J. Aust. Math. Soc. (First published online 2024), page 1 of 38*
doi:10.1017/S1446788723000381
*Provisional—final page numbers to be inserted when paper edition is published

RADU GROUPS ACTING ON TREES ARE CCR

LANCELOT SEMAL

(Received 7 March 2023; accepted 8 December 2023)

Communicated by George Willis

Abstract

We classify the irreducible unitary representations of closed simple groups of automorphisms of trees acting 2-transitively on the boundary and whose local action at every vertex contains the alternating group. As an application, we confirm Claudio Nebbia's CCR conjecture on trees for (d_0, d_1)-semi-regular trees such that $d_0, d_1 \in \Theta$, where Θ is an asymptotically dense set of positive integers.

2020 *Mathematics subject classification*: primary 22D12; secondary 20E08, 57M07.

Keywords and phrases: unitary representations, groups of automorphisms of semi-regular trees, type I groups, Nebbia's conjecture.

1. Introduction

In this document, topological groups are second-countable, locally compact groups are Hausdorff and the word 'representation' stands for a strongly continuous unitary representation on a separable complex Hilbert space. A locally compact group G is called **CCR** if the operator $\pi(f)$ is compact for all irreducible representations π of G and all $f \in L^1(G)$. For totally disconnected locally compact groups, this property is equivalent to the requirement that every irreducible representation of G is admissible; see [Neb99]. We recall that an irreducible representation π of a totally disconnected locally compact group G is **admissible** if for every compact open subgroup $K \leq G$, the space $\mathcal{H}_\pi^K$ of K-invariant vectors is finite dimensional. A very important property of CCR groups is that they are **type I** groups [BdlH20, Definition 6.E.7. and Proposition 6.E.11]. Loosely speaking, type I groups are the locally compact groups all of whose representations can be written as unique direct integrals of irreducible representations, thus reducing the study of arbitrary representations to considerations of irreducible representations. Concerning groups of automorphisms of trees, Nebbia's work highlighted surprising relations between the action on the boundary and the

The author is an F.R.S.-FNRS Research Fellow.

regularity of representation theory. To be more precise, he showed in [Neb99] that any closed unimodular CCR vertex-transitive subgroup $G \leq \mathrm{Aut}(T)$ of the group of automorphisms of a regular tree T necessarily acts transitively on the boundary ∂T. Further progress going in that direction was recently achieved by Houdayer and Raum [HR19] and, at a higher level of generality, by Caprace *et al.* [CKM23]. Among other things, they showed that a closed nonamenable type I subgroup acting minimally on a locally finite tree T acts 2-transitively on the boundary ∂T [CKM23, Corollary D]. Going in the other direction, Nebbia conjectured in [Neb99] that any closed subgroup of automorphisms of a regular tree acting transitively on the boundary is CCR. His conjecture naturally extends to the case of semi-regular trees.

CONJECTURE (CCR conjecture on trees [Neb99]). *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 3$ and let $G \leq \mathrm{Aut}(T)$ be a closed subgroup acting transitively on the boundary of T. Then G is CCR.*

We recall from [BM00, Lemma 3.1.1] that, for a locally finite tree T, closed subgroups $G \leq \mathrm{Aut}(T)$ are noncompact and act transitively on the boundary ∂T if and only if they act 2-transitively on ∂T. Furthermore, the existence of such a group implies that the tree is semi-regular. In particular, since compact groups are automatically CCR, the hypothesis of semi-regularity is not restrictive in the conjecture.

One of the first pieces of evidence supporting the conjecture was provided by Bernstein and Harish-Chandra's works. Among other things, they proved that rank one semi-simple algebraic groups over local-fields are uniformly admissible [Ber74, HC70]. We recall that a totally disconnected locally compact group G is **uniformly admissible** if for every compact open subgroup K, there exists a positive integer k_K such that $\dim(\mathcal{H}_\pi^K) < k_K$ for all irreducible representations π of G. In particular, uniformly admissible groups are CCR. Concerning nonlinear groups, the conjecture was supported by the complete classification of the irreducible representations of the full group of automorphisms of a semi-regular tree and more generally of closed subgroups acting transitively on the boundary and satisfying the Tits independence property [Ama03, Cho94, FTN91, Ol'77, Ol'82] (these classifications lead to the conclusion that they are uniformly admissible [Cio15]).

Our paper concerns closed subgroups acting 2-transitively on the boundary ∂T and whose local action at each vertex v contains the alternating group of corresponding degree. We recall that for each vertex $v \in V(T)$, the stabiliser $\mathrm{Fix}_G(v)$ of v acts on the set $E(v)$ of edges containing v. The image of $\mathrm{Fix}_G(v)$ in $\mathrm{Sym}(E(v))$ for this natural projection map is called the **local action** of G at v and we denote this group by $\underline{G}(v)$. When the degree of each vertex is greater than 6, these groups of automorphisms of trees have been extensively studied and classified by Radu in [Rad17]. For this reason, we call them **Radu groups**. It is not hard to realise that these groups are type I. Indeed, each Radu group G contains a cocompact subgroup H that is conjugate in $\mathrm{Aut}(T)$ to the semi-regular version of the universal group of Burger–Mozes $\mathrm{Alt}_{(i)}(T)^+$; see [Rad17, page 4]. Since H is both open and cocompact in G, [Kal70, Theorem 1] ensures that G is type I if and only if H is type I. However, when the degree of each

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

vertex is greater than 4, $\mathrm{Alt}_{(i)}(T)^+$ acts transitively on the boundary and satisfies the Tits independence property. It follows from [Ama03, Cio15] that H is a type I group, which proves that every Radu group is type I. The purpose of these notes is to go further. Inspired by Ol'shanskii's work and the recent progress achieved on the abstraction of his framework [Sem23], we give a classification of the irreducible representations of simple Radu groups and deduce a description of the irreducible representations of any Radu group. Among other things, we provide the following contribution to Nebbia's CCR conjecture.

THEOREM A. *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 6$. Then, Radu groups are uniformly admissible and hence CCR.*

To put this result into the perspective of Radu's paper, we recall that the local action $\underline{G}(v) \leq \mathrm{Sym}(E(v))$ at every vertex $v \in T$ of a closed subgroup $G \leq \mathrm{Aut}(T)$ that is 2-transitive on the boundary is a 2-transitive subgroup of $\mathrm{Sym}(E(v))$ [BM00, Lemma 3.1.1]. However, [Rad17, Proposition B.1 and Corollary B.2] ensure that

$$\Theta = \{d \geq 6 \mid \text{each finite 2-transitive subgroup of } \mathrm{Sym}(d) \text{ contains } \mathrm{Alt}(d)\}$$

is asymptotically dense in $\mathbb{N}$ and its ten smallest elements are 34, 35, 39, 45, 46, 51, 52, 55, 56 and 58. All together, this implies the following.

THEOREM B. *Nebbia's CCR conjecture on trees is confirmed for any (d_0, d_1)-semi-regular tree with $d_0, d_1 \in \Theta$ where Θ is the asymptotically dense subset of $\mathbb{N}$ defined above.*

We now come back to the classification of the irreducible representations of simple Radu groups. We recall that the irreducible representations of a closed automorphism group $G \leq \mathrm{Aut}(T)$ of a locally finite tree T split into three categories. An irreducible representation π of G is called:

- **spherical** if there exists a vertex $v \in V(T)$ such that π admits a nonzero $\mathrm{Fix}_G(v)$-invariant vector where $\mathrm{Fix}_G(v) = \{g \in G \mid gv = v\}$;
- **special** if it is not spherical and there exists an edge $e \in E(T)$ such that π admits a nonzero $\mathrm{Fix}_G(e)$-invariant vector where $\mathrm{Fix}_G(e) = \{g \in G \mid gv = v \text{ for all } v \in e\}$ is the fixator of the edge e;
- **cuspidal** if for every $e \in E(T)$, π does not admit a nonzero $\mathrm{Fix}_G(e)$-invariant vector.

The spherical and special representations are classified since the end of the 1970s at the level of generality of the conjecture, that is, for any closed noncompact subgroup $G \leq \mathrm{Aut}(T)$ acting transitively on the boundary of the tree; see [Mat77, Ol'77, Ol'82]. Furthermore, we recall that Matsumoto's work emphasises a strong connection between these kinds of representations and the irreducible representations of Hecke algebras. To be more precise, we recall that a group acting 2-transitively on the boundary is either type-preserving or admits a closed type-preserving subgroup of index 2 acting 2-transitively on the boundary. Since [CC15, Corollary 3.6] ensures that every closed type-preserving subgroup G acting 2-transitively on the boundary comes from a B–N pair, every spherical and every special representation of G also provide an irreducible

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

representation of the associated Hecke algebra $C_c(B\backslash G/B)$ of continuous compactly supported B-bi-invariant functions $f : G \to \mathbb{C}$, where $B = \mathrm{Fix}_G(e)$ is the pointwise fixator of an edge $e \in E(T)$. Matsumoto's works revealed that this correspondence is actually bijective; see [Mat77, Ch. 5, Section 6].

The cuspidal representations, however, are not classified at the level of generality of the conjecture. Nevertheless, a classification of the cuspidal representations was achieved for certain families of groups. Concerning nonlinear groups, Ol'shanskii obtained such a classification for any closed subgroup $G \leq \mathrm{Aut}(T)$ satisfying the Tits independence property by exploiting the independence of the action on the tree to deduce a particular factorisation of the compact open subgroups; see [Ama03, Ol'77]. Our paper takes advantage of the recent abstraction of his framework given by the notion of **Ol'shanskii's factorisation** (see [Sem23] or Definition 4.3 below) and the description of Radu groups [Rad17] to obtain a classification for their cuspidal representations. We start by considering a family of groups $G_{(i)}^{+}(Y_0, Y_1)$ indexed by finite subsets $Y_0, Y_1 \subseteq \mathbb{N}$; see Definition 3.3. It is shown in [Rad17] that these groups are abstractly simple when T is a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 4$ and that they exhaust the list of **simple Radu** groups when $d_0, d_1 \geq 6$. Furthermore, by exploiting the fact that these groups are determined by suitable local conditions, [Sem23] leads to the following result.

