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A spectral approach to quenched linear and higher-order response for partially hyperbolic dynamics

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Abstract. For smooth random dynamical systems we consider the quenched linear and higher-order response of equivariant physical measures to perturbations of the random dynamics. We show that the spectral perturbation theory of Gouëzel, Keller and Liverani [**28**, **33**], which has been applied to deterministic systems with great success, may be adapted to study random systems that possess good mixing properties. As a consequence, we obtain general linear and higher-order response results, as well as the differentiability of the variance in quenched central limit theorems (CLTs), for random dynamical systems (RDSs) that we then apply to random Anosov diffeomorphisms and random U(1) extensions of expanding maps. We emphasize that our results apply to random dynamical systems over a general ergodic base map, and are obtained without resorting to infinite-dimensional multiplicative ergodic theory.

Key words: linear response, random dynamical systems, partially hyperbolic dynamics
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1. *Introduction*

In this paper, we study quenched response theory for random dynamical systems (RDSs). The set-up is as follows. Take M to be a $\mathcal{C}^\infty$ Riemannian manifold with m being the measure induced by the associated volume form, take $(\Omega, \mathcal{F}, \mathbb{P})$ to be a Lebesgue space and, for some $r \geq 1$ and each $\epsilon \in (-1, 1)$, let $\mathcal{T}_\epsilon : \Omega \to \mathcal{C}^{r+1}(M, M)$ denote a one-parameter family of random maps with a 'measurable' dependence on ω. After fixing an invertible, $\mathbb{P}$-ergodic map $\sigma : \Omega \to \Omega$ from each $\mathcal{T}_\epsilon$, we obtain RDSs $(\mathcal{T}_\epsilon, \sigma)$ whose trajectories are random variables of the form

$$x, \mathcal{T}_{\epsilon,\omega}(x), \mathcal{T}^{(2)}_{\epsilon,\omega}(x), \ldots, \mathcal{T}^{(n)}_{\epsilon,\omega}(x), \ldots,$$

where $\mathcal{T}^{(n)}_{\epsilon,\omega}$ is short for the composition $\mathcal{T}_{\epsilon,\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{T}_{\epsilon,\omega}$. A family of probability measures $\{\mu_{\epsilon,\omega}\}_{\omega\in\Omega}$ on M is said to be *equivariant* for $(\mathcal{T}_\epsilon, \sigma)$ if $\mu_{\epsilon,\omega} \circ \mathcal{T}^{-1}_{\epsilon,\omega} = \mu_{\epsilon,\sigma\omega}$ for $\mathbb{P}$-almost every ω (see §2.1 for a precise definition). When $\mathcal{T}_\epsilon$ possesses some (partial) hyperbolicity and good mixing properties, one hopes to find a unique *physical* equivariant family of probability measures (a $\mathcal{T}$-equivariant family of measure $\{\mu_\omega\}_{\omega\in\Omega}$ is physical with respect to m if $n^{-1}\sum_{i=0}^{n-1} \delta_{\mathcal{T}_{\sigma^{-i}\omega}(x)} \to \mu_\omega$ for x in a (possibly ω-dependent) positive m-measure set with $\mathbb{P}$-probability 1), as such objects describe the m-almost every realized statistical behavior of the given RDS. Quenched response theory is concerned with questions of the regularity of the map $\epsilon \mapsto \{\mu_{\epsilon,\omega}\}_{\omega\in\Omega}$ and, in particular, how this regularity is inherited from that of $\epsilon \mapsto \mathcal{T}_\epsilon$. The one-parameter family of random maps $\epsilon \mapsto \mathcal{T}_\epsilon$ is said to exhibit quenched linear response if the measures $\{\mu_{\epsilon,\omega}\}_{\omega\in\Omega}$ vary differentiably with ϵ in an appropriate topology, with quenched higher-order (e.g. quadratic) responses being defined analogously.

Linear and higher-order response theory for deterministic (that is, non-random) systems is an established area of research, and there is a plethora of methods available for treating various systems (see [6] for a good review). Response theory has been developed for expanding maps in one and many dimensions [6, 7, 43], intermittent systems [2, 10, 34], Anosov diffeomorphisms [28, 41, 42], partially hyperbolic systems [16] and piecewise expanding interval maps [5, 9]. The tools and techniques one may apply to deduce response results are likewise numerous: there are arguments based on structural stability [41], standard pairs [16], the implicit function theorem [43] and on the spectral perturbation theory of Gouëzel, Keller and Liverani [28, 33] (and variants thereof, e.g., [24]).

On the other hand, the literature on quenched response theory for RDSs is relatively small and has only recently become an active research topic. With a few notable exceptions, most results for random systems have focused on the continuity of the equivariant random measure [3, 8, 22, 26, 37], although some more generally apply to the continuity of the Oseledets splitting and Lyapunov exponents associated to the Perron–Frobenius operator cocycle of the RDS [11, 14]. Quenched linear and higher-order response results are, to the best of our knowledge, limited to [44], where quenched linear and higher-order response is proved for general RDSs of $\mathcal{C}^k$ uniformly expanding maps, and to [20], wherein quenched linear response is proved for RDSs of Anosov maps near a fixed Anosov map. The relatively fewer results for response theory in the random case has been largely attributed to the difficulty in finding appropriate generalizations of the tools, techniques and constructions that have succeeded in the deterministic case. While the authors believe this sentiment is generally well founded, in this paper, we find that, for quenched linear and higher-order response problems, it is possible to directly generalize an approach from the deterministic case to the random case with surprisingly little trouble. In particular, by building on [37], we show that the application of Gouëzel–Keller–Liverani (GKL) spectral perturbation theory to response problems can be 'lifted' to the random case, which allows one to deduce corresponding quenched response from deterministic response 'for free'.

In the deterministic setting, the application of GKL perturbation theory to response problems is part of the more general 'functional analytic' approach to studying dynamical systems, which recasts the investigation of invariant measures and statistical properties of dynamical systems in functional analytic and operator theoretic terms. The key tool of this approach is the Perron–Frobenius operator, which, for a non-singular map $T \in \mathcal{C}^{r+1}(M, M)$ (a map $T : M \to M$ is *non-singular* with respect to m if $m(A) = 0$ implies that $m(T^{-1}(A)) = 0$), is denoted by $\mathcal{L}_T$ and defined for $f \in L^1(m)$ by

$$(\mathcal{L}_T f)(x) = \sum_{T(y)=x} \frac{f(y)}{|\det D_y T|}.$$

The key observation is that the statistical properties of T are often encoded in the spectral data of $\mathcal{L}_T$ provided that one considers the operator on an appropriate Banach space [4, 7, 23, 36]. Specifically, one desires a Banach space for which $\mathcal{L}_T$ is bounded and has a spectral gap (in addition to some other benign conditions), since then a unique physical invariant measure μ_T for T is often obtained as a fixed point of $\mathcal{L}_T$. One may then attempt to answer response theory questions by studying the regularity of the map $T \mapsto \mathcal{L}_T$ with a view towards deducing the regularity of $T \mapsto \mu_T$ via some spectral argument. The main obstruction to carrying out such a strategy is that $T \mapsto \mathcal{L}_T$ is usually not continuous in the relevant operator norm, and so standard spectral perturbation theory (e.g. Kato [32]) cannot be applied. Instead, however, one often has that $T \mapsto \mathcal{L}_T$ is continuous (or $\mathcal{C}^k$) in some weaker topology, and by applying GKL spectral perturbation theory it is then possible to deduce regularity results for $T \mapsto \mu_T$.

The main contribution of this paper is to show that the strategy detailed in the previous paragraph may still be applied in the random case to deduce quenched linear and higher-order response results. More precisely, with $\{(\mathcal{T}_\epsilon, \sigma)\}_{\epsilon \in (-1,1)}$ denoting the RDSs from earlier, the main (psuedo) theorem of this paper is the following (see Theorem 3.6 for a precise statement and §4 for our application to RDSs).

THEOREM A. *Suppose that $(\mathcal{T}_0, \sigma)$ exhibits ω-uniform exponential mixing on M and that, for $\mathbb{P}$-almost every ω, the hypotheses of GKL perturbation theory are 'uniformly' satisfied for the one-parameter families $\epsilon \mapsto \mathcal{T}_{\epsilon,\omega}$, as in the deterministic case. Then, whichever linear and higher-order response results that hold $\mathbb{P}$-almost every at $\epsilon = 0$ for the physical invariant probability measures of the one-parameter families $\epsilon \mapsto \mathcal{T}_{\epsilon,\omega}$ also hold in the quenched sense for the equivariant physical probability measures of the one-parameter family $\epsilon \mapsto \{(\mathcal{T}_\epsilon, \sigma)\}_{\epsilon \in (-1,1)}$ of RDSs.*

We note that, despite the mixing requirement placed on $(\mathcal{T}_0, \sigma)$ in Theorem A, we do not require that σ exhibit any mixing behaviour, other than being ergodic. The general strategy behind the proof of Theorem A is to consider for each $\epsilon \in (-1, 1)$ a 'lifted' operator obtained from the Perron–Frobenius operators $\{\mathcal{L}_{\mathcal{T}_{\epsilon,\omega}}\}_{\omega\in\Omega}$ associated to $\{\mathcal{T}_{\epsilon,\omega}\}_{\omega\in\Omega}$. Then, using the fact that the hypotheses of the GKL theorem (theorem 2.1) are satisfied 'uniformly' for the Perron–Frobenius operators $\epsilon \mapsto \mathcal{L}_{\mathcal{T}_{\epsilon,\omega}}$ and ω in some $\mathbb{P}$-full set, we deduce that the GKL theorem may be applied to the lifted operator. By construction, the fixed points of these lifted operators are exactly the equivariant physical probability measures of the corresponding RDS, and so we obtain the claimed linear and higher-order response via the conclusion of the GKL theorem. Using Theorem A, we easily obtain new quenched linear and higher-order response results for random Anosov maps

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(Theorem 4.8) and for random U(1) extensions of expanding maps (Theorem 4.10). We note that our examples consist of random maps that are uniformly close to a fixed system. However, this is not a strict requirement for the application of our theory and one could also consider 'non-local' examples, e.g., it is clear that the arguments in §4 are applicable to random systems consisting of arbitrary $\mathcal{C}^k$ expanding maps.

The structure of the paper is as follows. In §2, we introduce conventions that are used throughout the paper and review preliminary material related to RDSs and the GKL theorem. In §3, we consider random operator cocycles and their 'lifts' and then prove our main abstract result, Theorem 3.6, which is a version of the GKL theorem for the 'lifts' of certain operator cocycles. In §4, we discuss how Theorem 3.6 may be applied to study the quenched linear and higher-order response of general random $\mathcal{C}^{r+1}$ dynamical systems and then consider in detail the cases of random Anosov maps and random U(1) extensions of expanding maps.

In §5, as another application of Theorem 3.6, we show the differentiability of the variances in quenched CLTs for certain class of RDSs (including random Anosov maps and random U(1) extensions of expanding maps).

Lastly, Appendix A contains the proof of a technical lemma from §4.

2. *Preliminaries*

We adopt the following notational conventions.

(1) The symbol 'C' will, unless otherwise stated, be indiscriminately used to refer to many constants, which are uniform (or almost surely uniform) and whose value may change between usages. If we wish to emphasize that C depends on parameters $a_1, \dots, a_n$, then we may write $C_{a_1,\dots,a_n}$ instead.

(2) If X and Y are topological vector spaces such that X is continuously included into Y, then we will write $X \hookrightarrow Y$.

(3) If X and Y are Banach spaces, then we denote the set of bounded, linear operators from X to Y by $L(X, Y)$. When $X = Y$, we simply write $L(X)$.

(4) When X is a metric space, we denote the Borel σ-algebra on X by $\mathcal{B}_X$.

(5) If $A \in L(X)$, then we denote the spectrum of A by $\sigma(A)$ and the spectral radius by $\rho(A)$. We will frequently consider operators acting on a number of spaces simultaneously and, in such a situation, we may denote $\sigma(A)$ and $\rho(A)$ by $\sigma(A|X)$ and $\rho(A|X)$, respectively, for clarity.

2.1. *RDSs.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\sigma : \Omega \to \Omega$ be a measurably invertible, measure-preserving map. For a measurable space $(\Sigma, \mathcal{G})$, we say that a measurable map $\Phi : \mathbb{N}_0 \times \Omega \times \Sigma \to \Sigma$ is an *RDS* on Σ over the driving system σ if

$$\varphi_\omega^{(0)} = \mathrm{id}_\Sigma \quad \text{and} \quad \varphi_\omega^{(n+m)} = \varphi_{\sigma^m\omega}^{(n)} \circ \varphi_\omega^{(m)}$$

for each $n, m \in \mathbb{N}_0$ and $\omega \in \Omega$, with the notation $\varphi_\omega^{(n)} = \Phi(n, \omega, \cdot)$ and $\sigma\omega = \sigma(\omega)$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. A standard reference for RDSs is the monograph by Arnold [1]. It is easy to check that

$$\varphi_\omega^{(n)} = \varphi_{\sigma^{n-1}\omega} \circ \varphi_{\sigma^{n-2}\omega} \circ \cdots \circ \varphi_\omega \tag{1}$$

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with the notation $\varphi_\omega = \Phi(1, \omega, \cdot)$. Conversely, for each measurable map $\varphi : \Omega \times \Sigma \to \Sigma : (\omega, x) \mapsto \varphi_\omega(x)$, the measurable map $(n, \omega, x) \mapsto \varphi_\omega^{(n)}(x)$ given by (1) is an RDS. We call it an *RDS induced by φ over σ* and simply denote it by (φ, σ).

It is easy to see that if we define a skew-product map $\Theta : \Omega \times \Sigma \to \Omega \times \Sigma$ by $\Theta(\omega, x) = (\sigma\omega, \varphi_\omega(x))$ for each $(\omega, x) \in \Omega \times \Sigma$, then

$$\Theta^n(\omega, x) = (\sigma^n \omega, \varphi_\omega^{(n)}(x)) \quad \text{for all } n \in \mathbb{N}_0.$$

Rather than work with a single Θ-invariant measure on the product space $\Omega \times \Sigma$, we prefer to work with a family of *equivariant measures* supported on Σ fibers, the definition and existence of which we now recall from [1, §1.4]. A measure μ on $(\Omega \times \Sigma, \mathcal{F} \times \mathcal{G})$ is said to have marginal $\mathbb{P}$ on $(\Omega, \mathcal{F})$ if $\mu \circ \pi_\Omega^{-1} = \mathbb{P}$, where $\pi_\Omega : \Omega \times \Sigma \to \Omega$ is the projection onto the first coordinate. A probability measure μ on $(\Omega \times \Sigma, \mathcal{F} \times \mathcal{G})$ is Θ-invariant and has marginal $\mathbb{P}$ on $(\Omega, \mathcal{F})$ if and only if there is a measurable family of probability measures (a family of probability measures $\{\mu_\omega\}_{\omega\in\Omega}$ on $(\Sigma, \mathcal{G})$ is measurable if the map $\omega \mapsto \mu_\omega(A)$ is $(\mathcal{F}, \mathcal{B}_\mathbb{R})$-measurable for each $A \in \mathcal{G}$) $\{\mu_\omega\}_{\omega\in\Omega}$ such that $\mu(A) = \int_\Omega \int_\Sigma 1_A(\omega, x)\mu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$ for each $A \in \mathcal{F} \times \mathcal{G}$ and so that we have

$$\mu_\omega \circ \varphi_\omega^{-1} = \mu_{\sigma\omega} \quad \text{for almost every } \omega \in \Omega. \tag{2}$$

Hence, a measurable family of probability measures $\{\mu_\omega\}_{\omega\in\Omega}$ is said to be equivariant for (φ, σ) if it satisfies (2).

2.2. *The GKL theorem.* We recall the statement of the GKL theorem from [7] (although we note that the result first appeared in full generality in [28, 29] and in less generality in [33]). Fix an integer $N \geq 1$ and let E_j, $j \in \{0, \ldots, N\}$, be Banach spaces with $E_j \hookrightarrow E_{j-1}$ for each $j \in \{1, \ldots, N\}$. For a family of linear operators $\{A_\epsilon\}_{\epsilon\in[-1,1]}$ on these spaces, we consider the following conditions.

(GKL1) For all $i \in \{1, \ldots, N\}$ and $|\epsilon| \leq 1$,

$$\|A_\epsilon\|_{L(E_i)} \leq C.$$

(GKL2) There exists $M > 0$ such that $\|A_\epsilon^n\|_{L(E_0)} \leq CM^n$ for all $|\epsilon| \leq 1$ and $n \in \mathbb{N}$.

(GKL3) There exists $\alpha < M$ such that, for every $|\epsilon| \leq 1$, $f \in E_1$ and $n \in \mathbb{N}$,

$$\|A_\epsilon^n f\|_{E_1} \leq C\alpha^n \|f\|_{E_1} + CM^n \|f\|_{E_0}.$$

(GKL4) For every $|\epsilon| \leq 1$,

$$\|A_\epsilon - A_0\|_{L(E_N, E_{N-1})} \leq C|\epsilon|.$$

If $N \geq 2$, we have the following additional requirement.

