_1 moved to the right (left) ε units. Note that T_1 becomes L_1 of Section 2 with $c_1=c-\varepsilon$, $c_2=c+\varepsilon$ and T_2 becomes L_1 with $c_1=c+\varepsilon$, $c_2=c-\varepsilon$.

Consider now T_1. By Section 2, $-c$ is not a cluster point of the eigenvalues of S_1 if, and only if, $L^2-(c-\varepsilon)^2I$ with $c_1=c-\varepsilon$, $c_2=c+\varepsilon$ is nonoscillatory at both 0 and ∞. Thus to show $-c$ is a cluster point of eigenvalues, it suffices to show $L^2-(c-\varepsilon)^2I$ is oscillatory at one of $0,\infty$. Similar considerations apply to T_2 and c.

The following theorem utilizes these ideas.

Theorem 4.1. *Suppose in* (4.1) *that* $V_1=V_2=V$, p *is locally absolutely continuous, and*

(i) $p(x)=0(1/x), p'(x)=0(1/x^2)$ *as* $x\to\infty$,

(ii) $V(x)\to 0$ *and* $x^2V(x)\to-\infty$ *as* $x\to\infty$.

(iii) *In a neighbourhood of* 0, $p(x)\geqq|V(x)|+c$ *or* $p(x)\leqq -c-|V(x)|$. *Then* $-c$ *is a cluster point of eigenvalues and* c *is not.*

Proof. It follows from the above remarks that $-c$ is a cluster point of eigenvalues if (2.7), with $c_1=c-\varepsilon=d$, $c_2=c+\varepsilon$ $(0<\varepsilon<c)$ is oscillatory at infinity, i.e., if

$$-z''+\{p'+V^2+2(c-\varepsilon)V+p^2\}z=0 \tag{4.2}$$

is oscillatory. Now $V^2+p'+p^2=V^2+0(x^{-2})$ is small compared to $|V|$ by (i), (ii). Thus (4.2) is oscillatory at infinity by comparison with an oscillatory Euler equation, e.g., $-z''-(1/x^2)z=0$.

To see that c is not a cluster point of eigenvalues, we need to show L^2-d^2I is nonoscillatory at both 0 and infinity with $d=c_2=c-\varepsilon$, $c_1=c+\varepsilon$. To see that L^2-d^2I is nonoscillatory at 0 we apply Corollary 3.4. Then $p\pm v=p\pm(V+\varepsilon)$. Hence by (iii) either $p\pm v\geqq d$ or $p\pm v\leqq -d$; thus L^2-d^2I is nonoscillatory at 0. To show L^2-d^2I is nonoscillatory at infinity we use Corollary 3.1 with U as in (3.8); hence L^2-d^2I is nonoscillatory at infinity if

$$\begin{pmatrix} c_2-V & 0 \\ 0 & c_1+V\end{pmatrix}=\begin{pmatrix} c-\varepsilon-V & 0 \\ 0 & c+\varepsilon+V\end{pmatrix}\geqq(c-\varepsilon)I \tag{4.3}$$

near infinity. By (ii), $-2\varepsilon\leqq V(x)<0$ for x sufficiently large thus implying (4.3).

If conditions (i)–(iii) of Theorem 4.1 hold with $x^2V(x)\to-\infty$ as $x\to\infty$ replaced by $x^2V(x)\to\infty$ as $x\to\infty$, then c is a cluster point of eigenvalues and $-c$ is not.

Theorem 4.1 applies to Example 1 if either $r\neq 0$ or $|k|>|a|$.

In our final example we show both $\pm c$ may be cluster points.

Example 3. Suppose on $[1,\infty)$, $V_1(x)=V_2(x)=(a\sin x)/x^\delta$, $0<\delta<1$, $a\neq 0$ with p', $p=0(x^{-2})$ as $x\to\infty$. Suppose also $c>1/2$. Then c will be a cluster point of eigenvalues

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if for some $\varepsilon, 0<\varepsilon<c$, (2.6) is oscillatory at ∞ with $c_1=c+\varepsilon$, $c_2=d=c-\varepsilon$, i.e., if

$$-z''+(p'(x)+a^2x^{-2\delta}\sin^2 x-2a(c-\varepsilon)x^{-\delta}\sin x+p(x)^2)z=0 \tag{4.4}$$

is oscillatory at ∞. We analyse (4.4) by means of a result of Read [**19**] for partial differential equations which in the one-dimensional case applied to $-z''+q(x)z=0$, $b\leqq x<\infty$, is:

Theorem [19]. *Let h be a positive, locally absolutely continuous function on $[b,\infty)$ and let ε_0 satisfy $0<\varepsilon_0<1$. For $\lambda\geqq 0$ define*

$$E(\lambda)=\{x\geqq b: \int_b^x[-qh-(h')^2/4\varepsilon_0 h]\,dt\geqq\lambda\}. \tag{4.5}$$

If there exists an $\alpha>0$ and a sequence $\lambda_k\to\infty$ such that

$$(1-\varepsilon_0)(\lambda_k+\alpha)\int_{E(\lambda_k)} h^{-1}\geqq 1 \tag{4.6}$$

for each k, then $-z''+q(x)z=0$ is oscillatory at ∞.

To apply this theorem to (4.4), we take $h(x)=x^\delta+k\sin x$ where $k=4(c-\varepsilon)a\varepsilon_0^2$, $\varepsilon_0=1-\varepsilon$, and $\varepsilon<1$ is chosen so that $c>\varepsilon+1/2(1-\varepsilon)$. Take $b\geqq 1$ so that $h(x)\geqq x^\delta\varepsilon_0$ for $x\geqq b$. Then the terms of the integral in (4.5) which are not $0(1)$ as $x\to\infty$ are:

$$\int_b^x -a^2t^{-\delta}\sin^2 t\,dt=-a^2x^{1-\delta}/2(1-\delta)+0(1),$$

$$\int_b^x 2a(c-\varepsilon)kt^{-\delta}\sin^2 t\,dt=a(c-\varepsilon)kx^{1-\delta}/(1-\delta)+0(1),$$

$$\int_b^x -h'(t)^2/4\varepsilon_0 h(t)\,dt\geqq -\int_b^x(4\varepsilon_0^2t^\delta)^{-1}h'(t)^2\,dt=-k^2x^{1-\delta}/8\varepsilon_0^2(1-\delta)+0(1).$$

Since by choice of ε,

$$a(c-\varepsilon)k-a^2/2-k^2/8\varepsilon_0^2=a^2[2(c-\varepsilon)^2\varepsilon_0^2-1/2]>0,$$

we see that for each λ, $E(\lambda)$ contains an interval $[a_\lambda,\infty)$. Thus (4.6) holds for any $\alpha>0$ and sequence $\lambda_k\to\infty$. Therefore c is a cluster point. Similar arguments apply to give $-c$ a cluster point. It is unclear if $\pm c$ remain cluster points of eigenvalues if $c<1/2$.

In the above examples we have that infinitely many eigenvalues are in the gap as a result of a sufficiently large long range potential $V(x)$. For the physically important case $p(x)=k/x$, an examination of (2.5) shows the difficulty of constructing a potential whose singular behavior at zero produces infintely many eigenvalues in the gap.

Finally, we make a comment on the more general operator

$$Ty := \begin{pmatrix} \alpha_1^{-1} & 0 \\ 0 & \alpha_2^{-1} \end{pmatrix} Ly$$

where L is as in (1.1) and α_1, α_2 are positive functions. T acts in the Hilbert space of functions f satisfying $\int_0^\infty [\alpha_1 |f_1|^2 + \alpha_2 |f_2|^2] < \infty$. If we make the transformation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(x) = \begin{pmatrix} \eta(x) & 0 \\ 0 & \eta(x)^{-1} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}(t)$$

where $\eta(x) = [\alpha_2(x)/\alpha_1(x)]^{1/4}$ and $t = \int \sqrt{\alpha_1 \alpha_2}\, dx$, then a calculation shows $(\cdot = d/dt)$ $Ty = \lambda y$ reduces to

$$J\left\{\dot{z} - \frac{1}{\sqrt{\alpha_1 \alpha_2}} \begin{pmatrix} p + \eta'/\eta & (V_1 + c)\eta \\ (c_2 - V_2)\eta^{-1} & -p - \eta'/\eta \end{pmatrix} z \right\} = \lambda z.$$

Since $\int (\alpha_1 |y_1|^2 + \alpha_2 |y_2|^2)\, dx = \int (|z_1|^2 + |z_2|^2)\, dt$, an eigenvalue problem $Ty = \lambda y$ can be reduced to the type studied here.

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