THEOREM C. *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 4$. The cuspidal representations of $G_{(i)}^{+}(Y_0, Y_1)$ are in bijective correspondence given by induction with a family of irreducible representations of compact open subgroups. This correspondence is explicitly given by Theorem 4.12.*

Among other things, this theorem proves that the cuspidal representations of $G_{(i)}^{+}(Y_0, Y_1)$ are square-integrable. In the light of [HC70, Corollary of Theorem 2] and of the classification of spherical and special representations recalled in Section 2, this proves that the groups $G_{(i)}^{+}(Y_0, Y_1)$ are uniformly admissible, and hence CCR (the details are gathered in Section 5). To put these results into the perspective of Radu's classification, we recall that every Radu group G belongs to a finite chain $H_n \geq \cdots \geq H_0$ with $n \in \{0, 1, 2, 3\}$ such that $H_n = G$, $[H_t : H_{t-1}] = 2$ for all t and H_0 is conjugate in the group of type-preserving automorphisms $\mathrm{Aut}(T)^+$ to one of these $G_{(i)}^{+}(Y_0, Y_1)$ when $d_0, d_1 \geq 6$. However, Mackey's machinery allows one to describe the irreducible representation of a locally compact G in terms of the irreducible representations of any of its closed subgroups of index 2. In particular, when $d_0, d_1 \geq 6$, we obtain a description of the cuspidal representations of any Radu group from the cuspidal representations of the groups $G_{(i)}^{+}(Y_0, Y_1)$. We also deduce from Mackey's machinery that every Radu group is uniformly admissible. The author would like to underline that an application of Ol'shanskii's framework to Radu groups is already presented in [Sem23, Section 4] since these groups satisfy a generalisation of the Tits independence property (the property IP_k introduced in [BEW15]). However, except when the property IP_k is the Tits independence property ($k = 1$), the approach adopted in [Sem23] never leads to a classification of all the cuspidal representations.

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

1.1. Structure of the paper. In Section 2, we recall the classification of **spherical** and **special** representations of any closed noncompact subgroup $G \leq \mathrm{Aut}(T)$ acting transitively on the boundary; see [Mat77, Ol'77, Ol'82]. The purpose of Section 3 is to recall Radu's classification of **Radu groups** [Rad17] and the definition of $G^+_{(i)}(Y_0, Y_1)$. In Section 4, we recall the notion of **Ol'shanskii's factorisation** developed in [Sem23] and obtain a classification of the **cuspidal** representations of $G^+_{(i)}(Y_0, Y_1)$. The complete classification of the irreducible representations of $G^+_{(i)}(Y_0, Y_1)$ resulting from Sections 2 and 4 is then used in Section 5 to prove **uniform admissibility**. Finally, the purpose of the Appendix is to recall the details of the correspondence given by Mackey's machinery between the irreducible representations of a locally compact G and the irreducible representations of any of its closed subgroups of index 2. In particular, the Appendix provides a way to obtain the irreducible representations of any Radu group from the irreducible representations of the abstractly simple Radu groups $G^+_{(i)}(Y_0, Y_1)$. We also deduce from the Appendix that all Radu groups are uniformly admissible when $d_0, d_1 \geq 6$.

2. Spherical and special representations

Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 3$. We recall that a tree T is called (d_0, d_1)-**semi-regular** if there exists a bipartition $V(T) = V_0 \sqcup V_1$ of T such that every vertex of V_i has degree d_i and every edge of T contains exactly one vertex in each V_i. As explained in Section 1, the irreducible representations of any closed subgroup $G \leq \mathrm{Aut}(T)$ of the group of automorphisms of such a tree split into three categories. These representations are either **spherical**, **special** or **cuspidal**. The purpose of the present section is to recall the classification of spherical and special representations of any closed noncompact subgroup $G \leq \mathrm{Aut}(T)$ acting transitively on the boundary (in particular of any Radu group). This classification is a classical result known since the end of the 70's and we claim no originality. We refer to [Mat77, Ol'77, FTN91] for details.

The details of this classification are gathered in Theorems 2.2, 2.3, 2.5 below but we start with some preliminaries. Given a locally compact group G and a compact subgroup $K \leq G$, we say that (G, K) is a **Gelfand pair** if the convolution algebra $C_c(K\backslash G/K)$ of compactly supported, continuous K-bi-invariant functions on G is commutative. Let (G, K) be a Gelfand pair and let μ be the left-Haar measure of G renormalised so that $\mu(K) = 1$. A function $\varphi : G \to \mathbb{C}$ is called K**-spherical** if it is a K-bi-invariant continuous function with $\varphi(1_G) = 1$ and such that

$$\int_K \varphi(gkg')\, d\mu(k) = \varphi(g)\varphi(g') \quad \forall g, g' \in G.$$

THEOREM 2.1 [Lan85, Ch. IV, Section 3, Theorems 3 and 9]. *Let (G, K) be a Gelfand pair. For every irreducible representation π of G we have that* $\dim(\mathcal{H}^K_\pi) \leq 1$. *Furthermore, there is a bijective correspondence $\pi \to \varphi_\pi$ with inverse map given by the GNS construction between the equivalence classes of irreducible representations*

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

of G with nonzero K-invariant vectors and the K-spherical functions of positive type on G (the function φ_π is the function $\varphi_\pi(g) = \langle \pi(g)\xi, \xi\rangle$ corresponding to any unit vector $\xi \in \mathcal{H}_\pi^K$).

We now recall the details of the classification of spherical and special representations for any noncompact closed subgroup $G \leq \mathrm{Aut}(T)$ acting transitively on the boundary of T. We recall that these groups have either one or two orbits of vertices and we treat these cases separately.

THEOREM 2.2 [FTN91, Ch. II]. *Let T be a d-regular tree, let $v \in V(T)$ and let $G \leq \mathrm{Aut}(T)$ be a closed noncompact subgroup acting transitively on the vertices of T and the boundary ∂T. Then, $(G, \mathrm{Fix}_G(v))$ is a Gelfand pair and every spherical representation of G admits a nonzero $\mathrm{Fix}_G(v)$-invariant vector. Furthermore, the equivalence classes of spherical representations of G are in bijective correspondence with the interval $[-1; 1]$ via the map $\phi_v : \pi \mapsto \varphi_\pi(\tau_v)$ where τ_v is any element of G such that $d(\tau_v v, v) = 1$ and φ_π is the unique $\mathrm{Fix}_G(v)$-spherical function of positive type attached with π. Under this correspondence, the trivial representation corresponds to 1.*

The following theorem comes from [Mat77] but is formulated differently for the sake of consistency.

THEOREM 2.3 [Mat77, Ch. 5, Section 6]. *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 3$, let $v \in V(T)$, let v' be any vertex at distance one from v and let $G \leq \mathrm{Aut}(T)^+$ be a closed noncompact subgroup of type-preserving automorphisms acting transitively on the boundary ∂T. Then, there is exactly one spherical representation π_v of G with a nonzero $\mathrm{Fix}_G(v)$-invariant vector but no nonzero $\mathrm{Fix}_G(v')$-invariant vector. Furthermore, $(G, \mathrm{Fix}_G(v))$ is a Gelfand pair and apart from the two exceptional representations π_v and $\pi_{v'}$, every spherical representation of G admits a nonzero $\mathrm{Fix}_G(w)$-invariant vector for all $w \in V(T)$. In addition, the equivalence classes of spherical representations admitting a nonzero $\mathrm{Fix}_G(v)$-invariant vector are in bijective correspondence with the interval $[-1/(d'-1); 1]$ via the map $\phi_v : \pi \mapsto \varphi_\pi(\tau_v)$ where τ_v is an element of G such that $d(\tau_v v, v) = 2$ and d' is the degree of v'. Under this correspondence, the exceptional spherical representation π_v corresponds to $-1/(d'-1)$ and the trivial representation corresponds to 1. Finally, if π is a nonexceptional spherical representation of G,*

$$\phi_{v'}(\pi) = \frac{d(d'-1)}{d'(d-1)}\phi_v(\pi) + \frac{d-d'}{d'(d-1)}.$$

REMARK 2.4. The author wishes to point out that certain extrema of the real interval corresponding to spherical functions of positive types found in the literature are incorrect when G has two orbits of vertices. The incorrect values provided by [Ama03, BK88] for instance are based on earlier inaccuracies [IP83] that were already pointed out in [CMoS94, Remark 2, page 243]. The values provided in Theorem 2.3 can be computed from [Mat69].

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

To describe the special representations, let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 3$, let $e \in E(T)$ and let $G \leq \mathrm{Aut}(T)$ be an edge-transitive closed subgroup acting transitively on the boundary ∂T. We define $\mathcal{L}(e)$ as the subspace of $\mathrm{Fix}_G(e)$-right invariant square-integrable functions $\varphi : G \to \mathbb{C}$ satisfying

$$\int_{\mathrm{Fix}_G(v)} \varphi(gk)\, d\mu(k) = 0 \quad \forall g \in G \quad \forall v \in e.$$

Notice that $\mathcal{L}(e)$ is a closed left-invariant subspace of $L^2(G)$ and let σ be the unitary representation of G defined by $\sigma(t)\varphi(g) = \varphi(t^{-1}g)$ for all $g, t \in G$, for all $\varphi \in \mathcal{L}(e)$. If G is transitive on the vertices of T, we choose an inversion $h \in G$ of the edge e and consider the linear map $\nu : \mathcal{L}(e) \to \mathcal{L}(e)$ defined by $\nu(\varphi)(g) = \varphi(gh)$ for all $\varphi \in \mathcal{L}(e)$, for all $g \in G$. This map is well defined since for all $\varphi \in \mathcal{L}(e)$, for all $g \in G$, for all $v \in e$,

$$\begin{aligned}\int_{\mathrm{Fix}_G(v)} (\nu\varphi)(gk)\, d\mu(k) &= \int_{\mathrm{Fix}_G(v)} \varphi(gkh)\, d\mu(k) \\ &= \int_{\mathrm{Fix}_G(v)} \varphi(ghh^{-1}kh)\, d\mu(k) \\ &= \int_{\mathrm{Fix}_G(h^{-1}v)} \varphi(ghk)\, d\mu(k) = 0.\end{aligned}$$

On the other hand, since every element of $\mathcal{L}(e)$ is $\mathrm{Fix}_G(e)$-right invariant, note that ν is an involution and that it does not depend on our choice of inversion of the edge e. For each $\epsilon \in \{-1, 1\}$, let $\mathcal{L}(e)_\epsilon$ be the eigenspace of ν corresponding to the eigenvalue ϵ and let σ^ϵ be the restriction of σ to $\mathcal{L}(e)_\epsilon$. We are now ready to state the classification of special representations.