(GKL5) There exist linear operators $Q_1, \ldots, Q_{N-1}$ such that, for all $j \in \{1, \ldots, N - 1\}$ and $i \in \{j, \ldots, N\}$, we have $Q_j(E_i) \subseteq E_{i-j}$ and

$$\|Q_j\|_{L(E_i, E_{i-j})} \leq C, \tag{3}$$

and so that, for all $|\epsilon| \leq 1$ and $j \in \{2, \ldots, N\}$,

$$\left\| A_\epsilon - A_0 - \sum_{k=1}^{j-1} \epsilon^k Q_k \right\|_{L(E_N, E_{N-j})} \leq C|\epsilon|^j. \tag{4}$$

THEOREM 2.1. (The GKL theorem, [7, Theorem A.4]) *Fix an integer $N \geq 1$ and let E_j, $j \in \{0, \ldots, N\}$, be Banach spaces with $E_j \hookrightarrow E_{j-1}$ for each $j \in \{1, \ldots, N\}$. Suppose that $\{A_\epsilon\}_{\epsilon\in[-1,1]}$ satisfies (GKL1)–(GKL4) and if $N \geq 2$, then also (GKL5). For $z \notin \sigma(A_0|E_N)$, set $R_0(z) = (z - A_0)^{-1}$ and define*

$$S_\epsilon^{(N)}(z) = R_0(z) + \sum_{k=1}^{N-1} \epsilon^k \sum_{j=1}^{k} \sum_{\substack{l_1+\cdots+l_j=k \\ l_i \geq 1}} R_0(z) Q_{l_1} R_0(z) \cdots R_0(z) Q_{l_j} R_0(z). \tag{5}$$

In addition, for any $a > \alpha$, let

$$\eta = \frac{\log(a/\alpha)}{\log(M/\alpha)},$$

and, for $\delta > 0$, set

$$\mathcal{V}_{\delta,a}(A_0) = \{z \in \mathbb{C} : |z| \geq a \quad \text{and} \quad \operatorname{dist}(z, \sigma(A_0|E_j)) \geq \delta \quad \text{for all } j \in \{1, \ldots, N\}\}.$$

There exists $\epsilon_0 > 0$ so that $\mathcal{V}_{\delta,a}(A_0) \cap \sigma(A_\epsilon|E_1) = \emptyset$ for every $|\epsilon| \leq \epsilon_0$ and so that, for each $z \in \mathcal{V}_{\delta,a}(A_0)$,

$$\|(z - A_\epsilon)^{-1}\|_{L(E_1)} \leq C$$

and

$$\|(z - A_\epsilon)^{-1} - S_\epsilon^{(N)}(z)\|_{L(E_N, E_0)} \leq C|\epsilon|^{N-1+\eta}.$$

Remark 2.2. While the GKL theorem as stated in Theorem 2.1 is true, there is an error in the proof of the result in both [7, 28]. We refer the reader to [29] for details of the error and to the proof of [27, Theorem 3.3] for a corrected argument.

Remark 2.3. We emphasize that the inclusion $E_j \hookrightarrow E_{j-1}$ need not be compact in Theorem 2.1. In applications, one often needs good information on the spectrum of A_0 (such as quasi-compactness of $A_0 : E_1 \to E_1$, which is often shown by (GKL2), (GKL3) and the compactness of $E_1 \hookrightarrow E_0$). However, this freedom is essential in our application in §4 because $L^\infty(\Omega, E) \hookrightarrow L^\infty(\Omega, F)$ is not necessarily compact even if $E \hookrightarrow F$ is compact.

3. *A spectral approach to stability theory for operator cocycles*

Let X be a Banach space and let $\mathcal{S}_{L(X)}$ denote the σ-algebra generated by the strong operator topology on $L(X)$. If $A : \Omega \to L(X)$ is $(\mathcal{F}, \mathcal{S}_{L(X)})$-measurable, then we say that it is *strongly measurable*. For an overview of the properties of strong measurable maps, we refer the reader to [25, Appendix A]. The following lemma records the main properties of strongly measurable maps that we shall use.

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LEMMA 3.1. [25, Lemmas A.5 and A.6] *Suppose that X is a separable Banach space and that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a Lebesgue space. Then:*

(1) *the set of strongly measurable maps is closed under (operator) composition, i.e., if* $A_i : \Omega \to L(X)$, $i \in \{1, 2\}$, *are strongly measurable, then so is* $A_2 A_1 : \Omega \to L(X)$;

(2) *if* $A : \Omega \to L(X)$ *is strongly measurable and* $f : \Omega \to X$ *is* $(\mathcal{F}, \mathcal{B}_X)$*-measurable, then* $\omega \mapsto A_\omega f_\omega$ *is* $(\mathcal{F}, \mathcal{B}_X)$*-measurable too; and*

(3) *if* $A : \Omega \to L(X)$ *is such that* $\omega \mapsto A(\omega) f$ *is* $(\mathcal{F}, \mathcal{B}_X)$*-measurable for every* $f \in X$, *then A is strongly measurable.*

As a slight abuse of notation, for a given strongly measurable map $A : \Omega \to L(X)$, we denote an $(\mathcal{F} \times \mathcal{B}_X, \mathcal{B}_X)$-measurable map $(\omega, f) \mapsto A(\omega) f$ by A. In light of the previous lemma, we may now formally define the main objects of study for this section.

Definition 3.2. An RDS (A, σ) on X induced by a map $A : \Omega \mapsto L(X)$ is called an *operator cocycle* (or a *linear RDS*) if $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, $\sigma : \Omega \to \Omega$ is an invertible, ergodic, $\mathbb{P}$-preserving map, X is a separable Banach space and $A : \Omega \mapsto L(X)$ is strongly measurable. We say that (A, σ) is bounded if $A \in L^\infty(\Omega, L(X))$.

Throughout the rest of this paper, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and that $\sigma : \Omega \to \Omega$ is an invertible, ergodic, $\mathbb{P}$-preserving map. An operator cocycle (A, σ) is explicitly written as a measurable map

$$\mathbb{N}_0 \times \Omega \times X \to X : (n, \omega, f) \mapsto A^{(n)}(\omega) f, \quad A^{(n)}(\omega) := A(\sigma^{n-1}\omega) \circ \cdots \circ A(\omega).$$

We denote by X^* the dual space of X.

Definition 3.3. Let $\xi \in X^*$ be non-zero. We say that $A \in L(X)$ is ξ-Markov if $\xi(Af) = \xi(f)$ for every $f \in X$. We say that an operator cocycle (A, σ) is ξ-Markov if A is almost surely ξ-Markov.

Notice that our terminology in Definition 3.3 is non-standard: in the literature, a linear operator $A : X \to X$ is called Markov if $X = L^1(S, \mu)$ for a probability space (S, μ) and A is positive (i.e., $Af \geq 0$ μ-almost everywhere if $f \geq 0$ μ-almost everywhere) and ξ-Markov with $\xi(f) = \int_S f \, d\mu$ (cf. [35]). See also Definition 4.3 for a more general definition of positivity. We do not add the positivity condition to Definition 3.3 to make clear that the result in this section holds without it.

Definition 3.4. Suppose that (A, σ) is a ξ-Markov operator cocycle for some non-zero $\xi \in X^*$. We say that (A, σ) is ξ-mixing with rate $\rho \in [0, 1)$ if, for every $n \in \mathbb{N}$,

$$\operatorname*{ess\,sup}_{\omega \in \Omega} \| A^{(n)}(\omega)|_{\ker \xi} \| \leq C \rho^n. \tag{6}$$

Fix a non-zero $\xi \in X^*$. We define $\mathcal{X} \equiv \mathcal{X}_\xi$ as

$$\mathcal{X} = \{ f \in L^\infty(\Omega, X) : \xi(f) \text{ is almost surely constant} \}.$$

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Since $\mathcal{X}$ is a closed subspace of $L^{\infty}(\Omega, X)$, it is a Banach space with the usual norm. If (A, σ) is a bounded ξ-Markov operator cocycle, then we define $\mathbb{A} : \mathcal{X} \to \mathcal{X}$ by

$$(\mathbb{A} f)(\omega) = A(\sigma^{-1}(\omega)) f(\sigma^{-1}(\omega)). \tag{7}$$

We say that $\mathbb{A}$ is the *lift* of (A, σ). That $\mathbb{A} \in L(\mathcal{X})$ follows from Lemma 3.1 and the boundedness of (A, σ) (see [37] for a possible extension of the lift to the case when σ is not invertible). The following proposition is a natural generalization of [37, Proposition 2.3].

PROPOSITION 3.5. *Fix non-zero $\xi \in X^*$. If (A, σ) is a bounded, ξ-Markov, ξ-mixing operator cocycle with rate $\rho \in [0, 1)$, then 1 is a simple eigenvalue of $\mathbb{A}$ and $\sigma(\mathbb{A}|\mathcal{X}) \setminus \{1\} \subseteq \{z \in \mathbb{C} : |z| \leq \rho\}$.*

Proof. For each $c \in \mathbb{C}$, let

$$\mathcal{X}_c = \{f \in \mathcal{X} : \xi(f) = c \text{ almost surely}\}. \tag{8}$$

We note that $\mathcal{X}_c$ is non-empty since ξ is assumed to be non-zero. Since (A, σ) is a ξ-Markov operator cocycle, the lift $\mathbb{A}$ preserves $\mathcal{X}_c$. For any $f, g \in \mathcal{X}_c$, one has $f - g \in \mathcal{X}_0$ (i.e., $f - g \in \ker \xi$ almost surely), and so, as (A, σ) is ξ-mixing with rate ρ, we have, for every $n \in \mathbb{N}$ and almost every $\omega \in \Omega$, that

$$\begin{aligned} \|(\mathbb{A}^n f)(\omega) - (\mathbb{A}^n g)(\omega)\| &= \|A^{(n)}(\sigma^{-n}(\omega))(f_{\sigma^{-n}(\omega)} - g_{\sigma^{-n}(\omega)})\| \\ &\leq C\rho^n \|f_{\sigma^{-n}(\omega)} - g_{\sigma^{-n}(\omega)}\|. \end{aligned} \tag{9}$$

Upon taking the essential supremum, we see that $\mathbb{A}^n$ is a contraction mapping on $\mathcal{X}_c$ for large enough n. Since $\mathcal{X}_c$ is complete, it follows that $\mathbb{A}$ has a unique fixed point v_c in $\mathcal{X}_c$. Obviously, $v_c = cv_1$, and thus 1 is an eigenvalue of $\mathbb{A}$ on $\mathcal{X}$. Furthermore, $\mathcal{X} = \operatorname{span}\{v_1\} \oplus \mathcal{X}_0$ (indeed, for every $f \in \mathcal{X}$, we can write $f = f_1 + f_0$, where $f_1 = \xi(f)v_1 \in \operatorname{span}\{v_1\}$ and $f_0 = f - f_1 \in \mathcal{X}_0$, and note that $\operatorname{span}\{v_1\}$ and $\mathcal{X}_0$ are closed subspaces).

Since $\mathbb{A}$ preserves both $\operatorname{span}\{v_1\}$ and $\mathcal{X}_0$,

$$\sigma(\mathbb{A}|\mathcal{X}) = \sigma(\mathbb{A}| \operatorname{span}\{v_1\}) \sqcup \sigma(\mathbb{A}|\mathcal{X}_0),$$

where $\sqcup$ denotes a disjoint union. It is clear that $\sigma(\mathbb{A}| \operatorname{span}\{v_1\})$ consists of only a simple eigenvalue 1, while $\rho(\mathbb{A}|\mathcal{X}_0) \leq \rho$ since (A, σ) is ξ-mixing with rate ρ. Thus, $\sigma(\mathbb{A}|\mathcal{X}) \setminus \{1\} = \sigma(\mathbb{A}|\mathcal{X}_0) \subseteq \{z \in \mathbb{C} : |z| \leq \rho\}$. □

3.1. *Main result.*

Given a bounded, ξ-Markov, ξ-mixing operator cocycle (A, σ), we are interested in the question of stability (and differentiability) of the ξ-normalized fixed point v of $\mathbb{A}$. To this end, we formulate a number of conditions on operator cocycles that are reminiscent of the conditions of the GKL theorem.

Fix an integer $N \geq 1$ and let E_j, $j \in \{0, \ldots, N\}$, be Banach spaces. Let $\{(A_\epsilon, \sigma)\}_{\epsilon \in [-1,1]}$ be a family of operator cocycles on these spaces.

(QR0) (a) The Banach spaces $\{E_j\}_{j \in \{0,\ldots,N\}}$ satisfy $E_j \hookrightarrow E_{j-1}$ for each $j \in \{1, \ldots, N\}$. Moreover, E_N is separable and $\|\cdot\|_{E_j}$-dense in E_j for each $j \in \{0, \ldots, N\}$ (in particular, E_1 is separable).

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(b) There exists a non-zero functional $\xi \in E_0^*$ such that (A_ϵ, σ) is ξ-Markov on E_j for each $|\epsilon| \leq 1$, $j \in \{0, \ldots, N\}$ and so that (A_0, σ) is ξ-mixing on E_j for $j \in \{1, N\}$.

(QR1) For all $i \in \{1, \ldots, N\}$ and $|\epsilon| \leq 1$, we have $\operatorname{ess\,sup}_\omega \|A_\epsilon(\omega)\|_{L(E_i)} \leq C$.

(QR2) There exists $M > 0$ such that $\operatorname{ess\,sup}_\omega \|A_\epsilon^{(n)}(\omega)\|_{L(E_0)} \leq CM^n$ for all $|\epsilon| \leq 1$ and $n \in \mathbb{N}$.

(QR3) There exists $\alpha < M$ such that, for every $f \in E_1$, $|\epsilon| \leq 1$ and $n \in \mathbb{N}$,

$$\operatorname*{ess\,sup}_\omega \|A_\epsilon^{(n)}(\omega) f\|_{E_1} \leq C\alpha^n \|f\|_{E_1} + CM^n \|f\|_{E_0}.$$

(QR4) For every $|\epsilon| \leq 1$,

$$\operatorname*{ess\,sup}_\omega \|A_\epsilon(\omega) - A_0(\omega)\|_{L(E_N, E_{N-1})} \leq C|\epsilon|.$$

If $N \geq 2$, we have the following additional requirement.

(QR5) There exists linear operators $Q_1(\omega), \ldots, Q_{N-1}(\omega)$ for each ω such that, for all $j \in \{1, \ldots, N-1\}$ and $i \in \{j, \ldots, N\}$,

$$\operatorname*{ess\,sup}_\omega \|Q_j(\omega)\|_{L(E_i, E_{i-j})} \leq C$$

and such that, for all $|\epsilon| \leq 1$ and $j \in \{2, \ldots, N\}$,

$$\operatorname*{ess\,sup}_\omega \|A_\epsilon(\omega) - A_0(\omega) - \sum_{k=1}^{j-1} \epsilon^k Q_k(\omega)\|_{L(E_N, E_{N-j})} \leq C|\epsilon|^j.$$

We need not assume that $Q_1, \ldots, Q_{N-1}$ are measurable (recall that the essential supremum of a (not necessarily measurable) complex-valued function f on Ω is the infimum of $\sup_{\omega \in \Omega_0} |f(\omega)|$ over all $\mathbb{P}$-full measure sets Ω_0), which will make our application in §4 simpler.

Our main theorem for this section is the following.

THEOREM 3.6. *Fix an integer $N \geq 1$. Let E_j, $j \in \{0, \ldots, N\}$, be Banach spaces and let $\{(A_\epsilon, \sigma)\}_{\epsilon \in [-1,1]}$ be a family of operator cocycles on these spaces. Suppose that $\{(A_\epsilon, \sigma)\}_{\epsilon \in [-1,1]}$ satisfies* (QR0)–(QR4) *and, if $N \geq 2$, then also (QR5). Then there exists $\epsilon_0 \in (0, 1]$ such that (A_ϵ, σ) is ξ-mixing whenever $|\epsilon| < \epsilon_0$. Moreover, for each $\epsilon \in (-\epsilon_0, \epsilon_0)$, there is a unique $v_\epsilon \in L^\infty(\Omega, E_1)$ such that $A_\epsilon(\omega)v_\epsilon(\omega) = v_\epsilon(\sigma\omega)$ and $\xi(v_\epsilon(\omega)) = 1$ almost everywhere and so that*

$$\sup_{|\epsilon| < \epsilon_0} \operatorname*{ess\,sup}_\omega \|v_\epsilon(\omega)\|_{E_1} < \infty.$$

Lastly, there exists $\{v_0^{(k)}\}_{k=1}^{N-1} \subset L^\infty(\Omega, E_0)$ such that $\xi(v_0^{(k)}) = 0$ almost surely for each k and so that, for every $\eta \in (0, \log(1/\alpha)/\log(M/\alpha))$,

$$v_\epsilon = v_0 + \sum_{k=1}^{N-1} \epsilon^k v_0^{(k)} + O_\eta(\epsilon^{N-1+\eta}), \tag{10}$$

where $O_\eta(\epsilon^{N-1+\eta})$ is to be understood as an essentially bounded term in E_0 that possibly depends on η.