THEOREM 2.5 [FTN91, Ch. III, Section 2], [Mat77, Section 5.6]. *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 3$, let $e \in E(T)$ and let $G \leq \mathrm{Aut}(T)$ be a closed noncompact subgroup acting transitively on the boundary ∂T. Every special representation of G is square-integrable and admits a $\mathrm{Fix}_G(f)$-invariant vector for every $f \in E(T)$. Furthermore:*

(1) *If G acts transitively on $V(T)$, $(\sigma^{-1}, \mathcal{L}(e)_{-1})$ and $(\sigma^{+1}, \mathcal{L}(e)_{+1})$ are representatives of the two equivalence classes of special representations.*
(2) *If G has two orbits on $V(T)$, $(\sigma, \mathcal{L}(e))$ is a representative of the unique equivalence class of special representations.*

3. The Radu groups

Let T be a (d_0, d_1) semi-regular tree with $d_0, d_1 \geq 4$ and associated bipartition $V(T) = V_0 \sqcup V_1$, and let $\mathrm{Aut}(T)^+$ denote the group of type-preserving automorphisms of T that is the set of automorphisms of T leaving V_0 and V_1 invariants. The purpose of this section is to recall the classification of Radu groups [Rad17]. For this purpose,

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

we set

$$\mathcal{H}_T = \{G \leq \mathrm{Aut}(T) \mid G \text{ is closed and 2-transitive on } \partial T\}$$

and

$$\mathcal{H}_T^+ = \{G \leq \mathrm{Aut}(T)^+ \mid G \text{ is closed and 2-transitive on } \partial T\}.$$

If $d_0 \neq d_1$, notice that every automorphism of T is type-preserving, so $\mathcal{H}_T^+ = \mathcal{H}_T$. We recall that for each vertex $v \in V(T)$, the stabiliser $\mathrm{Fix}_G(v)$ of v acts on the set $E(v)$ of edges containing v, and that the image of $\mathrm{Fix}_G(v)$ in $\mathrm{Sym}(E(v))$ for this projection map (which we denote by $\underline{G}(v)$) is called the **local action** of G at v. Furthermore, we recall in the light of [BM00, Lemma 3.1.1], that every group $G \in \mathcal{H}_T^+$ is transitive on V_0 and V_1. Thus, all the groups $\underline{G}(v)$ with $v \in V_0$ (respectively $v \in V_1$) are permutation isomorphic to the same group $F_0 \leq \mathrm{Sym}(d_0)$ (respectively $F_1 \leq \mathrm{Sym}(d_1)$). A group $G \in \mathcal{H}_T$ such that $\underline{G}(v) \cong F_t \geq \mathrm{Alt}(d_t)$ for each vertex $v \in V_t(T)$ and for $t \in \{0, 1\}$ is called a **Radu group**. For each vertex $v \in V(T)$ and each positive integer $r \in \mathbb{N}$ let

$$S(v, r) = \{w \in V(T) \mid d(v, w) = r\}$$

be the set of vertices of T at distance r from v.

DEFINITION 3.1. A **legal colouring** $i : V(T) \to \mathbb{N}$ of T is the concatenation of a pair of maps

$$i_0 : V_0 \to \{1, \ldots, d_1\} \quad \text{and} \quad i_1 : V_1 \to \{1, \ldots, d_0\}$$

such that $i_0|_{S(v,1)} : S(v, 1) \to \{1, \ldots, d_1\}$ and $i_1|_{S(w,1)} : S(w, 1) \to \{1, \ldots, d_0\}$ are bijections for all $v \in V_1$ and $w \in V_0$.

Given a legal colouring i of T and an automorphism $g \in \mathrm{Aut}(T)$, the **local action** of g at a vertex $v \in V(T)$ is defined as

$$\sigma_{(i)}(g, v) = i|_{S(gv,1)} \circ g \circ (i|_{S(v,1)})^{-1} \in \begin{cases} \mathrm{Sym}(d_0) & \text{if } v \in V_0 \\ \mathrm{Sym}(d_1) & \text{if } v \in V_1. \end{cases}$$

REMARK 3.2. If $d_0 = d_1$, the tree T is a regular tree and this notion of legal colouring and local action of an element is different from the notion of legal colouring and local action used to define the universal Burger–Mozes groups in [BM00]. Indeed, with our definition, the closed subgroup $G \leq \mathrm{Aut}(T)$ of all automorphisms of trees $g \in G$ such that $\sigma_{(i)}(g, v) = \mathrm{id}$ for all $v \in V$ is not transitive on the set of vertices of T (not even transitive on V_0). It is good to note, however, that this colouring approach is compatible with that of Smith [Smi17].

Now, let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 4$ and let i be a legal colouring of T. For each vertex $v \in V(T)$ and each finite set $Y \subseteq \mathbb{N}$ let

$$S_Y(v) = \bigcup_{r \in Y} S(v, r)$$

and for each set of vertices $B \subseteq V(T)$ let

$$\mathrm{Sgn}_{(i)}(g, B) = \prod_{w \in B} \mathrm{sgn}(\sigma_{(i)}(g, w))$$

where $\mathrm{sgn}(\sigma_{(i)}(g, w))$ is the sign of the local action $\sigma_{(i)}(g, w)$ of the automorphism g at w for the legal colouring i.

DEFINITION 3.3. For all (possibly empty) finite sets Y_0, Y_1 of $\mathbb{N}$ and every legal colouring i of T, we set

$$G^+_{(i)}(Y_0, Y_1) = \left\{ g \in \mathrm{Aut}(T)^+ \,\middle|\, \begin{array}{l} \mathrm{Sgn}_{(i)}(g, S_{Y_0}(v)) = 1 \text{ for each } v \in V_{t_0} \\ \mathrm{Sgn}_{(i)}(g, S_{Y_1}(v)) = 1 \text{ for each } v \in V_{t_1} \end{array} \right\},$$

where $t_0 = \max(Y_0) \pmod 2$, $t_1 = (1 + \max(Y_1)) \pmod 2$ and $\max(\varnothing) = 0$.

REMARK 3.4. The choices of t_0 and t_1 are made in such a way that the vertices of $S_{Y_0}(v)$ with $v \in V_{t_0}$ at maximal distance from v and the vertices of $S_{Y_1}(w)$ with $w \in V_{t_1}$ at maximal distance from w have opposite types.

Note that $G^+_{(i)}(\varnothing, \varnothing) = \mathrm{Aut}(T)^+$ is the full group of type-preserving automorphisms and that $G^+_{(i)}(\{0\}, \{0\})$ is a subgroup of each $G^+_{(i)}(Y_0, Y_1)$. Furthermore, if T is a d-regular tree note that $G^+_{(i)}(\{0\}, \{0\})$ is conjugate to $U(\mathrm{Alt}(d))^+$ where $G^+ = G \cap \mathrm{Aut}(T)^+$ and $U(\mathrm{Alt}(d))$ is the universal Burger–Mozes group of the alternating group see [BM00]. As we recall below, when $d_0, d_1 \geq 6$, every simple Radu group is of the form $G^+_{(i)}(Y_0, Y_1)$ for some finite $Y_0, Y_1 \subseteq \mathbb{N}$ and some legal colouring i of T. Furthermore, every Radu group G belongs to a finite chain $H_n \geq \cdots \geq H_0$ with $n \in \{0, 1, 2, 3\}$ such that $H_n = G$, $[H_t : H_{t-1}] = 2$ for all t and H_0 is conjugate in $\mathrm{Aut}(T)^+$ to one of these $G^+_{(i)}(Y_0, Y_1)$. In particular, Mackey's machinery (see the Appendix) allows one to obtain a description of the irreducible representations of any Radu group from the irreducible representations of $G^+_{(i)}(Y_0, Y_1)$ and vice versa. We now recall more precisely the results proved in [Rad17] that are used in this paper.

THEOREM [Rad17, Theorem A]. *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 4$ and let i be a legal colouring of T. Then, for all finite subsets $Y_0, Y_1 \subseteq \mathbb{N}$ the group $G^+_{(i)}(Y_0, Y_1)$ belongs to $\mathcal{H}^+_T$ and is abstractly simple.*

To formulate the following results we let $G^{(\infty)}$ denote the intersection of all normal cocompact closed subgroups of the locally compact group G. Furthermore, we recall from [BM00, Proposition 3.1.2] that $H^{(\infty)}$ belongs to $\mathcal{H}^+_T$ and is topologically simple for all $H \in \mathcal{H}^+_T$ (in our cases, these groups are even abstractly simple). Finally, we let $\mathcal{G}^+_T(i)$ be the set of groups $G^+_i(Y_0, Y_1)$ with nonempty finite $Y_0, Y_1 \subseteq \mathbb{N}$ such that $y = \max(Y_t) \pmod 2$ for each $y \in Y_t$ with $y \geq \max(Y_{1-t})$ $(t \in \{0, 1\})$.