Remark 3.7. One is free to take $E_0 = E_1 = \cdots = E_N$ in Theorem 3.6, in which case the conditions (QR0)–(QR3) collapse into a single bound and (QR4)–(QR5) become standard operator norm inequalities. Hence, in this simple case, one recovers an expected Banach space perturbation result.

Remark 3.8. We note that Theorem 3.6 has been proved before for the cases where $N = 1$ and $N = 2$ in [20, 22], respectively.

Remark 3.9. The claim that 'there exists $\epsilon_0 \in (0, 1]$ such that (A_ϵ, σ) is ξ-mixing whenever $|\epsilon| < \epsilon_0$' is exactly the content of [20, Proposition 6] (as well as being an easy corollary of [14, Proposition 3.11]).

In fact, [20, Proposition 6] and [14, Proposition 3.11] tell us that the claim follows from (b) of (QR0), (QR3) and (QR4) with $N = 1$. Furthermore, upon examining these proofs, it is clear that something slightly stronger is true: in the setting of Theorem 3.6, for every $\kappa \in (\rho, 1)$, there exists $\epsilon_\kappa > 0$ such that, for all $\epsilon \in (-\epsilon_\kappa, \epsilon_\kappa)$,

$$\sup_{n\in\mathbb{N}} \kappa^{-n} \operatorname*{ess\,sup}_{\omega} \|A_\epsilon^n|_{\ker \xi}\|_{L(E_1)} \leq C. \tag{11}$$

3.2. *The proof of Theorem 3.6.* Before detailing the proof of Theorem 3.6, we introduce some basic constructs. For each $j \in \{0, \ldots, N\}$, let

$$\mathcal{E}_j = \{f \in L^\infty(\Omega, E_j) : \xi(f) \text{ is almost surely constant}\}.$$

Since $\xi \in E_j^*$ for each $j \in \{0, \ldots, N\}$ we observe that each $\mathcal{E}_j$ is a closed subspace of $L^\infty(\Omega, E_j)$ and, therefore, is a Banach space. Moreover, we have $\mathcal{E}_j \hookrightarrow \mathcal{E}_{j-1}$ for $j \in \{1, \ldots, N\}$. For each $j \in \{1, \ldots, N\}$, we may consider the lift $\mathbb{A}_{\epsilon,j}$ of the operator cocycle (A_ϵ, σ) on E_j, although we omit the subscript j and just write $\mathbb{A}_\epsilon$, which will be of no consequence.

The beginning of the proof of Theorem 3.6 is straightforward. First, we note that (QR1) implies that (A_0, σ) is bounded on E_j for $j \in \{1, N\}$ and so Proposition 3.5 may be applied to characterize the spectrum of $\mathbb{A}_0$ on $\mathcal{E}_1$ and $\mathcal{E}_N$. Let ρ be the rate of ξ-mixing in (QR0): that is, (A_0, σ) is ξ-mixing on E_j with rate ρ for each $j \in \{1, N\}$. Then it follows from (QR0) and Proposition 3.5 that 1 is a simple eigenvalue of $\mathbb{A}_0$, when considered on either space, and we have

$$\sigma(\mathbb{A}_0|\mathcal{E}_j) \setminus \{1\} \subseteq \{z \in \mathbb{C} : |z| \leq \rho\} \tag{12}$$

for $j \in \{1, N\}$. For each $j \in \{1, \ldots, N\}$, one may use basic functional analysis and the fact that $\mathcal{E}_N \hookrightarrow \mathcal{E}_j \hookrightarrow \mathcal{E}_1$ to deduce that 1 is a simple eigenvalue of $\mathbb{A}_0 : \mathcal{E}_j \to \mathcal{E}_j$ and that (12) holds. As a consequence, we find a ξ-normalized $v_0 \in \mathcal{E}_N$ that is the unique ξ-normalized fixed point of $\mathbb{A}_0 : \mathcal{E}_j \to \mathcal{E}_j$ for each $j \in \{1, \ldots, N\}$.

We now turn to constructing the ξ-normalized fixed points of $\mathbb{A}_\epsilon : \mathcal{E}_1 \to \mathcal{E}_1$. By Remark 3.9, we may find some $\kappa \in (\rho, 1)$ and $\epsilon_0 > 0$ such that (A_ϵ, σ) is ξ-mixing on E_1 with rate κ for every $\epsilon \in (-\epsilon_0, \epsilon_0)$. We note that each (A_ϵ, σ) is bounded on E_1 due to

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(QR1) and so, by Proposition 3.5, we find that 1 is a simple eigenvalue of $\mathbb{A}_\epsilon : \mathcal{E}_1 \to \mathcal{E}_1$ and that $\sigma(\mathbb{A}_\epsilon|\mathcal{E}_1) \setminus \{1\} \subseteq \{z \in \mathbb{C} : |z| \leq \kappa\}$. Thus, $\mathbb{A}_\epsilon : \mathcal{E}_1 \to \mathcal{E}_1$ has a unique ξ-normalized fixed point $v_\epsilon \in \mathcal{E}_1$ for each $\epsilon \in (-\epsilon_0, \epsilon_0)$, by Proposition 3.5. Moreover, by virtue of the uniform bound (11), we may strengthen the conclusion of Proposition 3.5: for all n sufficiently large, the family of maps $\{\mathbb{A}_\epsilon^n\}_{|\epsilon|<\epsilon_0}$ uniformly contract the set $\mathcal{X}_1$ from (8). Hence, we deduce the bound

$$\sup_{|\epsilon|<\epsilon_0} \operatorname*{ess\,sup}_{\omega} \|v_\epsilon(\omega)\|_{E_1} < \infty, \tag{13}$$

as required for Theorem 3.6.

Thus, to complete the proof of Theorem 3.6, it suffices to prove (10). It may be easily seen from the proof of Proposition 3.5 that the eigenprojection $\Pi_\epsilon \in L(\mathcal{E}_1)$ of $\mathbb{A}_\epsilon : \mathcal{E}_1 \to \mathcal{E}_1$ onto the eigenspace for 1 is defined for $f \in \mathcal{E}_1$ and $\epsilon \in (-\epsilon_0, \epsilon_0)$ by

$$\Pi_\epsilon(f) = \xi(f) v_\epsilon.$$

Since each v_ϵ is ξ-normalized, we consequently have

$$v_\epsilon = v_0 + (\Pi_\epsilon - \Pi_0) v_0. \tag{14}$$

If $\delta \in (0, 1-\kappa)$, then $D_\delta = \{z \in \mathbb{C} : |z - 1| = \delta\} \subseteq \mathbb{C} \setminus \sigma(\mathbb{A}_\epsilon|\mathcal{E}_1)$ for every $\epsilon \in (-\epsilon_0, \epsilon_0)$. Thus,

$$\Pi_\epsilon = \int_{D_\delta} (z - \mathbb{A}_\epsilon)^{-1} \, \mathrm{d}z. \tag{15}$$

Applying (15) to (14) yields

$$v_\epsilon = v_0 + \int_{D_\delta} ((z - \mathbb{A}_\epsilon)^{-1} - (z - \mathbb{A}_0)^{-1}) v_0 \, \mathrm{d}z. \tag{16}$$

The idea is to apply the GKL theorem to the lifts $\{\mathbb{A}_\epsilon\}_{\epsilon \in [-1,1]}$ with Banach spaces $\{\mathcal{E}_j\}_{0 \leq j \leq N}$ and then develop a Taylor expansion in (16). The hypothesis that (QR1)–(QR4) hold for $\{(A_\epsilon, \sigma)\}_{\epsilon \in [-1,1]}$ with constants almost surely independent of ω readily implies that the lifts $\{\mathbb{A}_\epsilon\}_{\epsilon \in [-1,1]}$ satisfy (GKL1)–(GKL4) for the spaces $\{\mathcal{E}_j\}_{0 \leq j \leq N}$. Hence, in the case where $N = 1$, we may apply Theorem 2.1 to the lifts $\{\mathbb{A}_\epsilon\}_{\epsilon \in [-1,1]}$.

However, the case where $N \geq 2$ is more delicate because the measurability of Q_j is not required in Theorem 3.6. Thus, we introduce the following functional space instead of $L^\infty(\Omega, E_j)$, where the objects are defined up to almost everywhere equality but we loosen the measurability requirement. For each $j \in \{0, \ldots, N\}$, let $B(\Omega, E_j)$ denote the set of (not necessarily measurable) bounded E_j-valued functions on Ω equipped with the uniform norm $\|\cdot\|_{B(\Omega, E_j)}$ that is defined by

$$\|f\|_{B(\Omega, E_j)} = \sup_{\omega \in \Omega} \|f(\omega)\|_{E_j}, \quad f \in B(\Omega, E_j),$$

and let

$$\mathcal{N}_j = \{f \in B(\Omega, E_j) : f = 0 \text{ almost surely}\}.$$

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Then $\mathcal{N}_j$ is a closed subspace of $B(\Omega, E_j)$ and, thus, we can form a quotient space

$$I^\infty(\Omega, E_j) = B(\Omega, E_j)/\mathcal{N}_j.$$

Since $B(\Omega, E_j)$ is a Banach space, $I^\infty(\Omega, E_j)$ is also a Banach space with respect to the quotient norm

$$\|f\|_{I^\infty(\Omega,E_j)} = \inf_{h\in\mathcal{N}_j} \|g - h\|_{B(\Omega,E_j)}, \quad f \in I^\infty(\Omega, E_j),$$

where g is a representative of f. As for $L^\infty(\Omega, E_j)$, under the identification of each element of $I^\infty(\Omega, E_j)$ with its representative, we have

$$\|f\|_{I^\infty(\Omega,E_j)} = \operatorname{ess\,sup}_{\omega} \|f(\omega)\|_{E_j}.$$

Thus, under the identification, we have $\|f\|_{L^\infty(\Omega,E_j)} = \|f\|_{I^\infty(\Omega,E_j)}$ for each $f \in L^\infty(\Omega, E_j)$. In particular, $L^\infty(\Omega, E_j)$ isometrically injects into $I^\infty(\Omega, E_j)$. Finally, let

$$\widetilde{\mathcal{E}}_j = \{f \in I^\infty(\Omega, E_j) : \xi(f) \text{ is almost surely constant}\}. \tag{17}$$

We simply write $\|f\|_{\widetilde{\mathcal{E}}_j}$ for $\|f\|_{I^\infty(\Omega,E_j)}$ if $f \in \widetilde{\mathcal{E}}_j$. Repeating the previous argument, one can show (GKL1)–(GKL4) for the lifts $\{\mathbb{A}_\epsilon\}_{\epsilon\in[-1,1]}$ with respect to the spaces $\{\widetilde{\mathcal{E}}_j\}_{0\le j\le N}$. We now deduce (GKL5) for the lifted systems.

PROPOSITION 3.10. *Assume the setting of Theorem 3.6 with $N \ge 2$. Then (GKL5) holds with operators $\mathbb{Q}_j$ defined by*

$$(\mathbb{Q}_j f)(\omega) = Q_j(\sigma^{-1}\omega) f_{\sigma^{-1}\omega},$$

where $j \in \{1, \dots, N-1\}$, $i \in \{j, \dots, N\}$ and $f \in \widetilde{\mathcal{E}}_i$.

Proof. It is straightforward to verify the required inequalities in (GKL5) from those in (QR5). Hence, it only remains to show that $\mathbb{Q}_j(\widetilde{\mathcal{E}}_i) \subset \widetilde{\mathcal{E}}_{i-j}$ for each $j \in \{1, \dots, N-1\}$ and $i \in \{j, \dots, N\}$. Let $j \in \{1, \dots, N\}$, $i \in \{j, \dots, N-1\}$ and fix $f \in \widetilde{\mathcal{E}}_i$. That $\mathbb{Q}_j f \in E_{i-j}$ almost everywhere follows immediately from (3), and so, to complete the proof, it is sufficient to show that $\xi(\mathbb{Q}_j f)$ is almost surely constant. Since E_N is $\|\cdot\|_{E_j}$-dense in E_j for almost every ω and each $\eta > 0$, we may find a $g_{\eta,\omega} \in E_N$ such that $\|f_{\sigma^{-1}\omega} - g_{\eta,\omega}\|_{E_j} \le \eta$. By (QR5), we have, for almost every ω, that

$$\begin{aligned} |\xi(\mathbb{Q}_j f)(\omega)| &= |\xi(Q_j(\sigma^{-1}\omega) f_{\sigma^{-1}\omega})| \\ &\le |\xi(Q_j(\sigma^{-1}\omega) g_{\eta,\omega})| + |\xi(Q_j(\sigma^{-1}\omega)(f_{\sigma^{-1}\omega} - g_{\eta,\omega})|. \end{aligned} \tag{18}$$

By (QR5) and as $\xi \in E_0^*$, we have $\lim_{\eta\to 0}|\xi(Q_j(\sigma^{-1}\omega)(f_{\sigma^{-1}\omega} - g_{\eta,\omega}))| = 0$ for almost every ω. On the other hand, since $g_{\eta,\omega} \in E_N$ by (QR5), again we have

$$\begin{aligned} &\xi(Q_j(\sigma^{-1}\omega) g_{\eta,\omega}) \\ &\quad = \lim_{\epsilon\to 0} \xi\bigg(\epsilon^{-j}(A_\epsilon(\sigma^{-1}\omega) - A_0(\sigma^{-1}\omega)) g_{\eta,\omega} - \sum_{k=1}^{j-1} \epsilon^{k-j} Q_k(\sigma^{-1}\omega) g_{\eta,\omega}\bigg). \end{aligned} \tag{19}$$

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Therefore, using the fact that (A_ϵ, σ) is ξ-Markov for every $\epsilon \in [-1, 1]$, we have $\xi(Q_1(\sigma^{-1}\omega)g_{\eta,\omega}) = 0$ almost surely. By a simple induction on (19), we find that $\xi(Q_j(\sigma^{-1}\omega)g_{\eta,\omega}) = 0$ almost surely too. Thus, in (18),

$$|\xi(\mathbb{Q}_j f)(\omega)| \le \limsup_{\eta\to 0}|\xi(Q_j(\sigma^{-1}\omega)g_{\eta,\omega})| + |\xi(Q_j(\sigma^{-1}\omega)(f_{\sigma^{-1}\omega} - g_{\eta,\omega})| = 0, \quad (20)$$

which completes the proof. □

By Proposition 3.10, we have (GKL5) for the lifts $\{\mathbb{A}_\epsilon\}_{\epsilon\in[-1,1]}$ with the spaces $\{\widetilde{\mathcal{E}}_j\}_{0\le j\le N}$ in the setting of Theorem 3.6 whenever $N \ge 2$, and so we can apply Theorem 2.1 in this case. As a consequence, we may now finish the proof of Theorem 3.6. Let $\eta \in (0, \log(1/\alpha)/\log(M/\alpha))$ and fix $a \in (\alpha, 1)$ so that $\eta = \log(a/\alpha)/\log(M/\alpha)$. Recall δ from (15) and notice that we may take δ to be as small as we like. Henceforth, we fix $\delta \in (0, 1 - a)$ and choose some $\delta_0 \in (0, \min\{\delta, 1 - a - \delta\})$. Upon recalling the statement of Theorem 2.1 and our earlier characterization of $\sigma(\mathbb{A}_0|\widetilde{\mathcal{E}}_j)$ for $j \in \{1, \ldots, N\}$ (see the paragraph following (12)),

$$D_\delta \subseteq \{z \in \mathbb{C} : |z| \ge s \text{ and } |z - 1| \ge \delta_0\} \subseteq \mathcal{V}_{\delta_0,a}(\mathbb{A}_0).$$

We now apply Theorem 2.1 to the lifts $\{\mathbb{A}_\epsilon\}_{\epsilon\in[-1,1]}$ with Banach spaces $\widetilde{\mathcal{E}}_j$, $j \in \{0, \ldots, N\}$, to deduce the existence of $\epsilon_\eta \in (0, \epsilon_0)$ such that, for every $\epsilon \in (-\epsilon_\eta, \epsilon_\eta)$, we have $\mathcal{V}_{\delta_0,a}(A_0) \cap \sigma(\mathbb{A}_\epsilon|\widetilde{\mathcal{E}}_1) = \emptyset$ and, for each $z \in \mathcal{V}_{\delta_0,a}(\mathbb{A}_0)$, that

$$\|(z - \mathbb{A}_\epsilon)^{-1} - \mathbb{S}_\epsilon^{(N)}(z)\|_{L(\widetilde{\mathcal{E}}_N,\widetilde{\mathcal{E}}_0)} \le C|\epsilon|^{N-1+\eta}, \quad (21)$$

where $\mathbb{S}_\epsilon^{(N)}(z)$ is defined as in (5). With (21) in hand, we may proceed with obtaining (10) via (16). In particular, for $z \in D_\delta$,

$$((z - \mathbb{A}_\epsilon)^{-1} - (z - \mathbb{A}_0)^{-1})v_0 = \sum_{k=1}^{N-1} \epsilon^k \sum_{m=1}^{k} \sum_{\substack{l_1+\cdots+l_m=k \\ l_i\ge 1}} \bigg(\prod_{i=1}^{m} (z - \mathbb{A}_0)^{-1}\mathbb{Q}_{l_i}\bigg)(z - \mathbb{A}_0)^{-1}v_0$$
$$+ ((z - \mathbb{A}_\epsilon)^{-1} - \mathbb{S}_\epsilon^{(N)})v_0. \quad (22)$$