THEOREM [Rad17, Theorem B]. *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 6$, let i be a legal colouring, and let $G \in \mathcal{H}^+_T$ be such that $\underline{G}(v) \cong F_0 \geq \mathrm{Alt}(d_0)$ and $\underline{G}(w) \cong F_1 \geq \mathrm{Alt}(d_1)$ for each $v \in V_0$, $w \in V_1$. Then, $G^{(\infty)}$ is conjugate in $\mathrm{Aut}(T)^+$ to an element of $\mathcal{G}^+_T(i)$ and $[G : G^{(\infty)}] \in \{1, 2, 4\}$.*

When T is a d-regular tree, a similar result holds for all $G \in \mathcal{H}_T - \mathcal{H}_T^+$.

THEOREM [Rad17, Corollary C]. *Let T be a d-regular tree with $d \geq 6$ and let i be a legal colouring and let $G \in \mathcal{H}_T - \mathcal{H}_T^+$ be such that $\underline{G}(v) \cong F \geq \mathrm{Alt}(d)$ for each $v \in V(T)$. Then, $G^{(\infty)}$ is conjugate to $G_{(i)}^+(Y, Y)$ for some finite subset Y of $\mathbb{N}$ and $[G : G^{(\infty)}] \in \{2, 4, 8\}$.*

Finally, the following theorem follows from Radu's description of Radu groups.

THEOREM 3.5. *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 6$ and let i be a legal colouring of T. Every Radu group G is contained in a finite chain $H_n \geq \cdots \geq H_0$ with $n \in \{0, 1, 2, 3\}$ such that $H_n = G$, $[H_t : H_{t-1}] = 2$ for all t and H_0 is conjugate in $\mathrm{Aut}(T)^+$ to some $G_{(i)}^+(Y_0, Y_1)$.*

4. Cuspidal representations of the simple Radu groups

Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 4$ and let $V(T) = V_0 \sqcup V_1$ be the associated bipartition. Let i be a legal colouring of T and let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets. We recall from Section 3 that $G_{(i)}^+(Y_0, Y_1)$ (Definition 3.3) is a simple Radu group and that every simple Radu group is of this form when $d_0, d_1 \geq 6$. The purpose of this section is to describe the irreducible representations of $G_{(i)}^+(Y_0, Y_1)$ and to show that this group is uniformly admissible, hence CCR.

We recall from Section 1 that the irreducible representations of $G_{(i)}^+(Y_0, Y_1)$ split into three categories. These are either **spherical**, **special** or **cuspidal**. A classification of the spherical and special representations of any subgroup $G \leq \mathrm{Aut}(T)$ acting 2-transitively on the boundary is already given in Section 2. In particular, this classification applies to $G_{(i)}^+(Y_0, Y_1)$. Our current purpose is to give a description of the cuspidal representations of these groups. As announced in Section 1, our idea is to take advantage of the framework developed in [Sem23] and the description of these groups provided by Radu. The abstraction of Ol'shanskii's framework expressed in [Sem23, Theorem A] provides a way to classify the representations at certain 'depths' where the depth is a generalisation of the 'spherical/special/cuspidal' framework, that is, larger depth means that more compact open subgroups have no nonzero invariant vector. This framework as well as the associated notion of depth depends on the existence and choice (assuming the existence) of a basis of neighbourhoods of the identity satisfying a certain factorisation property. Such a basis is given in [Sem23] for groups of automorphisms of trees satisfying the property IP_k as defined in [BEW15] and yielding to a classification of their cuspidal representations at sufficiently large depth depending on k. Since every Radu group G is known to satisfy the property IP_k for some sufficiently large k depending on G, this classification applies to the Radu groups. However, this approach only leads to a description of all the cuspidal representations of $G_{(i)}^+(Y_0, Y_1)$ when $k = 1$, that is, when G satisfies the Tits independence property, because for that choice of basis of neighbourhoods of the identity, the set of the associated representations at depth 0 contains some cuspidal representations (increasingly many as k grows). The purpose of this section is to go beyond this result by exhibiting

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

the basis of neighbourhoods of the identity of the group $G^+_{(i)}(Y_0, Y_1)$ that satisfies the factorisation property at all positive depths, and whose associated representations at depth 0 are exactly the spherical and special representations. This is done by relying on the local conditions given by the Definition 3.3 provided by Radu's work rather than the property IP_k and leads to a complete description of the cuspidal representations of $G^+_{(i)}(Y_0, Y_1)$ rather than a partial one; see Section 4.4.

4.1. Preliminaries. We start by recalling the axiomatic framework developed in [Sem23] (see that paper for details). Let G be a unimodular totally disconnected locally compact group, let $\mathcal{B}$ denote the set of compact open subgroups of G, $P(\mathcal{B})$ denote the power set of $\mathcal{B}$ and let

$$C : \mathcal{B} \to P(\mathcal{B})$$

be the map sending a compact open subgroup to its conjugacy class in G. Let $\mathcal{S}$ be a basis of neighbourhoods of the identity consisting of compact open subgroups of G and let $\mathcal{F}_{\mathcal{S}} = \{C(U) \mid U \in \mathcal{S}\}$. We equip $\mathcal{F}_{\mathcal{S}}$ with the partial order given by the reverse inclusion of representatives ($C(U) \leq C(V)$ if there exists $\tilde{U} \in C(U)$ and $\tilde{V} \in C(V)$ such that $\tilde{V} \subseteq \tilde{U}$). For a poset $(P, \leq)$ and an element $x \in P$, we recall that the **height** of x in $(P, \leq)$ is $L_x - 1$, where L_x is the maximal length of a strictly increasing chain in $P_{\leq x} = \{y \in P \mid y \leq x\}$ if such a maximal length exists and we say that the height is infinite otherwise.

Definition 4.1. A basis of neighbourhoods of the identity $\mathcal{S}$ consisting of compact open subgroups of G is called a **generic filtration** of G if the height of every element in $\mathcal{F}_{\mathcal{S}}$ is finite.

Remark 4.2. Every unimodular totally disconnected locally compact group has a generic filtration, namely the compact open subgroups U with $\mu(U) \leq 1$ for some Haar measure μ.

Every generic filtration $\mathcal{S}$ of G splits as a disjoint union $\mathcal{S} = \bigsqcup_{l \in \mathbb{N}} \mathcal{S}[l]$, where $\mathcal{S}[l]$ denotes the set of elements $U \in \mathcal{S}$ such that $C(U)$ has height l in $\mathcal{F}_{\mathcal{S}}$. The elements of $\mathcal{S}[l]$ are called the elements at **depth** l. Since $\mathcal{S}$ is a basis of neighbourhoods of the identity consisting of compact open subgroups of G, notice that for every irreducible representation π of G, there exists a group $U \in \mathcal{S}$ such that π admits a nonzero U-invariant vector. In particular, for every irreducible representation π of G, there exists a smallest nonnegative integer $l_\pi \in \mathbb{N}$ such that π admits nonzero U-invariant vectors for some $U \in \mathcal{S}[l_\pi]$. This l_π is called the **depth** of π with respect to $\mathcal{S}$. The key notion developed in [Sem23] is the notion of **factorisation** at depth l for a generic filtration $\mathcal{S}$, which we now recall.

Definition 4.3. Let G be a nondiscrete unimodular totally disconnected locally compact group, let $\mathcal{S}$ be a generic filtration of G and let l be a strictly positive integer. We say that $\mathcal{S}$ **factorises at depth** l if the following conditions hold.

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

(1) For all $U \in \mathcal{S}[l]$ and every V in the conjugacy class of an element of $\mathcal{S}$ such that $V \nsubseteq U$, there exists W in the conjugacy class of an element of $\mathcal{S}[l-1]$ such that

$$U \subseteq W \subseteq VU = \{vu \mid u \in U, v \in V\}.$$

(2) For all $U \in \mathcal{S}[l]$ and every V in the conjugacy class of an element of $\mathcal{S}$, the set

$$N_G(U, V) = \{g \in G \mid g^{-1}Vg \subseteq U\}$$

is compact.

Furthermore, the generic filtration $\mathcal{S}$ of G is said to **factorise$^+$ at depth** l if in addition for all $U \in \mathcal{S}[l]$ and every W in the conjugacy class of an element of $\mathcal{S}[l-1]$ such that $U \subseteq W$,

$$W \subseteq N_G(U, U) = \{g \in G \mid g^{-1}Ug \subseteq U\}.$$

Since G is unimodular, notice that the set $N_G(U, U)$ coincides with the normaliser $N_G(U)$ of U in G.

The relevance of this notion is given by [Sem23, Theorem A], which leads to a description of the irreducible representations at height l in terms of a family of irreducible representations of finite groups called $\mathcal{S}$-standard representations (Definition 4.11 below), if the generic filtration factorises$^+$ at depth l.

4.2. Generic filtration for $G^+_{(i)}(Y_0, Y_1)$. Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 4$ and let $V(T) = V_0 \sqcup V_1$ be the associated bipartition. Let i be a legal colouring of T and let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets. The purpose of this section is to provide a generic filtration for $G^+_{(i)}(Y_0, Y_1)$. To do this, let $\mathfrak{T}_0$ be the family of subtrees of T defined by

$$\mathfrak{T}_0 = \{B_T(v, r) \mid v \in V(T), r \geq 1\} \sqcup \{B_T(e, r) \mid e \in E(T), r \geq 0\}$$

and consider the basis of neighbourhoods of the identity given by the fixators of these trees

$$\mathcal{S}_0 = \{\mathrm{Fix}_G(\mathcal{T}) \mid \mathcal{T} \in \mathfrak{T}_0\}.$$

DEFINITION 4.4. A group $G \leq \mathrm{Aut}(T)$ is said to satisfy the hypothesis H_0 if for all $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}_0$,

$$\mathrm{Fix}_G(\mathcal{T}') \leq \mathrm{Fix}_G(\mathcal{T}) \quad \text{if and only if } \mathcal{T} \subseteq \mathcal{T}'. \qquad (H_0)$$

LEMMA 4.5 [Sem23, Lemma 4.12]. *Let $G \leq \mathrm{Aut}(T)$ be a closed nondiscrete unimodular subgroup satisfying the hypothesis H_0. Then, $\mathcal{S}_0$ is a generic filtration of G and the sets $\mathcal{S}_0[l]$ can be described as follows:*

- *if l is even, $\mathcal{S}_0[l] = \{\mathrm{Fix}_G(B_T(e, \frac{l}{2})) \mid e \in E(T)\}$;*
- *if l is odd, $\mathcal{S}_0[l] = \{\mathrm{Fix}_G(B_T(v, ((l+1)/2))) \mid v \in V(T)\}$.*

We come back to our case $G = G^+_{(i)}(Y_0, Y_1)$.