For each $k \in \{1, \ldots, N - 1\}$, we now define $v_0^{(k)} \in \widetilde{\mathcal{E}}_0$ by

$$v_0^{(k)} = \int_{D_\delta} \sum_{m=1}^{k} \sum_{\substack{l_1+\cdots+l_m=k \\ l_i\ge 1}} \bigg(\prod_{i=1}^{m} (z - \mathbb{A}_0)^{-1}\mathbb{Q}_{l_i}\bigg)(z - \mathbb{A}_0)^{-1}v_0 \, dz.$$

Furthermore, by the proof of Proposition 3.10, we have $\xi(v_0^{(k)}) = 0$ almost everywhere for all k. By integrating (22) over D_δ and recalling (16), we get

$$v_\epsilon = v_0 + \sum_{k=1}^{N-1} \epsilon^k v_0^{(k)} + \int_{D_\delta} ((z - \mathbb{A}_\epsilon)^{-1} - \mathbb{S}_\epsilon^{(N)})v_0 \, dz \quad (23)$$

in $\widetilde{\mathcal{E}}_0$. Moreover, since $D_\delta \subseteq \mathcal{V}_{\delta_0,a}(\mathbb{A}_0)$, it follows from (21) that

$$\Big\| \int_{D_\delta} ((z-\mathbb{A}_\epsilon)^{-1} - \mathbb{S}_\epsilon^{(N)}) v_0 \, dz \Big\|_{\widetilde{\mathcal{E}}_0} \leq \sup_{z\in\mathcal{V}_{\delta_0,s}(\mathbb{A}_0)} \|(z-\mathbb{A}_\epsilon)^{-1} - \mathbb{S}_\epsilon^{(N)}(z)\|_{L(\widetilde{\mathcal{E}}_N,\widetilde{\mathcal{E}}_0)} \|v_0\|_{\widetilde{\mathcal{E}}_N}$$
$$\leq C\|v_0\|_{\widetilde{\mathcal{E}}_N} |\epsilon|^{N-1+\eta}. \tag{24}$$

Finally, we show that $v_0^{(k)}$ lies in $\mathcal{E}_0 \subset L^\infty(\Omega, E_0)$ for each $k = 1, \ldots, N-1$. Recall that $v_\epsilon \in \mathcal{E}_0$ for every $\epsilon \in (-\epsilon_0, \epsilon_0)$. Thus, $\epsilon^{-1}(v_\epsilon - v_0)$ belongs to $\mathcal{E}_0$. Therefore, since $\mathcal{E}_0$ isometrically injects into $\widetilde{\mathcal{E}}_0$ (recall the argument above (17)), it follows from the Taylor expansion (23) and (24) that $\{\epsilon^{-1}(v_\epsilon - v_0)\}_{|\epsilon|<\epsilon_0}$ is a Cauchy sequence in $\mathcal{E}_0$. Denote its limit by v_0' so that $\epsilon^{-1}(v_\epsilon - v_0) - v_0'$ lies in $\mathcal{E}_0$ and $\|\epsilon^{-1}(v_\epsilon - v_0) - v_0'\|_{\mathcal{E}_0} \to 0$ as $\epsilon \to 0$. Then, by using again the fact that $\mathcal{E}_0$ isometrically injects into $\widetilde{\mathcal{E}}_0$, we deduce that v_0' equals the limit of $\{\epsilon^{-1}(v_\epsilon - v_0)\}_{|\epsilon|<\epsilon_0}$ in $\widetilde{\mathcal{E}}_0$. Hence, $v_0' = v_0^{(1)}$ by (23) and (24), which concludes that $v_0^{(1)}$ lies in $\mathcal{E}_0$. By considering $\epsilon^{-k}(v_\epsilon - v_0 - \sum_{j=1}^{k-1} \epsilon^j v_0^{(j)})$ instead of $\epsilon^{-1}(v_\epsilon - v_0)$, we can show via induction that $v_0^{(k)}$ also lies in $\mathcal{E}_0$ for each $k = 2, \ldots, N-1$. This completes the proof of Theorem 3.6 because (23) and (24) hold with $\mathcal{E}_j$ in place of $\widetilde{\mathcal{E}}_j$.

4. *Applications to smooth RDSs*

In this section, we shall apply Theorem 3.6 to smooth RDSs in order to obtain stability and differentiability results for their random equivariant probability measures. In particular, we will treat random Anosov maps and random U(1) extensions of expanding maps. The treatments of these settings have much in common, so we discuss some general, abstract details in earlier sections.

4.1. *Equivariant family of measures.*

Let M be a compact connected $\mathcal{C}^\infty$ Riemannian manifold and let m denote the associated Riemannian probability measure on M. Fix a Lebesgue space $(\Omega, \mathcal{F}, \mathbb{P})$ and an invertible, ergodic, $\mathbb{P}$-preserving map $\sigma : \Omega \to \Omega$. For some $r \geq 1$, let $\mathcal{T} : \Omega \to \mathcal{C}^{r+1}(M, M)$ denote a $(\mathcal{F}, \mathcal{B}_{\mathcal{C}^{r+1}(M,M)})$-measurable map. Recall from §2.1 that the RDS $(\mathcal{T}, \sigma)$ induced by $\mathcal{T}$ over σ is explicitly written as a measurable map

$$\mathbb{N}_0 \times \Omega \times X \ni (n, \omega, x) \mapsto \mathcal{T}_\omega^{(n)}(x), \quad \mathcal{T}_\omega^{(n)} := \mathcal{T}_{\sigma^{n-1}\omega} \circ \cdots \circ \mathcal{T}_\omega,$$

and, since σ is invertible, the equivariance of a measurable family of probability measures $\{\mu_\omega\}_{\omega\in\Omega}$ for $(\mathcal{T}, \sigma)$ is given as

$$\mu_\omega \circ \mathcal{T}_\omega^{-1} = \mu_{\sigma\omega} \quad \text{for almost every } \omega \in \Omega.$$

We aim to study the regularity of the dependence of $\{\mu_\omega\}_{\omega\in\Omega}$ on the map $\mathcal{T}$ as $\mathcal{T}$ is fiber-wise varied in a uniformly $\mathcal{C}^N$ way for some $N \leq r$. To do this, we shall realize equivariant families of probability measures as fixed points of (the lifts of) certain operator cocycles (linear RDSs) and then apply Theorem 3.6. In particular, we shall consider the Perron–Frobenius operator cocycle associated to the RDS $(\mathcal{T}, \sigma)$ on an appropriate Banach space. Recall that the *Perron–Frobenius operator* $\mathcal{L}_T$ associated to a non-singular (recall

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that a measurable map $T : M \to M$ is said to be non-singular (with respect to m) if $m(A) = 0$ implies that $m(T^{-1}(A)) = 0$) measurable map $T : M \to M$ is given by

$$\mathcal{L}_T f = \frac{\mathrm{d}[(fm) \circ T]}{\mathrm{d}m} \quad \text{for } f \in L^1(M, m), \tag{25}$$

where fm is a finite signed measure given by $(fm)(A) = \int_A f \, \mathrm{d}m$ for $A \in \mathcal{B}_M$ and $\mathrm{d}\mu/\mathrm{d}m$ is the Radon–Nikodym derivative of an absolutely continuous finite signed measure μ. Note that, for each M-valued random variable ψ whose distribution is fm for some density $f \in L^1(M, m)$, $T(\psi)$ has the distribution $(\mathcal{L}_T f)m$ (and, thus, $\mathcal{L}_T$ is also called the *transfer operator* associated with T). It is routine to verify that

$$\int_M (\mathcal{L}_T f) \cdot g \, \mathrm{d}m = \int_M f \cdot (g \circ T) \, \mathrm{d}m \quad \text{for } f \in L^1(M, m) \quad \text{and} \quad g \in L^\infty(M, m) \tag{26}$$

and that $\mathcal{L}_T$ is an m-Markov operator, where, in an abuse of notation, we let m denote the linear functional $f \in L^1(M, m) \mapsto \int f \, \mathrm{d}m$. In addition, $\mathcal{L}_T$ is positive: if $f \in L^1(M, m)$ satisfies $f \geq 0$ almost everywhere, then $\mathcal{L}_T f \geq 0$ almost everywhere.

Let $\mathcal{N}^{r+1}(M, M)$ denote the set of $T \in \mathcal{C}^{r+1}(M, M)$ satisfying $\det D_x T \neq 0$ for all $x \in M$. Notice that if $T \in \mathcal{N}^{r+1}(M, M)$, then T is automatically non-singular with respect to m and so $\mathcal{L}_T$ is a well-defined operator on $L^1(M, m)$. Additionally, for such T, we have $\mathcal{L}_T \in L(\mathcal{C}^r(M))$ with

$$(\mathcal{L}_T f)(x) = \sum_{T(y)=x} \frac{f(y)}{|\det D_y T|} \quad \text{for all } f \in \mathcal{C}^r(M).$$

Hence, from a measurable map $\mathcal{T} : \Omega \to \mathcal{N}^{r+1}(M, M)$, we obtain a map $\mathcal{L}_{\mathcal{T}} : \omega \mapsto \mathcal{L}_{\mathcal{T}_\omega} : \Omega \to L(\mathcal{C}^r(M))$, which is measurable by virtue of the following proposition (we postpone its proof until Appendix A because it is mundane but technical).

PROPOSITION 4.1. *The map $T \mapsto \mathcal{L}_T$ is continuous on $\mathcal{N}^{r+1}(M, M)$ with respect to the strong operator topology on $L(\mathcal{C}^r(M))$.*

Thus, if we demand that $\mathcal{T} \in \mathcal{N}^{r+1}(M, M)$ almost surely, then $(\mathcal{L}_{\mathcal{T}}, \sigma)$ is an m-Markov operator cocycle on $\mathcal{C}^r(M)$, which we shall call the *Perron–Frobenius operator cocycle* (on $\mathcal{C}^r(M)$) associated to $\mathcal{T}$. In order to apply the theory of §3, we require that the Perron–Frobenius operator cocycle is bounded and m-mixing. This later condition will entail some mixing hypotheses on our random systems. However, as in the deterministic case, in order to realize the mixing of the RDS in operator theoretic terms, we may be forced to consider the Perron–Frobenius operator cocycle on an alternative Banach space. Specifically, we shall seek Banach spaces $(X, \|\cdot\|_X)$ satisfying the following conditions.

(S1) $\mathcal{C}^r(M)$ is dense in X with $\mathcal{C}^r(M) \hookrightarrow X$.

(S2) The embedding $\mathcal{C}^r(M) \hookrightarrow (\mathcal{C}^\infty(M))^*$ given by the map $h \in \mathcal{C}^r(M) \mapsto (g \in \mathcal{C}^\infty(M) \mapsto \int gh \, \mathrm{d}m)$ continuously extends to an embedding $X \hookrightarrow (\mathcal{C}^\infty(M))^*$.

It is clear that any X satisfying (S1) must be separable. Moreover, we note that the functional $\varphi \in (\mathcal{C}^\infty(M))^* \mapsto \varphi(1_M)$ is continuous on $(\mathcal{C}^\infty(M))^*$ and yields m when pulled back via the embedding $\mathcal{C}^r(M) \hookrightarrow (\mathcal{C}^\infty(M))^*$ that is described in (S2). Hence, if

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(S2) holds so that we have an embedding $X \hookrightarrow (\mathcal{C}^\infty(M))^*$ that continuously extends the $\mathcal{C}^r(M) \hookrightarrow (\mathcal{C}^\infty(M))^*$, then m induces a continuous linear functional on X. In particular, we may speak of m-Markov operators in $L(X)$. The following proposition gives a sufficient condition for an m-Markov operator in $L(\mathcal{C}^r(M))$ to be extended to an m-Markov operator in $L(X)$.

PROPOSITION 4.2. *Let (A, σ) be a bounded, m-Markov operator cocycle on $\mathcal{C}^r(M)$ and let X be a Banach space satisfying (S1) and (S2). Suppose that*

$$\operatorname*{ess\,sup}_{\omega} \sup_{\substack{f\in\mathcal{C}^r(M)\\ \|f\|_X=1}} \|A(\omega)f\|_X < \infty. \tag{27}$$

Then A almost surely extends to a unique, bounded operator on X such that $\omega \mapsto A(\omega) : \Omega \to L(X)$ is strongly measurable. Consequently, (A, σ) is a bounded, m-Markov operator cocycle on X such that

$$\operatorname*{ess\,sup}_{\omega} \|A(\omega)\|_{L(X)} < \infty. \tag{28}$$

Proof. It is clear that A almost surely extends to a unique, bounded operator on X and that

$$\operatorname*{ess\,sup}_{\omega} \|A(\omega)\|_{L(X)} = \operatorname*{ess\,sup}_{\omega} \sup_{\substack{f\in\mathcal{C}^r(M)\\ \|f\|_X=1}} \|A(\omega)f\|_X < \infty.$$

That A is almost surely m-Markov in $L(X)$ follows straightforwardly from the fact that A is almost surely m-Markov in $L(\mathcal{C}^r(M))$ and that m uniquely extends to a continuous linear functional on X. Hence, it only remains to show that $\omega \mapsto A(\omega)$ is strongly measurable in $L(X)$. Suppose that $f \in X$. Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}^r(M)$ with limit f in X. For each n, the map $\omega \mapsto A(\omega)f_n$ is $(\mathcal{F}, \mathcal{B}_{\mathcal{C}^r(M)})$-measurable and so it must be $(\mathcal{F}, \mathcal{B}_X)$ measurable too due to (S1). Moreover, for almost every ω,

$$\lim_{n\to\infty} \|A(\omega)f_n - A(\omega)f\|_X = 0,$$

which is to say that $\omega \mapsto A(\omega)f$ is the almost everywhere pointwise limit (in X) of $(\mathcal{F}, \mathcal{B}_X)$-measurable functions. Hence, $\omega \mapsto A(\omega)f$ is $(\mathcal{F}, \mathcal{B}_X)$-measurable since $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space (in particular, complete). That $\omega \mapsto A(\omega)$ is strongly measurable in $L(X)$ then follows from Lemma 3.1 and the fact that $f \in X$ was arbitrary. □

Hence, by Propositions 4.1 and 4.2, if $\mathcal{T} : \Omega \to \mathcal{N}^{r+1}(M, M)$ is measurable and X satisfies (S1) and (S2), then the Perron–Frobenius operator cocycle $(\mathcal{L}_\mathcal{T}, \sigma)$ on $\mathcal{C}^r(M)$ can be extended to a bounded, m-Markov operator cocycle on X. Compare also (28) with (QR1).

The following proposition will help us to describe the relationship between the equivariant family of probability measures for $(\mathcal{T}, \sigma)$ and the fixed point of the lift of a bounded, m-mixing Perron–Frobenius operator cocycle $(\mathcal{L}_\mathcal{T}, \sigma)$.

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Definition 4.3. Assume that X satisfies (S1). $A \in L(X)$ is called *positive* if $A(X_+) \subset X_+$, where X_+ is the completion of $\{f \in \mathcal{C}^r(M) : f \geq 0\}$ in $\|\cdot\|_X$. An operator cocycle (A, σ) is called *positive* if A is almost surely positive. Furthermore, a distribution $f \in (\mathcal{C}^\infty(M))^*$ is called *positive* if $f(g) \geq 0$ for every $g \in \mathcal{C}^\infty(M)$ such that $g \geq 0$.