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

LEMMA 4.6. *Let $G = G^+_{(i)}(Y_0, Y_1)$ and $\mathcal{T}$ be a complete finite subtree of T. Then, $\mathrm{Fix}_G(\mathcal{T})$ does not fix any vertices outside of $\mathcal{T}$. In particular, $G^+_{(i)}(Y_0, Y_1)$ satisfies the hypothesis (H_0).*

PROOF. Since $H = G^+_{(i)}(\{0\}, \{0\})$ is a subgroup of $G^+_{(i)}(Y_0, Y_1)$, it suffices to prove that $\mathrm{Fix}_G(\mathcal{T})$ does not fix any vertices outside of $\mathcal{T}$. If $\mathcal{T}$ has a single vertex, the result is trivial. Now, suppose that $\mathcal{T}$ contains at least one edge. For each leaf w of $\mathcal{T}$, consider the unique vertex v of T that is adjacent to w and let

$$T(v, w) = \{x \in V(T) : d_T(x, v) < d_T(x, w)\}.$$

By choice of w, one has that $\mathcal{T} \subseteq T(v, w) \cup \{w\}$ and since H satisfies Tits' independence property, $\mathrm{Fix}_H(T(v, w))$ has a local action $\mathrm{Alt}(d - 1)$ at w, where w has degree d. In particular, since $d \geq 4$, $\mathrm{Fix}_H(\mathcal{T}) \subseteq \mathrm{Fix}_H(T(v, w) \cup \{w\}) = \mathrm{Fix}_H(T(v, w))$ does not fix any neighbour of w other than v and thus does not fix any vertices outside of $T(v, w) \cup \{w\}$. Since any vertex of T that does not belong to $\mathcal{T}$ must be outside of one of these $T(v, w) \cup \{w\}$, it follows that $\mathrm{Fix}_H(\mathcal{T})$ does not fix any vertices outside of $\mathcal{T}$ and the result follows. □

In particular, Lemma 4.5 ensures that $\mathcal{S}_0$ is a generic filtration of $G^+_{(i)}(Y_0, Y_1)$.

4.3. Factorisation. Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 4$ and let $V(T) = V_0 \sqcup V_1$ be the associated bipartition. Let i be a legal colouring of T and let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets. We have shown in Section 4.2 that $\mathcal{S}_0$ is a generic filtration of $G^+_{(i)}(Y_0, Y_1)$. The purpose of the present section is to prove that this generic filtration factorises$^+$ at all depths $l \geq 1$.

We start with some notation that is used in the proof (see Figure 1). For any two distinct vertices $v, w \in V(T)$, let $[v, w]$ be the unique geodesic between v and w. Suppose that $d(v, w) = n$, let $v = v_0, v_1, \ldots, v_n = w$ be the sequence of vertices corresponding to $[v, w]$ in T, let

$$p_{[v,w]} : [v, w] - \{v\} \longrightarrow [v, w] : v_i \mapsto v_{i-1}$$

and let

$$\begin{aligned} T(v, w) &= \{x \in V(T) \mid d_T(x, p_{[v,w]}(w)) < d_T(x, w)\} \\ &= \{x \in V(T) \mid d_T(x, v_{n-1}) < d_T(x, w)\}. \end{aligned}$$

The following intermediate result is the key ingredient needed to prove the factorisation of the generic filtration $\mathcal{S}_0$ of $G^+_{(i)}(Y_0, Y_1)$ at all depths $l \geq 1$.

PROPOSITION 4.7. *For all $l, l' \in \mathbb{N}$ such that $l \geq 1$ and $l' \geq l$, for all U in the conjugacy class of an element of $\mathcal{S}_0[l]$ and every V in the conjugacy class of an element of $\mathcal{S}_0[l']$ such that $V \nsubseteq U$, there exists a subgroup $W \in \mathcal{S}_0[l - 1]$ such that $U \subseteq W \subseteq VU$.*

PROOF. To shorten the proof and to make the argument clearer, parts of the reasoning are proved in Lemmas 4.8 and 4.9 below. Since the proof is quite long and technical, we first give an idea of its structure. We begin the proof by identifying the group W

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

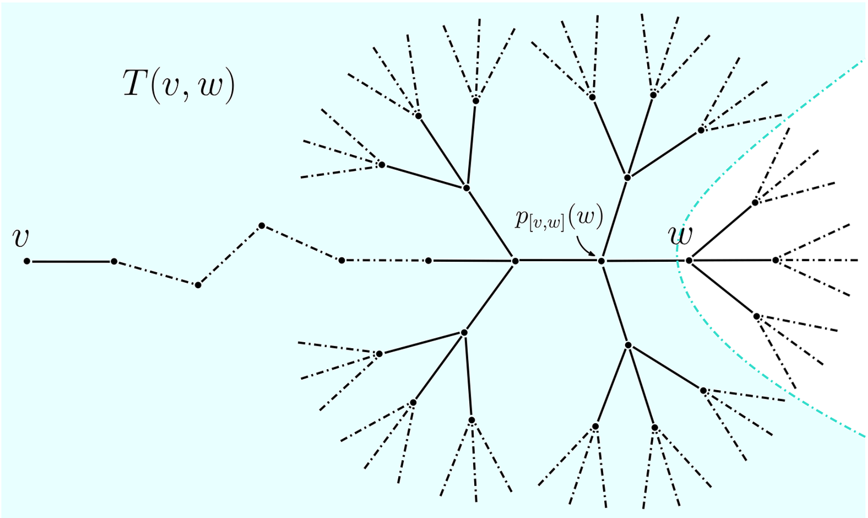

FIGURE 1. The set $T(v,w)$. Colour available online.

from U and V. We then prove that every element of W decomposes as a product of an element of V and an element of U. The proof of this decomposition is the technical part. It is achieved by a compactness argument taking advantage from the fact that $G_{(i)}^{+}(Y_0, Y_1)$ is defined by local action conditions.

Let $G = G_{(i)}^{+}(Y_0, Y_1)$. As announced at the beginning of the proof, we start by identifying W. Notice that $\mathfrak{T}_0$ is stable under the action of G. Furthermore, for every $g \in \mathrm{Aut}(T)^+$ and for every subtree $\mathcal{T}$ of T, we have that $g\mathrm{Fix}_G(\mathcal{T})g^{-1} = \mathrm{Fix}_G(g\mathcal{T})$. In particular, there exist $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}_0$ such that $U = \mathrm{Fix}_G(\mathcal{T})$ and $V = \mathrm{Fix}_G(\mathcal{T}')$. Since $V \not\subseteq U$, notice that $\mathcal{T} \not\subseteq \mathcal{T}'$. If l is even, Lemma 4.5 ensures that $\mathcal{T} = B_T(e, \frac{l}{2})$ for some edge $e \in E(T)$. Furthermore, since $\mathcal{T} \not\subseteq \mathcal{T}'$ and since $l' \geq l$, there exists a unique vertex $v \in e$ such that $\mathcal{T}' \subseteq T(v,w) \cup B_T(v, \frac{l}{2})$, where w denotes the other vertex of e. In this case, we let $\mathcal{T}_W = B_T(v, \frac{l}{2})$. However, if l is odd, Lemma 4.5 ensures that $\mathcal{T} = B_T(w, (l+1)/2)$ for some vertex $w \in V(T)$. Furthermore, since $\mathcal{T} \not\subseteq \mathcal{T}'$ and since $l' \geq l$, there exists a unique vertex $v \in B_T(w,1) - \{w\}$ such that $\mathcal{T}' \subseteq T(v,w) \cup B_T(\{v,w\}, (l-1)/2)$. In that case, we let $\mathcal{T}_W = B_T(\{v,w\}, (l-1)/2)$. In both cases, we set $W = \mathrm{Fix}_G(\mathcal{T}_W)$. Notice by construction that $W \in \mathcal{S}_0[l-1]$ and that $U \subseteq W$ (since $\mathcal{T}_W \subseteq \mathcal{T}$). Our purpose is therefore to show that $W \subseteq VU$. To this end, let $\alpha \in W$ and let us show the existence of an element $\alpha_0 \in U$ such that $\alpha|_{\mathcal{T}'} = \alpha_0|_{\mathcal{T}'}$. We start by explaining why the existence of α_0 settles the proof. Indeed, if α_0 exists, notice that the automorphism $\alpha_1 = \alpha_0^{-1} \circ \alpha$ is an element of G for which $\alpha_1|_{\mathcal{T}'} = id|_{\mathcal{T}'}$. In particular, we have that $\alpha_1 \in \mathrm{Fix}_G(\mathcal{T}')$, $\alpha_0 \in \mathrm{Fix}_G(\mathcal{T})$ and by construction $\alpha = \alpha_0 \circ \alpha_1$, which proves that $W \subseteq UV$. Applying the inverse map on both sides of the inclusion, we obtain that $W \subseteq VU$ which settles the proof.