PROPOSITION 4.4. *Let X be a Banach space satisfying (S1) and (S2) and let (A, σ) be a bounded, m-Markov operator cocycle on X. Suppose that (A, σ) is positive and m-mixing and that h is the unique m-normalized fixed point of the lift $\mathbb{A} : \mathcal{X} \to \mathcal{X}$ on $\mathcal{X} \subset L^\infty(\Omega, X)$ (recall (7) for its definition). Then there exists a measurable family of Radon probability measures $\{\mu_\omega\}_{\omega\in\Omega}$ such that $h(\omega)(g) = \int g \, \mathrm{d}\mu_\omega$ for every $g \in \mathcal{C}^\infty(M)$ and almost every ω.*

Proof. Notice that the set

$$\mathcal{D} = \{f \in L^\infty(\Omega, X) : m(f) = 1 \text{ and } f \in X_+ \text{ almost surely}\}$$

is almost surely invariant under $A(\omega)$ since (A, σ) is bounded, positive and m-Markov. Hence, we may carry out the construction of h in Proposition 3.5 with $\mathcal{D}$ in place of $\mathcal{X}_1$ to conclude that $h \in X_+$ almost surely. Thus, there exists $\{f_k\}_{k\in\mathbb{N}} \subseteq L^\infty(\Omega, \mathcal{C}^r(M))$ such that $f_k(\omega) \geq 0$ and $\int f_k(\omega) \, \mathrm{d}m = 1$ for every k and so that $\lim_{k\to\infty} f_k(\omega) = h(\omega)$ in X for almost every ω. As $X \hookrightarrow (\mathcal{C}^\infty(M))^*$, it follows that $\lim_{k\to\infty} f_k(\omega) = h(\omega)$ in the sense of distributions as well. Thus, for any positive $g \in \mathcal{C}^\infty(M)$,

$$h(\omega)(g) = \lim_{k\to\infty} f_k(\omega)(g) = \lim_{k\to\infty} \int f_k(\omega) \cdot g \, \mathrm{d}m \tag{29}$$

(recall the embedding of $\mathcal{C}^r(M)$ in (S2)). As $f_k(\omega)$ and g are positive, it follows from (29) that $h(\omega)(g) \geq 0$ for every such g. Hence, $h(\omega)$ is a positive distribution for almost every ω. On the other hand, as is well known, for any positive $f \in (\mathcal{C}^\infty(M))^*$, one can find a positive Radon measure μ_f such that $f(g) = \int g \, \mathrm{d}\mu_f$ for every $g \in \mathcal{C}^\infty(M)$. We denote by μ_ω the positive Radon measure corresponding to $h(\omega)$.

To see that μ_ω is a probability measure for almost every ω, we note that, by (29) and as $\int f_k(\omega) \, \mathrm{d}m = 1$ for every k,

$$\mu_\omega(M) = h(\omega)(1_M) = \lim_{k\to\infty} \int f_k(\omega) \, \mathrm{d}m = 1.$$

Finally, $\{\mu_\omega\}_{\omega\in\Omega}$ is a measurable family on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ because for, any $A \in \mathcal{B}_M$, by using (29) again,

$$\mu_\omega(A) = h(\omega)(1_A) = \lim_{k\to\infty} \int_A f_k(\omega) \, \mathrm{d}m$$

for almost every ω, while, for every k, $\omega \mapsto f_k(\omega) : \Omega \to \mathcal{C}^r(M)$ is measurable and $f \mapsto \int_A f \, \mathrm{d}m : \mathcal{C}^r(M) \mapsto \mathbb{C}$ is continuous, so that $\omega \mapsto \int_A f_k(\omega) \, \mathrm{d}m$ is measurable too. □

Hence, if X satisfies (S1) and (S2) and if the Perron–Frobenius operator cocycle $(\mathcal{L}_\mathcal{T}, \sigma)$ on X is m-mixing, then we obtain a measurable family of Radon probability measures

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$\{\mu_\omega\}_{\omega\in\Omega}$ such that $h : \omega \mapsto (g \mapsto \int g \, d\mu_\omega)$ is in $L^\infty(\Omega, X)$. Furthermore, $\{\mu_\omega\}_{\omega\in\Omega}$ is equivariant because it follows from (29) that, for any $A \in \mathcal{B}_M$ and almost every ω,

$$\mu_\omega(T_\omega^{-1}(A)) = h(\omega)\big(1_{T_\omega^{-1}(A)}\big) = \lim_{k\to\infty} \int f_k(\omega) \cdot 1_A \circ T_\omega \, dm.$$

Due to (26), the continuity of $\mathcal{L}_{T_\omega} : X \to X$ and the fact that h is the fixed point of the lift of $(\mathcal{L}_\mathcal{T}, \sigma)$, this coincides with

$$\lim_{k\to\infty} \int \mathcal{L}_{T_\omega} f_k(\omega) \cdot 1_A \, dm$$
$$= \lim_{k\to\infty} \mathcal{L}_{T_\omega} f_k(\omega)(1_A) = \mathcal{L}_{T_\omega} h(\omega)(1_A) = h(\sigma\omega)(1_A) = \mu_{\sigma\omega}(A).$$

4.2. *The conditions (QR4) and (QR5).* In this section, we discuss a sufficient condition for a family of Perron–Frobenius operator cocycles $\{(\mathcal{L}_{\mathcal{T}_\epsilon}, \sigma)\}_{\epsilon\in[-1,1]}$ to satisfy (QR4) and (QR5). We emphasize that these conditions hold rather independently of how the underlying random dynamics $(\mathcal{T}_\epsilon, \sigma)$ behave (see Proposition 4.5 for a precise statement), so we treat (QR4) and (QR5) here as a final preparation before specializing to our applications. For simplicity, throughout this section, we assume that M is a d-dimensional torus $\mathbb{T}^d$. One may straightforwardly remove this assumption by considering a partition of unity. (Refer to, e.g., [7, 28]; see also Appendix A).

Notice that (QR4) and (QR5) are conditions for a single iteration $\mathcal{L}_{T_{\epsilon,\omega}}$ (not for $\mathcal{L}_{T_{\epsilon,\sigma^{n-1}\omega}} \circ \cdots \circ \mathcal{L}_{T_{\epsilon,\omega}}, n \in \mathbb{N}$), and so clear observations may be found in the non-random setting. Fix $r \geq 1$ and $1 \leq s \leq r$, and consider $T \in \mathcal{C}^N([-1, 1], \mathcal{C}^{r+1}(\mathbb{T}^d, \mathbb{T}^d))$. Let $1 \leq N \leq s$ be an integer and let E_j, $j \in \{0, \ldots, N\}$, be Banach spaces with $E_j \hookrightarrow E_{j-1}$ for each $j \in \{1, \ldots, N\}$ satisfying the following conditions.

(P1) The condition (S1) holds with E_j in place of X for each $j \in \{0, \ldots, N\}$.

(P2) The condition (S2) holds with E_j in place of X for each $j \in \{0, \ldots, N\}$.

(P3) There are constants $C > 0$ and $0 \leq \rho \leq r - N$ such that

$$\|uf\|_{E_j} \leq C\|u\|_{\mathcal{C}^{\rho+j}}\|f\|_{E_j} \quad \text{for each } u, f \in \mathcal{C}^r(\mathbb{T}^d) \quad \text{and} \quad j \in \{0, \ldots, N\}.$$

(P4) There is a constant $C > 0$ such that

$$\left\| \frac{\partial}{\partial x_l} f \right\|_{E_{j-1}}$$
$$\leq C\|f\|_{E_j} \quad \text{for each } f \in \mathcal{C}^r(\mathbb{T}^d), l \in \{1, \ldots, d\} \quad \text{and} \quad j \in \{1, \ldots, N\}.$$

Observe that all conditions (P1)–(P4) are not for the operators $\mathcal{L}_{T_\epsilon}$, $\epsilon \in [-1, 1]$, with $T_\epsilon := T(\epsilon)$, but for the spaces E_j, $j \in \{0, \ldots, N\}$, so the following proposition is quite useful in our applications. Note that if

$$\|\mathcal{L}_{T_\epsilon} f\|_{E_j} < C_\epsilon\|f\|_{E_j} \quad \text{for each } f \in \mathcal{C}^r(\mathbb{T}^d), j \in \{0, \ldots, N\} \quad \text{and} \quad |\epsilon| \leq 1, \tag{30}$$

then it follows from Proposition 4.2 that $\mathcal{L}_{T_\epsilon}$ is a bounded operator on E_j for each $j \in \{0, \ldots, N\}$ and $|\epsilon| \leq 1$.

PROPOSITION 4.5. *Let N be a positive integer, let $T \in \mathcal{C}^N([-1,1], \mathcal{C}^{r+1}(\mathbb{T}^d, \mathbb{T}^d))$ and let E_j, $j \in \{0, \ldots, N\}$, be Banach spaces with $E_j \hookrightarrow E_{j-1}$ for each $j \in \{1, \ldots, N\}$ satisfying (P1)–(P4). Suppose that $T_\epsilon \in \mathcal{N}^{r+1}(\mathbb{T}^d, \mathbb{T}^d)$ for each $\epsilon \in [-1,1]$ and that (30) holds. Then $\epsilon \mapsto \mathcal{L}_{T_\epsilon} f$ is in $\mathcal{C}^j([-1,1], E_{i-j})$ for each $j \in \{1, \ldots, N\}$, $i \in \{j, \ldots, N\}$ and $f \in E_i$.*

Before starting the proof of Proposition 4.5, we discuss a consequence of Proposition 4.5 with respect to the conditions (QR4) and (QR5). Let $\{(\mathcal{T}_\epsilon, \sigma)\}_{\epsilon\in[-1,1]}$ be a family of RDSs such that, for almost every ω, the map $\epsilon \mapsto T_{\epsilon,\omega} := \mathcal{T}_\epsilon(\omega)$ is in $\mathcal{C}^N([-1,1], \mathcal{C}^{r+1}(\mathbb{T}^d, \mathbb{T}^d))$. Let E_j, $j \in \{0, \ldots, N\}$, be Banach spaces with $E_j \hookrightarrow E_{j-1}$ for each $j \in \{1, \ldots, N\}$ satisfying (P1)–(P4). We suppose the following.

(P5) $T_{\epsilon,\omega} \in \mathcal{N}^{r+1}(\mathbb{T}^d, \mathbb{T}^d)$ for each $\epsilon \in [-1,1]$ and almost every ω. Furthermore, (27) holds with E_j and $\mathcal{L}_{T_{\epsilon,\omega}}$ in place of X and $A(\omega)$ for every $j \in \{0, \ldots, N\}$ and $|\epsilon| \le 1$.

Then it follows, from Proposition 4.2, that the Perron–Frobenius operator cocyles $(\mathcal{L}_{\mathcal{T}_\epsilon}, \sigma)$, $\epsilon \in [-1,1]$, can be extended to bounded operator cocycles on each E_j and that (QR1) holds for these operator cocycles by virtue of (28).

For each $j \in \{0, \ldots, N\}$, $i \in \{j, \ldots, N\}$ and almost every ω, it follows from Proposition 4.5 that we can define $Q_j(\omega) : E_i \to E_{i-j}$ by

$$Q_j(\omega) f = \frac{1}{j!} \left(\frac{d^j}{d\epsilon^j} \mathcal{L}_{T_{\epsilon,\omega}} f \right)_{\epsilon=0} \quad \text{for } f \in E_i.$$

By the definition, it is straightforward to see that, for all $\epsilon \in [-1,1]$ and $2 \le j \le N$,

$$\operatorname*{ess\,sup}_\omega \|\mathcal{L}_{T_{\epsilon,\omega}} - \mathcal{L}_{T_{0,\omega}}\|_{L(E_N, E_{N-1})} \le C|\epsilon|$$

and

$$\operatorname*{ess\,sup}_\omega \|\mathcal{L}_{T_{\epsilon,\omega}} - \mathcal{L}_{T_{0,\omega}} - \sum_{k=1}^{j-1} \epsilon^k Q_k(\omega)\|_{L(E_N, E_{N-j})} \le C|\epsilon|^j.$$

To summarize the above argument, we conclude the following corollary.

COROLLARY 4.6. *Suppose that (P1)–(P5) hold for the family of Perron–Frobenius operator cocyles $\{(\mathcal{L}_{\mathcal{T}_\epsilon}, \sigma)\}_{\epsilon\in[-1,1]}$ on Banach spaces E_j, $j \in \{0, \ldots, N\}$. Then (QR1), (QR4) and (QR5) hold.*

We now return to the proof of Proposition 4.5.

Proof of Proposition 4.5. Fix $1 \le \sigma \le N$, $1 \le \rho \le r$, $f \in \mathcal{C}^\sigma([-1,1], \mathcal{C}^\rho(\mathbb{T}^d))$, $g \in \mathcal{C}^\rho(\mathbb{T}^d)$ and $1 \le l \le d$ for the time being (notice that this f is different from f in the statement of Proposition 4.5 in the sense that this f depends on $\epsilon \in [-1,1]$). We simply denote $d^a/d\epsilon^a f \in \mathcal{C}^{\sigma-a}([-1,1], \mathcal{C}^\rho(\mathbb{T}^d))$ by $f^{(a)}$ for each integer $a \in [0, \sigma]$. We also simply denote by $\partial_l g$ the partial derivative of g with respect to the lth coordinate and let

$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_d$ for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. Then, for each $\epsilon \in [-1, 1]$ and $x \in \mathbb{T}^d$,

$$\partial_l(f_\epsilon^{(1)})(x) = \partial_\epsilon \partial_l \tilde{f}(\epsilon, x) = \partial_l \partial_\epsilon \tilde{f}(\epsilon, x),$$

where $\tilde{f} : [-1, 1] \times \mathbb{T}^d \to \mathbb{T}^d$ is given by $\tilde{f}(\epsilon, x) = f_\epsilon(x)$. (Since $f^{(1)} \in \mathcal{C}^0([-1, 1], \mathcal{C}^1(\mathbb{T}^d))$, it is straightforward to see that the first equality holds and that $(\epsilon, x) \mapsto \partial_l(f_\epsilon^{(1)})(x)$ is continuous. The second equality also immediately follows from these observations together with the Schwarz–Clairaut theorem on equality of mixed partials.) In particular,

$$f^{(1)} = (\epsilon \mapsto \partial_\epsilon \tilde{f}(\epsilon, \cdot)) \quad \text{in } \mathcal{C}^{\sigma-1}([-1, 1], \mathcal{C}^\rho(\mathbb{T}^d)). \tag{31}$$

Furthermore, it is also straightforward to see that the map $\epsilon \mapsto \partial_l f_\epsilon$ is in $\mathcal{C}^\sigma([-1, 1], \mathcal{C}^{\rho-1}(\mathbb{T}^d))$, which we denote by $\partial_l f$ as a slight abuse of notation, and that

$$(\partial_l f)^{(1)} = \partial_l(f^{(1)}) \quad \text{in } \mathcal{C}^{\sigma-1}([-1, 1], \mathcal{C}^{\rho-1}(\mathbb{T}^d)), \tag{32}$$

$$(\det DT)^{(1)} = \det DT^{(1)} \quad \text{in } \mathcal{C}^{N-1}([-1, 1], \mathcal{C}^{r-1}(\mathbb{T}^d)). \tag{33}$$

Moreover, we denote by $T_{(l)} \in \mathcal{C}^N([-1, 1], \mathcal{C}^{r+1}(\mathbb{T}^d))$ the map $\epsilon \mapsto (x \mapsto T_{(l),\epsilon}(x))$, where $T_{(l),\epsilon}(x)$ is the lth coordinate of $T_\epsilon(x) \in \mathbb{T}^d$ (under the identification of $\mathbb{T}^d$ with $\mathbb{R}^d$). Finally, we define a map $\mathbf{L} f : [-1, 1] \to \mathcal{C}^\rho(\mathbb{T}^d)$ by

$$(\mathbf{L} f)_\epsilon = \mathcal{L}_{T_\epsilon} f_\epsilon \quad \text{for } \epsilon \in [-1, 1],$$

which is well defined by virtue of (30). The following is the key lemma for the proof of Proposition 4.5.

LEMMA 4.7. *For each $f \in \mathcal{C}^\sigma([-1, 1], \mathcal{C}^\rho(\mathbb{T}^d))$ with $\sigma \in [1, N]$ and $\rho \in [1, r]$, $(\mathbf{L} f)^{(1)}$ exists in $\mathcal{C}^{\sigma-1}([-1, 1], \mathcal{C}^{\rho-1}(\mathbb{T}^d))$ and is of the form*

$$(\mathbf{L} f)^{(1)} = \mathbf{L}\bigg(\sum_{|\alpha| \le 1} J_{0,\alpha} \cdot \partial^\alpha f + \sum_{|\alpha| \le 1} J_{1,\alpha} \cdot \partial^\alpha f^{(1)} \bigg), \tag{34}$$

where $J_{k,\alpha}$ is in $\mathcal{C}^{N-1}([-1, 1], \mathcal{C}^{r-1}(\mathbb{T}^d))$ is a polynomial function of $\partial^\beta T_{(l)}$, $\partial^\beta T_{(l)}^{(1)}$ ($1 \le l \le d$, $|\beta| \le 2$) and $(\det DT)^{-1}$ for each $k = 0, 1$ and multi-index α with $|\alpha| \le 1$.

Proof. Observe that $\det D_x T_\epsilon > 0$ for all $|\epsilon| \le 1$ and $x \in \mathbb{T}^d$ or $\det D_x T_\epsilon < 0$ for all $|\epsilon| \le 1$ and $x \in \mathbb{T}^d$ because $T_\epsilon \in \mathcal{N}^{r+1}(\mathbb{T}^d, \mathbb{T}^d)$ for each $|\epsilon| \le 1$ and $\epsilon \mapsto T_\epsilon$ is continuous. We consider only the former case because the other case is similar. Also, we show (34) only around $\epsilon = 0$ to keep our notation simple (the general case can be literally treated). First, we note that there is $\epsilon_0 > 0$, $B \in \mathbb{N}$, a finite covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of $\mathbb{T}^d$ and $\mathcal{C}^{r+1}$ maps $(T_\epsilon)_{\lambda,b}^{-1} : U_\lambda \to (T_\epsilon)_{\lambda,b}^{-1}(U_\lambda)$ for $|\epsilon| < \epsilon_0$, $\lambda \in \Lambda$ and $b \in \{1, \ldots, B\}$ such that, for each $|\epsilon| < \epsilon_0$, $\lambda \in \Lambda$, $b \in \{1, \ldots, B\}$ and $g \in \mathcal{C}^r(\mathbb{T}^d)$,

$$T_\epsilon \circ (T_\epsilon)_{\lambda,b}^{-1}(x) = x \quad \text{on } U_\lambda \tag{35}$$

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and

$$\mathcal{M}_1(\epsilon, g)(x) := \sum_{T_\epsilon(y)=x} g(y) = \sum_{b=1}^{B} g \circ (T_\epsilon)^{-1}_{\lambda,b}(x) \quad \text{on } U_\lambda \tag{36}$$

(because $T_0 \in \mathcal{N}^{r+1}(\mathbb{T}^d, \mathbb{T}^d)$ and $\mathbb{T}^d$ is compact; see Appendix A for details). Note also that if we define $\mathcal{M}_2 : [-1, 1] \times \mathcal{C}^r(\mathbb{T}^d) \to \mathcal{C}^r(\mathbb{T}^d)$ by

$$\mathcal{M}_2(\epsilon, g) = \frac{g}{\det DT_\epsilon} \quad \text{for } \epsilon \in [-1, 1],\ g \in \mathcal{C}^r(\mathbb{T}^d),$$

then, for each $f \in \mathcal{C}^\sigma([-1, 1], \mathcal{C}^r(\mathbb{T}^d))$ and $|\epsilon| < \epsilon_0$,

$$(\mathbf{L}f)_\epsilon = \mathcal{M}_1(\epsilon, \mathcal{M}_2(\epsilon, f_\epsilon)).$$

Notice that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are linear with respect to $g \in \mathcal{C}^r(\mathbb{T}^d)$. Hence, it follows from the chain rule for (Fréchet) derivatives that

$$(\mathbf{L}f)^{(1)}_\epsilon = \partial_\epsilon \mathcal{M}_1(\epsilon, \mathcal{M}_2(\epsilon, f_\epsilon)) + \mathcal{M}_1(\epsilon, \partial_\epsilon \mathcal{M}_2(\epsilon, f_\epsilon) + \mathcal{M}_2(\epsilon, f^{(1)}_\epsilon)) \tag{37}$$

if the derivatives exist, where $\partial_\epsilon = \partial/\partial\epsilon$.

Now we calculate $\partial_\epsilon \mathcal{M}_1$ and $\partial_\epsilon \mathcal{M}_2$. First, we show that

$$\partial_\epsilon \mathcal{M}_1(\epsilon, g) = \mathcal{M}_1\biggl(\epsilon, -\sum_{l=1}^{d} \partial_l g \cdot \frac{\sum_{k=1}^{d} (\operatorname{adj}(DT_\epsilon))_{l,k} \cdot T^{(1)}_{(k),\epsilon}}{\det DT_\epsilon}\biggr), \tag{38}$$

where $\operatorname{adj}(A)$ is the adjugate matrix (i.e., the transpose of the cofactor matrix) of a matrix A. By (31) (with $\epsilon \mapsto \mathcal{M}_1(\epsilon, g)$ in place of f), (36) and the chain rule for derivatives,

$$\partial_\epsilon \mathcal{M}_1(\epsilon, g)(x) = \partial_\epsilon \widetilde{\mathcal{M}}_{1,g}(\epsilon, x) = \sum_{b=1}^{B} \sum_{l=1}^{d} \partial_l g \circ (T_\epsilon)^{-1}_{b,\lambda}(x) \cdot \partial_\epsilon ((T_\epsilon)^{-1}_{b,\lambda})_{(l)}(x), \quad x \in U_\lambda, \tag{39}$$

where $((T_\epsilon)^{-1}_{b,\lambda})_{(l)}(x)$ is the lth coordinate of $(T_\epsilon)^{-1}_{b,\lambda}(x)$ and $\widetilde{\mathcal{M}}_{1,g}(\epsilon, x) = \mathcal{M}_1(\epsilon, g)(x)$. On the other hand, by differentiating the lth coordinate of (35) for $1 \le l \le d$, we get

$$T^{(1)}_{(\ell),\epsilon}(y) + \sum_{k=1}^{d} \partial_k T_{(\ell),\epsilon}(y) \cdot \partial_\epsilon ((T_\epsilon)^{-1}_{b,\lambda})_{(k)}(x) = 0, \quad y = (T_\epsilon)^{-1}_{b,\lambda}(x),\ x \in U_\lambda.$$

In the matrix form (under the identification of $\mathbb{T}^d$ with $\mathbb{R}^d$), this can be written as

$$T^{(1)}_\epsilon(y) + D_y T_\epsilon[\partial_\epsilon (T_\epsilon)^{-1}_{b,\lambda}(x)] = 0, \quad y = (T_\epsilon)^{-1}_{b,\lambda}(x),\ x \in U_\lambda,$$

where we see $T^{(1)}_\epsilon(y)$ and $\partial_\epsilon (T_\epsilon)^{-1}_{b,\lambda}(x)$ as column vectors. Thus, since $A^{-1} = (\det A)^{-1} \operatorname{adj}(A)$ for any invertible matrix A,

$$\partial_\epsilon (T_\epsilon)^{-1}_{b,\lambda}(x) = -(\det D_y T_\epsilon)^{-1} \operatorname{adj}(D_y T_\epsilon)[T^{(1)}_\epsilon(y)], \quad y = (T_\epsilon)^{-1}_{b,\lambda}(x),\ x \in U_\lambda. \tag{40}$$

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Equation (38) immediately follows from (39) and (40). Furthermore, by the quotient rule for derivatives and (33),

$$\partial_\epsilon \mathcal{M}_2(\epsilon, g) = -\frac{g \cdot \det DT_\epsilon^{(1)}}{(\det DT_\epsilon)^2} \tag{41}$$

and

$$\partial_l(\mathcal{M}_2(\epsilon, g)) = \frac{\partial_l g}{\det DT_\epsilon} - \frac{g \cdot \partial_l(\det DT_\epsilon)}{(\det DT_\epsilon)^2}. \tag{42}$$

The conclusion immediately follows from (37), (38), (41) and (42). □

Now we complete the proof of Proposition 4.5. We first consider the case when $f \in \mathcal{C}^r(\mathbb{T}^d)$. We will show by induction that, for each $1 \le k \le j$, $(\mathbf{L}f)^{(k)}$ exists and is of the form

$$(\mathbf{L}f)^{(k)} = \mathbf{L}\bigg(\sum_{|\alpha|\le k} \widehat{J}_{k,\alpha} \cdot \partial^\alpha f\bigg), \tag{43}$$

where $\widehat{J}_{k,\alpha}$ is in $\mathcal{C}^{N-k}([-1,1], \mathcal{C}^{r-k}(\mathbb{T}^d))$ is a polynomial function of $\partial^\beta T_{(l)}^{(k')}$ $(1 \le l \le d$, $0 \le k' \le k$, $|\beta| \le k+1)$ and $(\det DT)^{-1}$ for each multi-index α with $|\alpha| \le k$. Equation (43) for $k = 1$ is an immediate consequence of Lemma 4.7 (notice that f in Lemma 4.7 depended on ϵ while f here does not). Suppose that $k \ge 2$ and (43) holds with $k-1$ instead of k. Then, by Lemma 4.7,

$$\begin{aligned}(\mathbf{L}f)^{(k)} = \mathbf{L}\bigg(&\sum_{|\alpha|\le 1} J_{0,\alpha} \cdot \partial^\alpha\bigg(\sum_{|\beta|\le k-1} \widehat{J}_{k-1,\beta} \cdot \partial^\beta f\bigg) \\ &+ \sum_{|\alpha|\le 1} J_{1,\alpha} \cdot \partial^\alpha\bigg(\sum_{|\beta|\le k-1} \widehat{J}^{(1)}_{k-1,\beta} \cdot \partial^\beta f\bigg)\bigg).\end{aligned}$$

Therefore, (43) immediately follows from (32) and (33). Furthermore, $\epsilon \mapsto d^j/d\epsilon^j \mathcal{L}_{T_\epsilon} f = (\mathbf{L}f)^{(j)}_\epsilon$ exists as an element of $\mathcal{C}^0([-1,1], E_{i-j})$ by (P3) and the fact that $s - j \ge 0$.

Next, we consider the general case, i.e., the case when $f \in E_i$. By (P1), one can find $\{f_n\}_{n\ge 1} \subset \mathcal{C}^r(\mathbb{T}^d)$ such that $\|f - f_n\|_{E_i} \to 0$ as $n \to \infty$. By the result in the previous paragraph, $(\mathbf{L}f_n)^{(k)}$ exists as an element of a Banach space $\mathcal{C}^0([-1,1], E_{i-k})$ for all $1 \le k \le j$. On the other hand, it follows from (P3), (P4), (30) and (43) that

$$\sup_{\epsilon\in[-1,1]} \|(\mathbf{L}f_n)^{(k)}_\epsilon - (\mathbf{L}f_m)^{(k)}_\epsilon\|_{E_{i-k}} \le C \sum_{|\alpha|\le k} \sup_{\epsilon\in[-1,1]} \|\widehat{J}_{k,\alpha,\epsilon}\|_{\mathcal{C}^{r-k}} \|f_n - f_m\|_{E_i} \to 0$$

as $n, m \to \infty$. In particular, $\lim_{n\to\infty}(\mathbf{L}f_n)^{(j)}$ exists. In a similar manner, we can show that $\mathbf{L}$ is a bounded operator from $\mathcal{C}^0([-1,1], E_i)$ to $\mathcal{C}^j([-1,1], E_{i-j})$, so that $\lim_{n\to\infty}(\mathbf{L}f_n)^{(j)} = (\mathbf{L}f)^{(j)}$ in $\mathcal{C}^0([-1,1], E_{i-j})$. In conclusion, $\mathbf{L}f : \epsilon \mapsto \mathcal{L}_{T_\epsilon} f$ is in $\mathcal{C}^j([-1,1], E_{i-j})$. □

4.3. *Random Anosov maps.* Let M be a compact, connected $\mathcal{C}^\infty$ Riemannian manifold with dimension d. In this section, we consider RDSs consisting of Anosov maps lying in a small $\mathcal{C}^{r+1}(M, M)$-neighborhood of a fixed, topologically transitive Anosov

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diffeomorphism $T \in \mathcal{C}^{r+1}(M, M)$ for some $r \geq 1$. The setting we consider is very similar to that of [20, §4]; however, we obtain more general conclusions than those of [20]. For the remainder of this section, we fix a topologically transitive Anosov diffeomorphism $T \in \mathcal{C}^{r+1}(M, M)$. Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and that $\sigma : \Omega \to \Omega$ is a measurably invertible, ergodic, measure-preserving map. For every $\eta > 0$, we define

$$\mathcal{O}_\eta(T) = \{S \in \mathcal{C}^{r+1}(M, M) : d_{\mathcal{C}^{r+1}}(S, T) < \eta\}.$$

Recall that if η is sufficiently small, then $\mathcal{O}_\eta(T) \subset \mathcal{N}^{r+1}(M, M)$ and every $S \in \mathcal{O}_\eta(T)$ is an Anosov diffeomorphism. A map $\mathcal{T} : \Omega \to \mathcal{C}^{r+1}(M, M)$ will be said to be measurable if it is $(\mathcal{F}, \mathcal{B}_{\mathcal{C}^{r+1}(M,M)})$-measurable.

We consider RDSs induced by measurable maps $\mathcal{T} : \Omega \to \mathcal{O}_\eta(T)$ for some small, fixed η, over σ. Our main result for this section concerns the stability properties of the equivariant family of probability measures associated to such systems. We will formulate our result in the setting developed by Gouëzel and Liverani in [28]. In particular, in [28] it is shown that, when a topologically transitive Anosov map is smoothly perturbed, the Sinai–Ruelle–Bowen measure varies with similar regularity in certain anisotropic Banach spaces.

A small technical comment is required before proceeding: in [28] the usual metric on M is replaced by an adapted metric for T (which will also be adapted for $S \in \mathcal{O}_\eta(T)$ provided that η is sufficiently small); we shall do the same here. We denote by m the Riemannian probability measure induced by the adapted metric on M. For each $q \geq 0$, $p \in \mathbb{N}_0$ with $p \leq r$, one obtains a space $B_{p,q}(T)$ by taking the completion of $\mathcal{C}^r(M)$ with respect to anisotropic norms $\|\cdot\|_{p,q}$ as defined in [28, §3] (actually, the norms in [28, §3] are defined on the real Banach space $\mathcal{C}^r(M, \mathbb{R})$. Here, we consider the complexification, which is of no consequence). Since the map T is fixed, we will just write $B_{p,q}$ in place of $B_{p,q}(T)$. Our main result for this section is the following theorem.

THEOREM 4.8. *Let $N, p \in \mathbb{N}$ and $q \geq 0$ satisfying that $p + q < r - N$. Then there exists $\eta_0 > 0$ such that every measurable $\mathcal{T} : \Omega \to \mathcal{O}_{\eta_0}(T)$ has an equivariant measurable family of Radon probability measures $\{\mu_\omega^{\mathcal{T}}\}_{\omega\in\Omega}$ and $h_{\mathcal{T}} \in L^\infty(\Omega, B_{p+N,q})$ such that $h_{\mathcal{T}}(\omega)(g) = \int g \, d\mu_\omega^{\mathcal{T}}$ for each $g \in \mathcal{C}^\infty(M)$ and almost every ω. In addition, if $\{\mathcal{T}_\epsilon : \Omega \to \mathcal{O}_{\eta_0}(T)\}_{\epsilon\in[-1,1]}$ is a family of measurable maps such that there is a bounded subset $\mathcal{K}$ of $\mathcal{C}^N([-1, 1], \mathcal{C}^{r+1}(M, M))$ (recall that $\mathcal{C}^{r+1}(M, M)$ is a $\mathcal{C}^{r+1}$ Banach manifold and so, for $k \leq r + 1$, we may talk of $\mathcal{C}^k$ curves taking values in $\mathcal{C}^{r+1}(M, M)$) satisfying that $\epsilon \mapsto \mathcal{T}_\epsilon(\omega)$ lies in $\mathcal{K}$ for almost every ω, then the map $\epsilon \mapsto h_{\mathcal{T}_\epsilon}(\omega)$ is in $\mathcal{C}^{N-1}([-1, 1], B_{p,q+N})$ for almost every ω.*

We will use Theorem 3.6 to prove Theorem 4.8, with the help of Propositions 4.2 and 4.4 and of Corollary 4.6. Therefore, we should check (P1)–(P5) and (QR0)–(QR3) for appropriate Banach spaces. We start with the basic properties of the $B_{p,q}$ spaces from [28].

(1) By the definition of $\|\cdot\|_{p,q}$, it is straightforward to see that $\|\partial_l f\|_{p,q} \leq \|f\|_{p+1,q-1}$ for each $f \in \mathcal{C}^r(M)$ and $1 \leq l \leq d$. Furthermore, $B_{p+1,q-1} \hookrightarrow B_{p,q}$.
(2) [28, Lemma 2.1] If $p + q < r$, then the unit ball in $B_{p+1,q-1}$ is relatively compact in $B_{p,q}$.

(3) [28, Lemma 3.2] $\|uf\|_{p,q} \leq C\|u\|_{\mathcal{C}^{p+q}}\|f\|_{p,q}$ for each $u \in \mathcal{C}^{p+q}(M)$ and $f \in \mathcal{C}^r(M)$. In particular, if $p+q < r$, then $\mathcal{C}^r(M) \hookrightarrow B_{p,q}$ (see also [28, Remark 4.3]).
(4) [28, Proposition 4.1] We have $B_{p,q} \hookrightarrow (\mathcal{C}^q(M))^*$. Specifically, for each $h \in \mathcal{C}^r(M)$, one obtains a distribution $\tilde{h} \in (\mathcal{C}^q(M))^*$ defined by $\tilde{h}(g) = \int hg \, \mathrm{d}m$. The map $h \mapsto \tilde{h}$ continuously extends from $\mathcal{C}^r(M)$ to $B_{p,q}$ and yields the required inclusion.