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

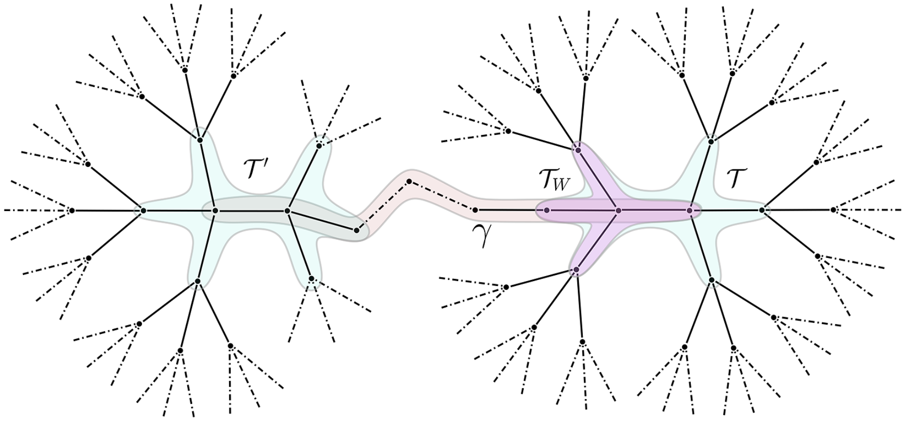

FIGURE 2. The tree $\mathcal{T}_W$ and the geodesic γ. Colour available online.

Now, let us prove the existence of α_0. As announced at the beginning of the proof, we are going to use a compactness argument taking advantage of the fact that $G_{(i)}^+(Y_0, Y_1)$ is defined by local actions conditions. More precisely, we define a descending chain of nonempty compact sets $\Omega_n \subseteq \mathrm{Aut}(T)^+$ and an increasing chain of finite subtrees $\mathcal{R}_n$ of T such that $T = \bigcup_{n\in\mathbb{N}} \mathcal{R}_n$ and such that for all $h \in \Omega_n$:

- $h \in \mathrm{Fix}_G(\mathcal{T})$ and $h|_{\mathcal{T}'} = \alpha|_{\mathcal{T}'}$;
- $\mathrm{Sgn}_{(i)}(h, S_{Y_0}(v)) = 1$ for all v in $V_{t_0} \cap \mathcal{R}_n$;
- $\mathrm{Sgn}_{(i)}(h, S_{Y_1}(v)) = 1$ for all v in $V_{t_1} \cap \mathcal{R}_n$.

We recall that in the above, $t_0 = \max(Y_0) \bmod 2$, $t_1 = (1 + \max(Y_1)) \pmod 2$ and $\max(\varnothing) = 0$. Let us first show that this settles the existence of α_0. Since the Ω_n form a descending chain of nonempty compact sets in a Hausdorff space, we obtain $\bigcap_{n\in\mathbb{N}} \Omega_n \neq \varnothing$. Let $\alpha_0 \in \bigcap_{n\in\mathbb{N}} \Omega_n$. Since $\alpha_0 \in \Omega_0$, notice that $\alpha_0|_{\mathcal{T}} = \mathrm{id}\,|_{\mathcal{T}}, \alpha_0|_{\mathcal{T}'} = \alpha|_{\mathcal{T}'}$. To see that α_0 is as desired, all that remains is to show that $\alpha_0 \in G_{(i)}^+(Y_0, Y_1)$. However, for every $v \in V_t(T)$, there exists a positive integer $n \in \mathbb{N}$ such that $v \in \mathcal{R}_n$ and since $\alpha_0 \in \Omega_n$, we have that $\mathrm{Sgn}_{(i)}(\alpha_0, S_{Y_t}(v)) = 1$. This proves that α_0 is as desired.

What remains is to define the descending chain of nonempty compact sets $\Omega_n \subseteq \mathrm{Aut}(T)^+$. Suppose that $\max(Y_0) \leq \max(Y_1)$ (the proof for $\max(Y_1) \leq \max(Y_0)$ is similar). Let γ be the smallest geodesic of T that contains both the centre of $\mathcal{T}$ and the centre of $\mathcal{T}'$, and is oriented from $\mathcal{T}$ to $\mathcal{T}'$ (note that the centre is either a vertex or an edge depending on the values of l and l'). Since $\mathcal{T} \nsubseteq \mathcal{T}'$ and since $l' \geq l$, notice that γ contains at least two vertices (see Figure 2). The increasing chain of finite subtrees $\mathcal{R}_n$ of T such that $T = \bigcup_{n\in\mathbb{N}} \mathcal{R}_n$ that we are going to use is $\mathcal{R}_n = B_T(\gamma, n)$. We let

$$\Omega_{-1} = \{h \in \mathrm{Aut}(T)^+ \mid h|_{\mathcal{T}} = \mathrm{id}\,|_{\mathcal{T}} \text{ and } h|_{\mathcal{T}'} = \alpha|_{\mathcal{T}'}\}.$$

Since $\alpha \in \mathrm{Fix}_G(\mathcal{T}_W)$ and since $\mathcal{T}_W$ contains all vertices of $\mathcal{T} \cap \mathcal{T}'$, notice that Ω_{-1} is not empty. Now, since $\max(Y_0) \leq \max(Y_1)$, notice that there exists a unique $r \in \mathbb{N}$ such

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

that $\max(Y_0) + 2r \leq \max(Y_1) \leq \max(Y_0) + 2r + 1$ (where one of these inequalities is an equality). We let

$$\Omega_0 = \left\{ h \in \Omega_{-1} \;\middle|\; \begin{array}{l} \mathrm{Sgn}_{(i)}(g, S_{Y_0}(v)) = 1 \text{ for each } v \in B_T(\gamma, 2r) \cap V_{t_0} \\ \mathrm{Sgn}_{(i)}(g, S_{Y_1}(v)) = 1 \text{ for each } v \in B_T(\gamma, 0) \cap V_{t_1} \end{array} \right\}.$$

Lemma 4.8 below ensures that this set is not empty. From there, we define the sets Ω_n by induction on n. For every $n \geq 1$, let h_n be an element of Ω_{n-1} and let

$$\Omega_n = \left\{ h \in \Omega_{n-1} \;\middle|\; \begin{array}{c} h|_{B_T(\gamma, n-1+\max(Y_1))} = h_n|_{B_T(\gamma, n-1+\max(Y_1))} \\ \mathrm{Sgn}_{(i)}(h, S_{Y_0}(w)) = 1 \text{ for all } w \in B_T(\gamma, n+2r) \cap V_{t_0} \\ \mathrm{Sgn}_{(i)}(h, S_{Y_1}(w)) = 1 \text{ for all } w \in B_T(\gamma, n) \cap V_{t_1} \end{array} \right\}.$$

For this induction to make sense, it is important for Ω_n to be not empty for all $n \geq 1$. This is proved by Lemma 4.9 below, which ensures that Ω_n is a nonempty compact set. The result follows. □

Our current purpose is to prove Lemmas 4.8 and 4.9. To this end, we introduce some formalism that is used in both proofs. For all $v \in V(T)$, we need an automorphism $h_{(v)} \in \mathrm{Aut}(T)^+$ that is used to create an element of ω_{n+1} from an element of Ω_n. We start by choosing four functions:

$$\phi_0 : \{1, \ldots, d_0\} \longrightarrow \mathrm{Sym}(d_0) : k \mapsto \phi_0(k);$$

$$\phi_1 : \{1, \ldots, d_1\} \longrightarrow \mathrm{Sym}(d_1) : k \mapsto \phi_1(k);$$

$$\tilde{\phi}_0 : \{1, \ldots, d_0\} \times \{1, \ldots, d_0\} \longrightarrow \mathrm{Sym}(d_0) : (k, l) \mapsto \tilde{\phi}_0(k, l);$$

$$\tilde{\phi}_1 : \{1, \ldots, d_1\} \times \{1, \ldots, d_1\} \longrightarrow \mathrm{Sym}(d_1) : (k, l) \mapsto \tilde{\phi}_1(k, l),$$

such that $\phi_t(k)$ is an odd permutation of $\mathrm{Sym}(d_t)$ that fixes k and $\tilde{\phi}_t(k, l)$ is an odd permutation of $\mathrm{Sym}(d_t)$ that fixes k and l.

If $v \in V(T) - \gamma$, we choose $w \in \gamma$ and let $h_{(v)} \in \mathrm{Aut}(T)^+$ be such that:

(1) $h_{(v)} \in \mathrm{Fix}_{\mathrm{Aut}(T)^+}(T(p_{[w,v]}(v), v))$;
(2) $\sigma_{(i)}(h_{(v)}, v) = \phi_t(i(p_{[w,v]}(v)))$, where $t \in \{0, 1\}$ is such that $v \in V_t$.

Notice that for all $v \in V(T) - \gamma$ and every $w, w' \in \gamma$, $p_{[w,v]}(v) = p_{[w',v]}(v)$ (recall the definition of $p_{[w,v]}$ from Section 4.3) so that our choice of $w \in \gamma$ does not change the two properties that $h_{(v)}$ must satisfy (see Figure 3).

If $v \in \gamma$, we have two cases. Remember that γ has at least two vertices. If v is an end of γ, let w be the unique vertex of γ that is adjacent to v and choose an automorphism $h_{(v)} \in \mathrm{Aut}(T)^+$ such that:

(1) $h_{(v)} \in \mathrm{Fix}_{\mathrm{Aut}(T)^+}(T(w, v))$;
(2) $\sigma_{(i)}(h_{(v)}, v) = \phi_t(i(w))$, where $t \in \{0, 1\}$ is such that $v \in V_t$.

However, if v is not an end of γ, let w_1, w_2 be the two neighbours of v that belong to γ and choose an automorphism $h_{(v)} \in \mathrm{Aut}(T)^+$ such that:

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

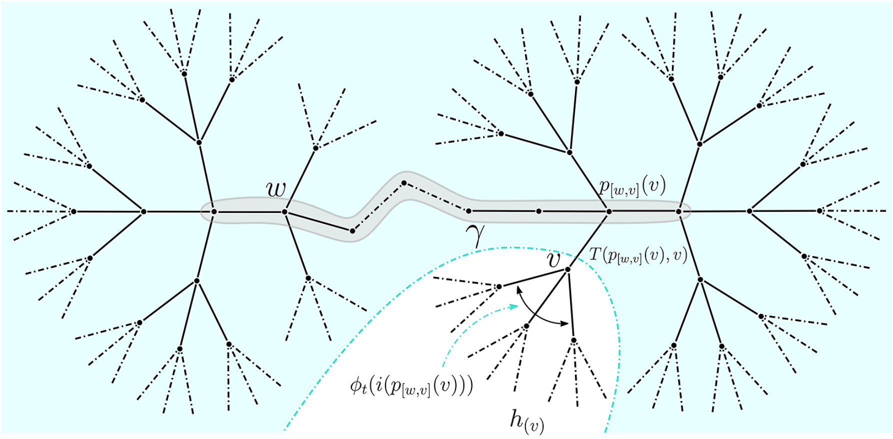

FIGURE 3. The automorphism $h_{(v)}$. Colour available online.