We also remark that there exist injections $B_{p,q} \to B_{p-1,q}$ and $B_{p,q} \to B_{p,q'}$ for $q' > q$ due to [28, Remark 4.2]. By the fourth item of the above list, the functional $h \mapsto \int h \, \mathrm{d}m$ on $\mathcal{C}^r(M)$ extends to a continuous functional on $B_{p,q}$, which we shall also denote by m. The following result summarizes some facts from [13, 28] pertaining to the boundedness and mixing of the Perron–Frobenius operator associated to maps in $\mathcal{O}_\eta(T)$ for small η. We refer the reader to [28, Lemma 2.2] and the discussion at the beginning of [28, §7] for the first and second items and to [13, Proposition 2.10] for the third item (see also [19, §3]).

PROPOSITION 4.9. *There exists $0 < \eta_0 \leq \eta$ such that, for any $p \in \mathbb{N}_0$ and $q \geq 0$ with $p+q < r$, we have the following.*

(1) *For every sequence $\{T_i\}_{i\in\mathbb{N}} \subseteq \mathcal{O}_{\eta_0}(T)$ and $n \in \mathbb{N}$,*
$$\|\mathcal{L}_{T_n} \circ \cdots \circ \mathcal{L}_{T_1}\|_{L(B_{p,q})} \leq C_{p,q}.$$
(2) *There exists $\alpha_{p,q} \in [0,1)$ such that, for every $\{T_i\}_{i\in\mathbb{N}} \subseteq \mathcal{O}_{\eta_0}(T)$, $n \in \mathbb{N}$ and $f \in B_{p+1,q}$,*
$$\|(\mathcal{L}_{T_n} \circ \cdots \circ \mathcal{L}_{T_1})f\|_{p+1,q} \leq C_{p,q}\alpha_{p,q}^n\|f\|_{B_{p+1,q}} + C_{p,q}\|f\|_{B_{p,q+1}}.$$
(3) *There exists a constant $\lambda_{p,q} \in [0,1)$ such that, for every sequence $\{T_i\}_{i\in\mathbb{Z}} \subseteq \mathcal{O}_{\eta_2}(T)$ and $n \in \mathbb{N}$,*
$$\|\mathcal{L}_{T_n} \circ \cdots \circ \mathcal{L}_{T_1}|_{V_{p,q}}\|_{L(B_{p,q})} \leq C_{p,q}\lambda_{p,q}^n,$$
where $V_{p,q} = \ker(m|_{B_{p,q}}) = \{h \in B_{p,q} \mid m(h) = 0\}$.

Fix $q \geq 0$ and $p \in \mathbb{N}$ with $p+q < r-N$. Let $E_j = B_{p+j,q+N-j}$, $j \in \{0,\ldots,N\}$. Then the conditions (P1)–(P4) on these Banach spaces immediately follow from the above list (recall that each E_j is the completion of $\mathcal{C}^r(M)$ with respect to $\|\cdot\|_{E_j}$). Furthermore, fix a bounded subset $\mathcal{K}$ of $\mathcal{C}^N([-1,1], \mathcal{C}^{r+1}(M,M))$ and let $\{\mathcal{T}_\epsilon : \Omega \to \mathcal{O}_{\eta_0}(T)\}_{\epsilon\in[-1,1]}$ be a family of measurable maps such that $\epsilon \mapsto \mathcal{T}_\epsilon(\omega)$ lies in $\mathcal{K}$ for almost every ω. Then, by virtue of Proposition 4.2, the first part of Proposition 4.9 and the above list, Perron–Frobenius operator cocycles $\{(\mathcal{L}_{\mathcal{T}_\epsilon}, \sigma)\}_{\epsilon\in[-1,1]}$ associated with the random dynamics $\{(\mathcal{T}_\epsilon, \sigma)\}_{\epsilon\in[-1,1]}$ are precisely defined on these Banach spaces and they satisfy (P5) and (QR0) with $\xi = m$ (see the remark following (S2)) except the m-mixing property. In fact, (QR2), (QR3) and the mixing of $\{(\mathcal{L}_{\mathcal{T}_\epsilon}, \sigma)\}_{\epsilon\in[-1,1]}$ on E_j for $j \in \{1, N\}$ are consequences of each item of Proposition 4.9, respectively. By Corollary 4.6, (QR1), (QR4) and (QR5) also hold for $\{(\mathcal{L}_{\mathcal{T}_\epsilon}, \sigma)\}_{\epsilon\in[-1,1]}$ on E_j (see also [28, Lemma 7.1] and [28, §9]).

By Proposition 4.4 (and the remark following it), for each $\epsilon \in [-1,1]$, $\mathcal{T}_\epsilon$ has an equivariant measurable family of Radon probability measures $\{\mu_\omega^{\mathcal{T}_\epsilon}\}_{\omega\in\Omega}$ and $h_{\mathcal{T}_\epsilon} \in L^\infty(\Omega, E_N) = L^\infty(\Omega, B_{p+N-1,q})$ such that $h_{\mathcal{T}_\epsilon}(\omega)(g) = g\mathrm{d}\mu_\omega^{\mathcal{T}_\epsilon}$ for each $g \in$

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$\mathcal{C}^\infty(M)$ and almost every ω. Furthermore, we apply Theorem 3.6 to deduce the claim that $\epsilon \mapsto h_{\mathcal{T}_\epsilon}(\omega)$ is in $\mathcal{C}^{N-1}([-1, 1], E_0) = \mathcal{C}^{N-1}([-1, 1], B_{p-1,q+N})$ for almost every ω, which completes the proof of Theorem 3.6.

4.4. *Random U(1) extensions of expanding maps.* In this section, we will apply Theorem 3.6 to quenched linear response problems for random U(1) extensions of expanding maps. Let $\mathcal{U}$ be the set of $\mathcal{C}^\infty$ endomorphisms $T : \mathbb{T}^2 \to \mathbb{T}^2$ on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ of the form

$$T : \begin{pmatrix} x \\ s \end{pmatrix} \mapsto \begin{pmatrix} E(x) \\ s + \tau(x) \bmod 1 \end{pmatrix}, \tag{44}$$

where $E : \mathbb{S}^1 \to \mathbb{S}^1$ is a $\mathcal{C}^\infty$ orientation-preserving endomorphism on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $\tau : \mathbb{S}^1 \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function (T is called the *U(1) extension* of E over τ). U(1) extensions of expanding maps can be seen as toy models of (piecewise) hyperbolic flows such as geodesic flows on manifolds with negative curvature or dispersive billiard flows (via suspension flows of hyperbolic maps; see [**31**, **40**]), and have been intensively studied by several authors (see, e.g., [**15**, **21**, **38**, **39**]). When we want to emphasize the dependence of E and τ in (44) on T, we write them as E_T and τ_T. Fix $T \in \mathcal{U}$ and assume that E is an expanding map on $\mathbb{S}^1$ in the sense that $\min_{x\in\mathbb{S}^1} E'(x) > 1$. Let r be a positive integer. For every $\eta > 0$, we define

$$\mathcal{O}_\eta(T) = \{S \in \mathcal{U} \mid d_{\mathcal{C}^{r+1}}(S, T) < \eta\}.$$

Note that $\mathcal{U} \subset \mathcal{N}^{r+1}(\mathbb{T}^2, \mathbb{T}^2)$ and that, if η is sufficiently small, then E_S is an expanding map for every $S \in \mathcal{O}_\eta(T)$.

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and that $\sigma : \Omega \to \Omega$ is a measurably invertible, ergodic, $\mathbb{P}$-preserving map on $(\Omega, \mathcal{F}, \mathbb{P})$. When $\tau_T(x) = \alpha$ for any $x \in \mathbb{S}^1$ with some constant α, then obviously T does not admit any mixing physical measure because the rotation $s \mapsto s + (1/2\pi)\alpha$ mod one has no mixing physical measure. However, it is known that if τ satisfies a generic condition, called the *partial captivity* condition, then T admits a unique absolutely continuous invariant probability measure for which correlation functions of T decay exponentially fast (in particular, T is mixing). The partial captivity condition was first introduced by Faure [**21**] and was proved to be generic in [**38**]. Furthermore, it was shown in [**39**, Theorem 1.6] that, if T satisfies the partial captivity condition, then there is an $\eta_0 > 0$ and an $m_0 \in \mathbb{N}$, only depending on T (see the comment above Proposition 4.11 for more a precise choice of η_0 and m_0), such that, if $r \geq m_0$, then, for any measurable map $\mathcal{T} : \Omega \to \mathcal{O}_{\eta_0}(T)$, the RDS $(\mathcal{T}, \sigma)$ induced by $\mathcal{T}$ over σ admits a unique equivariant measurable family of absolutely continuous probability measures $\{\mu_\omega^{\mathcal{T}}\}_{\omega\in\Omega}$ such that the Radon–Nikodym derivative of $\mu_\omega^{\mathcal{T}}$ is in the usual Sobolev space $H^r(\mathbb{T}^2)$ of regularity r for $\mathbb{P}$-almost every ω and that quenched correlation functions of $(\mathcal{T}, \sigma)$ for $\{\mu_\omega^{\mathcal{T}}\}_{\omega\in\Omega}$ decay exponentially fast.

Assume that T satisfies the transversality condition, and fix such an $\eta_0 > 0$ and an $m_0 \in \mathbb{N}$. Assume also that $r \geq m_0 + 1$. The main result in this section is the following theorem.

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THEOREM 4.10. *Let N be positive integers such that $N \le r - m_0$. If $\{\mathcal{T}_\epsilon : \Omega \to \mathcal{O}_{\eta_0}(T)\}_{\epsilon\in[-1,1]}$ is a family of measurable maps such that there is a bounded subset $\mathcal{K}$ of $\mathcal{C}^N([-1,1], \mathcal{C}^{r+1}(\mathbb{T}^2, \mathbb{T}^2))$ satisfying that $\epsilon \mapsto \mathcal{T}_\epsilon(\omega)$ lies in $\mathcal{K}$ for $\mathbb{P}$-almost every $\omega \in \Omega$, then the map $\epsilon \mapsto \mu_\omega^{\mathcal{T}_\epsilon}$ is in $\mathcal{C}^{N-1}([-1,1], H^{r-N}(\mathbb{T}^2))$ for $\mathbb{P}$-almost every $\omega \in \Omega$.*

We recall the basic properties of the Sobolev spaces $H^m(\mathbb{T}^2)$ with regularity $m \in \mathbb{N}_0$. Recall that $\|f\|_{H^m}^2 = \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{L^2}^2$.

(1) By the definition of $\|\cdot\|_{H^m}$, it is straightforward to see that $\|\partial_l f\|_{H^m} \le \|f\|_{H^{m+1}}$ for each $f \in \mathcal{C}^{m+1}(\mathbb{T}^2)$ and $1 \le l \le d$ and that $\|uf\|_{H^m} \le C\|u\|_{\mathcal{C}^m}\|f\|_{H^m}$ for each $u, f \in \mathcal{C}^m(\mathbb{T}^2)$.

(2) By Kondrachov's embedding theorem, $H^{m+1}(\mathbb{T}^2) \hookrightarrow H^m(\mathbb{T}^2)$ and the unit ball in $H^{m+1}(\mathbb{T}^2)$ is relatively compact in $H^m(\mathbb{T}^2)$.

(3) $\mathcal{C}^{m'}(\mathbb{T}^2)$ is dense in $H^m(\mathbb{T}^2)$ for each $m' \ge m$ because $\mathcal{C}^{m'}(\mathbb{T}^2) \subset H^m(\mathbb{T}^2) \subset \mathcal{C}^{m-1}(\mathbb{T}^2)$ by Sobolev's embedding theorem.

(4) By the Cauchy–Schwarz inequality, we have $H^m(\mathbb{T}^2) \hookrightarrow (\mathcal{C}^0(\mathbb{T}^2))^*$ by $h \mapsto \tilde{h}$ given by $\tilde{h}(g) = \int hg\, dm$ for $g \in \mathcal{C}^0(\mathbb{T}^2)$.

Let $\lambda_0 := (\inf_{S\in\mathcal{O}_{\eta_0}} \min_{x\in\mathbb{S}^1} E_S'(x))^{-1}$, which is less than one by taking η_0 small, if necessary. Fix $\lambda \in (\lambda_0^{1/2}, 1)$. Let m_0 be a sufficiently large integer given in [**39**, Theorem 1.5]. Let $N \le r - m_0$ be a positive integer. By taking η_0 small, if necessary, we assume that $(\inf_{S\in\mathcal{O}_{\eta_0}} \min_{x\in\mathbb{S}^1} E_S'(x))^{-1} < \lambda$. Fix a family of measurable maps $\{\mathcal{T}_\epsilon : \Omega \to \mathcal{O}_{\eta_0}(T)\}_{\epsilon\in[-1,1]}$ such that there is a bounded subset $\mathcal{K}$ of $\mathcal{C}^N([-1,1], \mathcal{C}^{r+1}(\mathbb{T}^2, \mathbb{T}^2))$ satisfying that $\epsilon \mapsto \mathcal{T}_\epsilon(\omega)$ lies in $\mathcal{K}$ for $\mathbb{P}$-almost every $\omega \in \Omega$. Let $(\mathcal{L}_{\mathcal{T}_\epsilon}, \sigma)$ be the Perron–Probenius cocycle induced by $(\mathcal{T}_\epsilon, \sigma)$. Then it follows from [**39**, §4] that $\mathcal{L}_{\mathcal{T}_\epsilon}$ almost surely extends to a unique, bounded operator on $H^m(\mathbb{T}^2)$ such that $\omega \mapsto \mathcal{L}_{\mathcal{T}_\epsilon}(\omega) : \Omega \to L(H^m(\mathbb{T}^2))$ is strongly measurable for each $\epsilon \in [-1, 1]$. The following estimates were proved in [**12**, §§2.3 and 2.4].

PROPOSITION 4.11. *There is a constant $\rho \in (0, 1)$ (which may depend on T and η_0) such that, for all $|\epsilon| \le 1$, the following hold.*

(1) *For each $m \ge 0$ and $n \ge 1$,*

$$\operatorname*{ess\,sup}_\omega \|\mathcal{L}_{\mathcal{T}_\epsilon}^{(n)}(\omega)\|_{L(H^m(\mathbb{T}^2))} \le C.$$

(2) *For each $m \ge m_0$, $n \ge 1$ and $f \in H^m(\mathbb{T}^2)$,*

$$\operatorname*{ess\,sup}_\omega \|\mathcal{L}_{\mathcal{T}_\epsilon}^{(n)}(\omega) f\|_{H^{m+1}} \le C\lambda^n \|f\|_{H^{m+1}} + C\|f\|_{H^m}.$$

(3) *For each $m \ge m_0$, $n \ge 1$ and $f \in H^m(\mathbb{T}^2)$ with $\int_{\mathbb{T}^2} f\, dm = 0$,*

$$\operatorname*{ess\,sup}_\omega \|\mathcal{L}_{\mathcal{T}_\epsilon}^{(n)}(\omega) f\|_{H^m} \le C\rho^n \|f\|_{H^m}.$$

We now prove Theorem 4.10. Let $m = r - N$. For $j \in \{0, \ldots, N\}$, set $E_j = H^{m+j}(\mathbb{T}^2)$. Then, in the same manner as in the proof of Theorem 4.8, we can apply Theorem 3.6, with the help of Propositions 4.2 and 4.4 and of Corollary 4.6, to deduce the claim that

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$\epsilon \mapsto \mu_\omega^{\mathcal{T}_\epsilon}$ is in $\mathcal{C}^{N-1}([-1,1], E_0) = \mathcal{C}^{N-1}([-1,1], H^{r-N}(\mathbb{T}^2))$ for almost every ω, which completes the proof of Theorem 4.10.

5. *Application to the differentiability of random dynamical variances*

In this section, as another application of Theorem 3.6, we show the differentiability of the variances in quenched CLTs for certain class of RDSs (including random Anosov maps and random U(1) extensions of expanding maps). See, e.g., [**45**] and reference therein for the background of this topic.

The quenched CLT, stated precisely below, was first shown by Dragičević *et al.* [**18**] for random piecewise expanding maps and was later extended to random piecewise hyperbolic maps by the same authors [**19**]. The argument in these results was extracted in an abstract form in [**12**] and the quenched CLT for random (1) extensions of expanding maps was established as its application. In this section, we employ a mixture of the settings of §§4.1 and 4.2 and [**12**]. Specifically, to apply Theorem 3.6, we assume the following.

- There are a real number $r \ge 1$, a positive integer $N \le r$, a bounded subset $\mathcal{K}$ of $\mathcal{C}^N([-1,1], \mathcal{C}^{r+1}(M,M))$ and a family of measurable maps $\{\mathcal{T}_\epsilon : \Omega \to \mathcal{N}^{r+1}(M,M)\}_{\epsilon\in[-1,1]}$ such that $\epsilon \mapsto \mathcal{T}_\epsilon(\omega)$ lies in $\mathcal{K}$ for almost every ω.
- $\{E_j\}_{j\in\{0,\ldots,N\}}$ is a family of Banach spaces with $E_j \hookrightarrow E_{j-1}$ for each $j \in \{1,\ldots,N\}$ for which (P1)–(P5) of §4.2 holds for the family of the Perron–Frobenius operator cocycles $\{(\mathcal{L}_{\mathcal{T}_\epsilon}, \sigma)\}_{\epsilon\in[-1,1]}$ (and so, by Corollary 4.6, it satisfies (QR1), (QR2), (QR4) and (QR5)).
- $\{(\mathcal{L}_{\mathcal{T}_\epsilon}, \sigma)\}_{\epsilon\in[-1,1]}$ satisfies (QR0) and (QR3).