(1) $h_{(v)} \in \mathrm{Fix}_{\mathrm{Aut}(T)^+}(T(w_1, v) \cup T(w_2, v))$;
(2) $\sigma_{(i)}(h_{(v)}, v) = \tilde{\phi}_t(i(w_1), i(w_2))$, where $t \in \{0, 1\}$ is such that $v \in V_t$.

We are now ready to prove Lemmas 4.8 and 4.9.

LEMMA 4.8. *The set*

$$\Omega_0 = \left\{ h \in \Omega_{-1} \;\middle|\; \begin{array}{l} \mathrm{Sgn}_{(i)}(g, S_{Y_0}(v)) = 1 \textit{ for each } v \in B_T(\gamma, 2r) \cap V_{t_0} \\ \mathrm{Sgn}_{(i)}(g, S_{Y_1}(v)) = 1 \textit{ for each } v \in B_T(\gamma, 0) \cap V_{t_1} \end{array} \right\}.$$

is not empty.

PROOF. We recall that $\max(Y_0) \le \max(Y_1)$, that $r \in \mathbb{N}$ is the unique integer such that $\max(Y_0) + 2r \le \max(Y_1) \le \max(Y_0) + 2r + 1$, that $t_0 = \max(Y_0) \pmod 2$ and that $t_1 = (\max(Y_1) + 1) \pmod 2$. Remember from the proof of Proposition 4.7 that Ω_{-1} is not empty and let $h_0 \in \Omega_{-1}$. We are going to modify the element h_0 with the automorphisms $h_{(v)}$ to get an element of Ω_0. A concrete example of the procedure is given on a 4-regular tree with $Y_0 = \{0\}$ and $Y_1 = \{1, 2\}$ by Figures 4, 5 and 6. In these figures:

- the hollow vertices are those considered by the current and previous steps;
- the circled vertices are those for which we wish to change the sign $\mathrm{Sgn}_{(i)}(g, S_{Y_0}(v))$ or $\mathrm{Sgn}_{(i)}(g, S_{Y_1}(v))$ (depending on the step) without affecting the sign of other hollow vertices;
- the vertices boxed in squares are the vertices for which a change of the local action is applied to achieve the desired change of sign (note that for our choice $Y_0 = \{0\}$, these vertices are also the circled vertices).

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

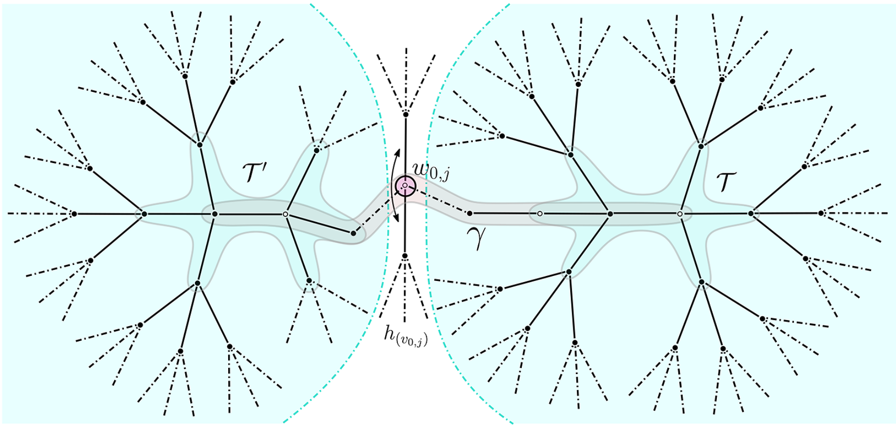

FIGURE 4. Step I of the proof of Lemma 4.8. Colour available online.

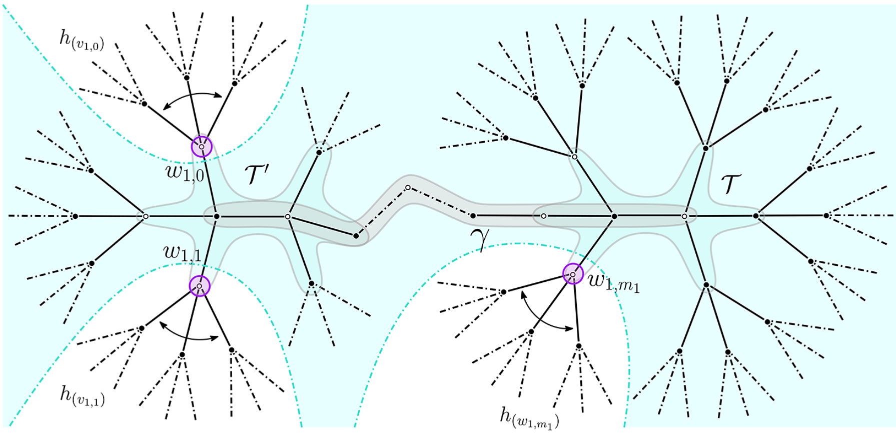

FIGURE 5. Step II of the proof of Lemma 4.8. Colour available online.

Let $\{w_{0,0}, \ldots, w_{0,m_0}\}$ be the set of vertices $w \in B_T(\gamma, 0) \cap V_{t_0}$ such that $\mathrm{Sgn}_{(i)}(h_0, S_{Y_0}(w)) = -1$. For all $j = 0, 1, \ldots, m_0$, we choose a vertex

$$v_{0,j} \in \bigcap_{w \in \gamma - \{w_{0,j}\}} T(w_{0,j}, w)$$

such that $d(v_{0,j}, w_{0,j}) = \max(Y_0)$. In particular, notice that $v_{0,j} \in S_{Y_0}(w_{0,j})$ but that $v_{0,j} \notin S_{Y_0}(w)$ for every $w \in B_T(\gamma, 0) \cap V_{t_0} - \{w_{0,j}\}$. Furthermore, since $\mathrm{Sgn}_{(i)}(h_0, S_{Y_0}(w_{0,j})) = -1$, $h_0|_{\mathcal{T}} = \mathrm{id}\,|_{\mathcal{T}}$, $h_0|_{\mathcal{T}'} = \alpha|_{\mathcal{T}'}$, and due to the form of $\mathcal{T}$ and $\mathcal{T}'$, the vertices $v_{0,j}$ must be such that the automorphisms $h_{(v_{0,0})}, \ldots, h_{(v_{0,m_0})}$ fix $\mathcal{T} \cup \mathcal{T}'$ pointwise. Indeed, if $v_{0,j}$ is outside the trees $\mathcal{T}$ and $\mathcal{T}'$ or is a leaf of either of these trees, the conclusion is clear. Furthermore, $v_{0,j}$ can not be an interior vertex of $\mathcal{T}$ or

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

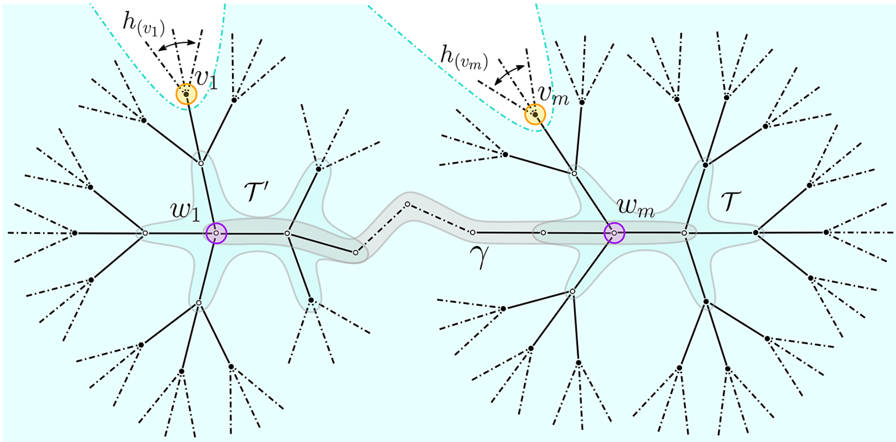

FIGURE 6. Step III of the proof of Lemma 4.8. Colour available online.