To apply the result in [**12**] we further assume the following.

- The inclusion $E_1 \hookrightarrow E_0$ is compact.

Then, with the help of (Q), for any $\epsilon \in [-1,1]$, (P5), (P3), (QR0) (b) and (QR3) we verify hypotheses (A1), (A2), (UG) and (LY) of [**12**, §2.1], respectively, for $\mathcal{L}_\omega = \mathcal{L}_{\mathcal{T}_{\epsilon,\omega}}$, $\mathscr{M} = \mathcal{N}^{r+1}(M,M)$, $\mathscr{D} = \mathcal{C}^{r+1}(M)$, $\mathscr{B} = E_1$, $\mathscr{E} = \mathcal{C}^\rho(M)$ and $\mathscr{B}_+ = E_0$ with some $\rho \le r$ (where (P1) and (P2) are used to ensure that $\mathcal{L}_\omega$ is well defined in the manner of [**12**]). Examples satisfying the above conditions include the random Anosov maps and random U(1) extensions of expanding maps considered in §§4.3 and 4.4.

Therefore, under this setting, it follows from [**12**, Theorem 2.6] that a quenched CLT holds for $(\mathcal{T}_\epsilon, \sigma)$ for each $\epsilon \in [-1,1]$, in the following sense. Let $g \in L^\infty(\Omega, \mathcal{C}^r(M))$. Let $\{\mu_{\epsilon,\omega}\}_{\omega\in\Omega}$ be the equivariant measurable family of Radon probability measures such that there exists $h_\epsilon \in L^\infty(\Omega, E_N)$ satisfying $h_\epsilon(\omega)(u) = \int u \,\mathrm{d}\mu_{\epsilon,\omega}$ for each $u \in \mathcal{C}^\infty(M)$ and almost every ω (refer to Proposition 4.4). Let g_ϵ be the centering of g with respect to $\{\mu_{\epsilon,\omega}\}_{\omega\in\Omega}$: that is,

$$g_{\epsilon,\omega} := g_\omega - \int_{\mathbb{T}^d} g_\omega \,\mathrm{d}\mu_{\epsilon,\omega}.$$

We define the *variance* (the *drift coefficient*) V_ϵ of $(\mathcal{T}_\epsilon, \sigma, \mu_\epsilon, g_\epsilon)$ by

$$V_\epsilon := \int_\Omega \bigg(\int_{\mathbb{T}^d} g_{\epsilon,\omega}^2 \,\mathrm{d}\mu_{\epsilon,\omega} + 2\sum_{n=1}^\infty \int_{\mathbb{T}^d} g_{\epsilon,\omega} \cdot (g_{\epsilon,\sigma^n\omega} \circ \mathcal{T}_{\epsilon,\omega}^{(n)}) \,\mathrm{d}\mu_{\epsilon,\omega} \bigg) \mathbb{P}(\mathrm{d}\omega).$$

(The limit exists due to the exponential decay of correlations, which follows from Remark 3.9 in a standard manner.) Suppose that $V_\epsilon > 0$ (this is a generic condition for random expanding maps; refer to [17]). Then, for $\mathbb{P}$-almost every $\omega \in \Omega$, $(S_n g_\epsilon)_\omega/\sqrt{n}$ converges in distribution to a normal distribution with mean zero and variance V_ϵ as $n \to \infty$: that is, for any $a \in \mathbb{R}$,

$$\lim_{n\to\infty} \mu_\omega\bigg(\frac{(S_n g_\epsilon)_\omega}{\sqrt{n}} \leq a\bigg) = \frac{1}{\sqrt{2\pi V_\epsilon}} \int_{-\infty}^{a} e^{-z^2/2V_\epsilon} dz,$$

where $(S_n g_\epsilon)_\omega = \sum_{j=0}^{n-1} g_{\epsilon,\sigma^j\omega} \circ \mathcal{T}_{\epsilon,\omega}^{(j)}$.

We now show that the regularity of the variance V_ϵ is subjected to the regularity of the dynamics $\mathcal{T}_\epsilon$. Recall that $M > 0$ and $\alpha < M$ in (QR2) and (QR3).

THEOREM 5.1. *$V : \epsilon \mapsto V_\epsilon$ is of class $\mathcal{C}^{N-1+\eta}$ for every $\eta \in (0, \log(1/\alpha)/\log(M/\alpha))$. In particular, V is differentiable if $N \geq 2$.*

Remark 5.2. As seen below, the proof of Theorem 5.1 uses only Theorem 3.6, not any result in [12]. Hence, (Q) and the assumption $V_\epsilon > 0$ are not necessary for the $\mathcal{C}^{N-1+\eta}$-regularity of V.

Proof. We use the idea that appeared in [28, Remark 2.10] for deterministic Anosov maps. Recall the notation $\mathbb{A}_\epsilon$, $\widetilde{\mathcal{E}}_j$ and $\mathcal{N}_j$ for $j \in \{0, \ldots, N\}$ given in §3.2 for $A_\epsilon = \mathcal{L}_{\mathcal{T}_\epsilon}$. Then V_ϵ can be written as

$$V_\epsilon = \int_\Omega \bigg[(g_\epsilon \cdot h_\epsilon)(\omega)(g_{\epsilon,\omega}) + 2\sum_{n=1}^{\infty} (\mathbb{A}_\epsilon^n (g_\epsilon \cdot h_\epsilon))(\omega)(g_{\epsilon,\omega})\bigg] \,\mathrm{d}\mathbb{P},$$

where $g_\epsilon \cdot h_\epsilon \in \widetilde{\mathcal{E}}_N$ is given by $(g_\epsilon \cdot h_\epsilon)(\omega)(u) := \int_M g_{\epsilon,\omega} \cdot u \,\mathrm{d}\mu_{\epsilon,\omega}$ for each $\omega \in \Omega$ and $u \in \mathcal{C}^\infty(M)$. (Recall the duality (26) and the σ-invariance of $\mathbb{P}$.) Notice also that

$$m(g_\epsilon \cdot h_\epsilon)(\omega) = (g_\epsilon \cdot h_\epsilon)(\omega)(1_M) = \int_M g_{\epsilon,\omega} \,\mathrm{d}\mu_{\epsilon,\omega} = 0 \quad \mathbb{P}\text{-almost surely}$$

by the definition of g_ϵ, so that

$$\mathbb{A}_\epsilon^n (g_\epsilon \cdot h_\epsilon) = (\mathbb{A}_\epsilon|_{\mathcal{N}_N})^n (g_\epsilon \cdot h_\epsilon) \quad \text{and} \quad \rho(\mathbb{A}_\epsilon|_{\mathcal{N}_N}) < 1$$

for any sufficiently small $\epsilon \geq 0$ (recall Remark 3.9).

Now recall the Neumann series expansion $\sum_{n=0}^\infty A^n = (\mathrm{Id} - A)^{-1}$ for a bounded linear operator A with $\rho(A) < 1$. Applying it to $A = \mathbb{A}_\epsilon|_{\mathcal{N}_N}$, we get

$$V_\epsilon = \int_\Omega [-(g_\epsilon \cdot h_\epsilon)(\omega)(g_{\epsilon,\omega}) + 2((\mathrm{Id} - \mathbb{A}_\epsilon|_{\mathcal{N}_N})^{-1}(g_\epsilon \cdot h_\epsilon))(\omega)(g_{\epsilon,\omega})] \,\mathrm{d}\mathbb{P}. \tag{45}$$

On the other hand, it follows from (18) and (19) that, for any $\Phi \in \widetilde{\mathcal{E}}_N$, there exists $\Phi^{(k)} \in \widetilde{\mathcal{E}}_0$ with $k = 1, \ldots, N-1$ such that

$$\bigg\|(\mathrm{Id} - \mathbb{A}_\epsilon)^{-1}\Phi - (\mathrm{Id} - \mathbb{A}_0)^{-1}\Phi - \sum_{k=1}^{N-1} \epsilon^k \Phi^{(k)}\bigg\|_{\widetilde{\mathcal{E}}_0} \leq C\|\Phi\|_{\widetilde{\mathcal{E}}_N} |\epsilon|^{N-1+\eta}.$$

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This immediately implies the $\mathcal{C}^{N-1+\eta}$-regularity of the second term of the right-hand side of (45). Moreover, Theorem 3.6 leads to the $\mathcal{C}^{N-1+\eta}$-regularity of the first term of the right-hand side of (45) via the $\mathcal{C}^{N-1+\eta}$-regularities of h_ϵ because $(g_\epsilon \cdot h_\epsilon)(\omega)(g_{\epsilon,\omega}) = h_\epsilon(\omega)(g_\omega^2) - h_\epsilon(\omega)(g_\omega)^2$. This completes the proof. □

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A. *Appendix. Proof of Proposition 4.1*

Let $T \in \mathcal{N}^{r+1}(M, M)$. Then it follows from [30, Corollary 1] that T is a covering map. Hence, by a basic property of covering spaces, there is a discrete topological space Γ such that, for every $x \in M$, there is a neighborhood U_x of x such that $T^{-1}(\{x\})$ is homeomorphic to Γ and $T^{-1}(U_x)$ is homeomorphic to $U_x \times \Gamma$. In other words, $T^{-1}(U_x)$ is a union of disjoint open sets $\{\widetilde{U}_{b,x}\}_{b=1}^B$ such that $T : \widetilde{U}_{b,x} \to U_x$ is a homeomorphism for each $b = 1, \ldots, B$, where B is the cardinality of Γ. If $B = \infty$, then, since $|\det DT|$ is bounded uniformly away from zero due to the compactness of M,

$$m(T^{-1}(U_x)) \geq B \cdot \inf_{y\in M} |\det DT(y)|\, m(U_x) = \infty,$$

which contradicts M having finite m-measure. Hence, $B < \infty$. Furthermore, there is a small neighborhood $\mathcal{U}$ of T in $\mathcal{N}^{r+1}(M, M)$ such that, for each $S \in \mathcal{U}$ and $x \in M$, there are disjoint open sets $\{\widetilde{U}^S_{b,x}\}_{b=1}^B$ such that $S : \widetilde{U}^S_{b,x} \to U_x$ is a $\mathcal{C}^{r+1}$ diffeomorphism for each b and that, for each $y \in U_x$,

$$d_M((S|_{\widetilde{U}^S_{b,x}})^{-1}(y), (T|_{\widetilde{U}_{b,x}})^{-1}(y)) \to 0 \tag{A.1}$$

as $S \to T$ in $\mathcal{N}^{r+1}(M, M)$.

Since M is compact, there is a finite subfamily $\{U_\lambda\}_{\lambda\in\Lambda}$ (with $|\Lambda| < \infty$) of the open covering $\{U_x\}_{x\in M}$ of M. As per the previous paragraph, for each $\lambda \in \Lambda$, there are disjoint open sets $\{\widetilde{U}_{b,\lambda}\}_{b=1}^B$ such that $T : \widetilde{U}_{b,\lambda} \to U_\lambda$ is a $\mathcal{C}^{r+1}$ diffeomorphism for each $b = 1, \ldots, B$. Notice that, for each $\lambda \in \Lambda$, $x \in U_\lambda$ and a complex-valued function f on M, it holds that

$$\sum_{T(y)=x} f(y) = \sum_{b=1}^{B} (f \circ (T|_{\widetilde{U}_{b,\lambda}})^{-1})(x). \tag{A.2}$$

Let $\{K_\lambda\}_{\lambda\in\Lambda}$ be a closed covering of M such that $K_\lambda \subset U_\lambda$ and let $\{\rho_\lambda\}_{\lambda\in\Lambda}$ be a partition of unity of M subordinate to the covering $\{K_\lambda\}_{\lambda\in\Lambda}$ (that is, ρ_λ is a $\mathcal{C}^\infty$ function on M with values in $[0, 1] \subset \mathbb{R}$ such that the support of ρ_λ is contained in K_λ for each $\lambda \in \Lambda$ and $\sum_{\lambda\in\Lambda} \rho_\lambda(x) = 1$ for each $x \in M$). Then, in view of (A.2), we get that, for each $f \in \mathcal{C}^r(M)$ and $x \in M$,

$$\mathcal{L}_T f(x) = \sum_{\lambda\in\Lambda} \rho_\lambda(x) \cdot \sum_{T(y)=x} \frac{f(y)}{|\det DT(y)|} = \sum_{\lambda\in\Lambda} \sum_{b=1}^{B} \rho_\lambda(x) \cdot \bigg(\frac{f}{|\det DT|} \circ (T|_{\widetilde{U}_{b,\lambda}})^{-1}\bigg)(x)$$

and, for each $S \in \mathcal{U}$,

$$\|\mathcal{L}_T f - \mathcal{L}_S f\|_{\mathcal{C}^r} \leq \sum_{\lambda\in\Lambda} \sum_{b=1}^{B} \|\rho_\lambda\|_{\mathcal{C}^r} \bigg\| \frac{f}{|\det DT|} \circ (T|_{\widetilde{U}_{b,\lambda}})^{-1} - \frac{f}{|\det DS|} \circ (S|_{\widetilde{U}^S_{b,\lambda}})^{-1} \bigg\|_{\mathcal{C}^r(K_\lambda)}.$$

Therefore, since both $|\Lambda|$ and B are finite and $\||\det DT|^{-1} - |\det DS|^{-1}\|_{\mathcal{C}^r} \to 0$ as $S \to T$ in $\mathcal{N}^{r+1}(M, M)$, it suffices to show that, for each $\lambda \in \Lambda$, $b = 1, \ldots, B$ and $f \in \mathcal{C}^r(M)$,

$$\|f \circ (T|_{\widetilde{U}_{b,\lambda}})^{-1} - f \circ (S|_{\widetilde{U}^S_{b,\lambda}})^{-1}\|_{\mathcal{C}^r(K_\lambda)} \to 0 \quad \text{as } S \to T \text{ in } \mathcal{N}^{r+1}(M, M). \tag{A.3}$$

Fix $\lambda \in \Lambda$, $b = 1, \ldots, B$ and $f \in \mathcal{C}^r(M)$. By taking K_λ small, if necessary, we can assume that K_λ is included in a local chart of M, so we assume that K_λ is a closed subset of $\mathbb{R}^d$, where $d = \dim M$. We use the notation $\partial_i(\cdot)$, $\partial^\alpha(\cdot)$ and $\mathrm{adj}(\cdot)$ given in the proof of Proposition 4.5. Recall that $T|_{\widetilde{U}_{b,\lambda}} : \widetilde{U}_{b,\lambda} \to U_\lambda$ is a $\mathcal{C}^{r+1}$ diffeomorphism, so, by the inverse function theorem and the fact that $A^{-1} = (\det A)^{-1}\mathrm{adj}(A)$ for any invertible matrix A,

$$D((T|_{\widetilde{U}_{b,\lambda}})^{-1})(x) = DT(y) = (\det DT(y))^{-1}\mathrm{adj}(DT(y))$$

with $y = (T|_{\widetilde{U}_{b,\lambda}})^{-1}(x)$ for any $x \in U_\lambda$. Since each entry of $\mathrm{adj}(DT(y))$ (the transpose of the cofactor matrix of $DT(y)$) is a polynomial of $\partial_i T$ $(1 \leq i \leq d)$, by the chain rule for derivatives, we conclude that, for each $i = 1, \ldots, d$,

$$\partial_i(f \circ (T|_{\widetilde{U}_{b,\lambda}})^{-1}) = \sum_{l=1}^{d} (J_l \cdot \partial_l f) \circ (T|_{\widetilde{U}_{b,\lambda}})^{-1} \quad \text{on } U_\lambda,$$

where J_l is a polynomial function of $\partial_i T_j$ $(1 \leq i, j \leq d)$ and $(\det DT)^{-1}$. Applying this formula repeatedly, we get that, for each multi-index α with $|\alpha| \leq r$,

$$\partial^\alpha(f \circ (T|_{\widetilde{U}_{b,\lambda}})^{-1}) = \sum_{|\beta|\leq|\alpha|} (J_{\alpha,\beta} \cdot \partial^\beta f) \circ (T|_{\widetilde{U}_{b,\lambda}})^{-1} \quad \text{on } U_\lambda,$$

where $J_{\alpha,\beta} = J^T_{\alpha,\beta}$ is a polynomial function of $\partial^\gamma T_j$ $(1 \leq j \leq d,\ |\gamma| \leq |\beta|)$ and $(\det DT)^{-1}$. Now (A.3) immediately follows from (A.1), and this completes the proof.

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