$\mathcal{T}'$, since $\mathrm{Sgn}_{(i)}(h_0, S_{Y_0}(w_{0,j})) = -1$ and $h_0 \in \Omega_{-1}$. In particular, if $v_{0,j}$ was an interior vertex of $\mathcal{T}$, $w_{0,j}$ would be too far inside $\mathcal{T}$ to have $\mathrm{Sgn}_{(i)}(h_0, S_{Y_0}(w_{0,j})) = -1$ as h_0 fixes pointwise a large enough ball around $w_{0,j}$ and if $v_{0,j}$ was an interior vertex of $\mathcal{T}'$, $w_{0,j}$ would be too far inside $\mathcal{T}'$ to have $\mathrm{Sgn}_{(i)}(h_0, S_{Y_0}(w_{0,j})) = -1$ as h_0 agrees with α on a large enough ball around $w_{0,j}$. This proves as desired that the automorphisms $h_{(v_{0,0})}, \ldots, h_{(v_{0,m_0})}$ fix $\mathcal{T} \cup \mathcal{T}'$ pointwise. In particular, the automorphism

$$h_{0,0} = h_0 \circ h_{(v_{0,0})} \circ \cdots \circ h_{(v_{0,m_0})}$$

satisfies $h_{0,0}|_{\mathcal{T}} = h_0|_{\mathcal{T}} = \mathrm{id}\,|_{\mathcal{T}}, h_{0,0}|_{\mathcal{T}'} = h_0|_{\mathcal{T}'} = \alpha|_{\mathcal{T}'}$ and

$$\mathrm{Sgn}_{(i)}(h_{0,0}, S_{Y_0}(w)) = 1 \quad \forall w \in \gamma \cap V_{t_0}.$$

If $r \neq 0$, we iterate this procedure. For every $1 \leq v \leq 2r$, let $\{w_{v,0}, \ldots, w_{v,m_v}\}$ be the set of vertices $w \in B_T(\gamma, v) \cap V_{t_0}$ such that

$$\mathrm{Sgn}_{(i)}(h_{v-1,0}, S_{Y_0}(w)) = -1.$$

For all $j = 0, 1, \ldots, m_v$, we choose a vertex

$$v_{v,j} \in \bigcap_{w \in B_T(\gamma,v) \cap V_{t_0} - \{w_{v,j}\}} T(w_{v,j}, w)$$

such that $d(v_{v,j}, w_{v,j}) = \max(Y_0)$. Hence, notice that $v_{v,j} \in S_{Y_0}(w_{v,j})$ but that $v_{v,j} \notin S_{Y_0}(w)$ for every $w \in B_T(\gamma, v) \cap V_{t_0} - \{w_{v,j}\}$. Furthermore, notice that the automorphisms $h_{(v_{v,0})}, \ldots, h_{(v_{v,m_v})}$ fix $\mathcal{T} \cup \mathcal{T}'$ pointwise. In particular, the automorphism

$$h_{v,0} = h_{v-1,0} \circ h_{(v_{v,0})} \circ \cdots \circ h_{(v_{v,m_v})}$$

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

satisfies that $h_{\nu,0}|_{\mathcal{T}} = h_{\nu-1,0}|_{\mathcal{T}} = \mathrm{id}\,|_{\mathcal{T}}, h_{\nu,0}|_{\mathcal{T}'} = h_{\nu-1,0}|_{\mathcal{T}'} = \alpha|_{\mathcal{T}'}$ and

$$\mathrm{Sgn}_{(i)}(h_{\nu,0}, S_{Y_0}(w)) = 1 \quad \forall w \in B_T(\gamma,\nu) \cap V_{t_0}.$$

Consider the element $h_{2r,0}$ that we have just constructed. This element behaves as desired for the condition given by Y_0 on the vertices of $B_T(\gamma, 2r) \cap V_{t_0}$. However, nothing ensures that the condition given by Y_1 on the vertices of $B_T(\gamma, 0) \cap V_{t_1}$ is yet satisfied. We now take care of this task. Let $\{w_0, \ldots, w_m\}$ be the set of vertices $w \in B_T(\gamma, 0) \cap V_{t_1}$ such that $\mathrm{Sgn}_{(i)}(h_{2r,0}, S_{Y_1}(w)) = -1$. For all $j = 0, 1, \ldots, m$, let $v_j \in \bigcap_{w\in\gamma-\{w_j\}} T(w_j, w)$ such that $d(v_j, w_j) = \max(Y_1)$. In particular, notice that $v_j \in S_{Y_1}(w_j)$ but that $v_j \notin S_{Y_1}(w)$ for every $w \in B_T(\gamma, 0) \cap V_{t_1} - \{w_j\}$. Furthermore, since $\max(Y_0) + 2r \leq \max(Y_1)$, notice from Remark 3.4 that $v_j \notin S_{Y_0}(v)$ for every $v \in B_T(\gamma, 2r) \cap V_{t_0}$. However, just as before, the automorphisms $h_{(v_0)}, \ldots, h_{(v_m)}$ fix $\mathcal{T} \cup \mathcal{T}'$ pointwise. In particular, $h_1 = h_{2r,0} \circ h_{(v_0)} \circ \cdots \circ h_{(v_m)}$ satisfies:

- $h_1|_{\mathcal{T}} = h_{2r,0}|_{\mathcal{T}} = \mathrm{id}\,|_{\mathcal{T}}$ and $h_1|_{\mathcal{T}'} = h_{2r,0}|_{\mathcal{T}'} = \alpha|_{\mathcal{T}'}$;
- $\mathrm{Sgn}_{(i)}(h_1, S_{Y_0}(w)) = 1$ for all $w \in B_T(\gamma, 2r) \cap V_{t_0}$;
- $\mathrm{Sgn}_{(i)}(h_1, S_{Y_1}(w)) = 1$ for all $w \in B_T(\gamma, 0) \cap V_{t_1}$.

This proves that $h_1 \in \Omega_0$ and therefore that Ω_0 is not empty. □

LEMMA 4.9. *For all $n \geq 1$, the set*

$$\Omega_n = \left\{ h \in \Omega_{n-1} \;\middle|\; \begin{array}{c} h|_{B_T(\gamma,n-1+\max(Y_1))} = h_n|_{B_T(\gamma,n-1+\max(Y_1))} \\ \mathrm{Sgn}_{(i)}(h, S_{Y_0}(w)) = 1 \quad \forall w \in B_T(\gamma, n+2r) \cap V_{t_0} \\ \mathrm{Sgn}_{(i)}(h, S_{Y_1}(w)) = 1 \quad \forall w \in B_T(\gamma, n) \cap V_{t_1} \end{array} \right\}.$$

is a nonempty compact subset of $\mathrm{Aut}(T)^+$.

PROOF. We show that Ω_n is not empty by induction. Lemma 4.8 ensures that Ω_0 is not empty. Suppose that Ω_{n-1} is not empty and let $h_n \in \Omega_{n-1}$ be the automorphism appearing in the definition of Ω_n. Just as in the proof of Lemma 4.8, we are going to modify h_n with the automorphisms $h_{(v)}$ to obtain an element of Ω_n. A concrete example of the procedure is given by Figures 7 and 8 on a 4-regular tree with $Y_0 = \{0\}$ and even $\max(Y_1)$ (with the same conventions as before and where the vertices concerned by the current step are highlighted in dotted areas). Let $\{\tilde{w}_0, \ldots, \tilde{w}_k\}$ be the set of vertices w that belong to $B_T(\gamma, n+2r) \cap V_{t_0}$ and such that $\mathrm{Sgn}_{(i)}(h_n, S_{Y_0}(w)) = -1$. For each $j = 0, 1, \ldots, k$, we choose a vertex $\tilde{v}_j \in \bigcap_{w\in B_T(\gamma,n+2r)\cap V_{t_0}-\{\tilde{w}_j\}} T(\tilde{w}_j, w)$ such that $d(\tilde{v}_j, \tilde{w}_j) = \max(Y_0)$. Hence, notice that $\tilde{v}_j \notin S_{Y_0}(w)$ for all $w \in B_T(\gamma, n+2r) \cap V_{t_0} - \{\tilde{w}_j\}$. Furthermore, since $\max(Y_1) - 1 \leq 2r + \max(Y_0)$, notice from Remark 3.4 that $\tilde{v}_j \notin S_{Y_1}(w)$ for all $w \in B_T(\gamma, n-1) \cap V_{t_1}$. Furthermore, since $\mathrm{Sgn}_{(i)}(h_n, S_{Y_0}(\tilde{w}_j)) = -1$, $h_n|_{\mathcal{T}} = \mathrm{id}\,|_{\mathcal{T}}$, $h_n|_{\mathcal{T}'} = \alpha|_{\mathcal{T}'}$, and due to the form's of $\mathcal{T}$ and $\mathcal{T}'$, notice that the automorphisms $h_{(\tilde{v}_0)}, \ldots, h_{(\tilde{v}_k)}$ fix $\mathcal{T} \cup \mathcal{T}'$ pointwise. Indeed, if $\tilde{v}_j$ is outside the trees $\mathcal{T}$ and $\mathcal{T}'$ or is a leaf of either of these trees, the conclusion is clear. However, $\tilde{v}_j$ cannot be an interior vertex of $\mathcal{T}$ or $\mathcal{T}'$, since $\mathrm{Sgn}_{(i)}(h_n, S_{Y_0}(\tilde{w}_j)) = -1$ and $h_n \in \Omega_{n-1}$. In particular, if $\tilde{v}_j$ was an interior vertex of $\mathcal{T}$,

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

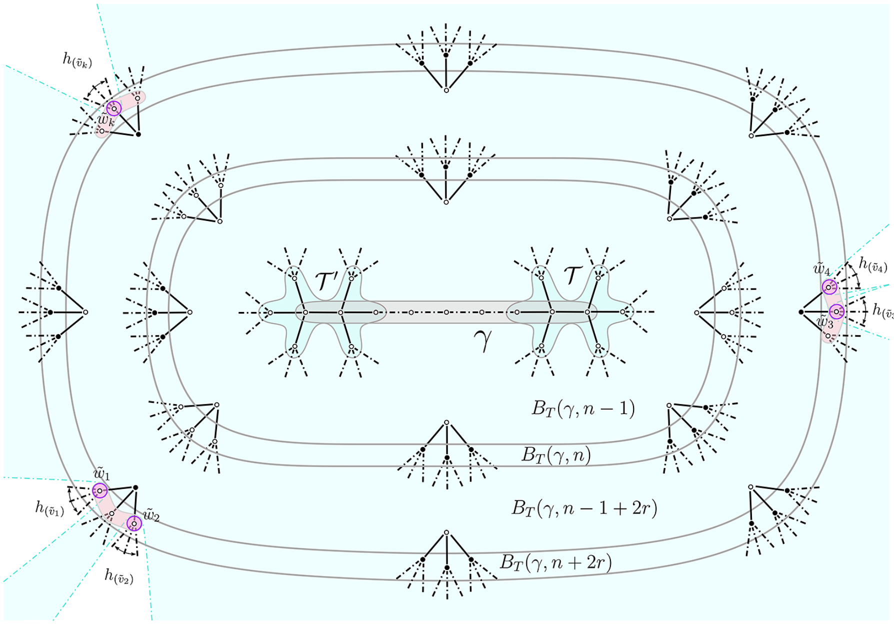

FIGURE 7. Step I of the proof of Lemma 4.9. Colour available online.

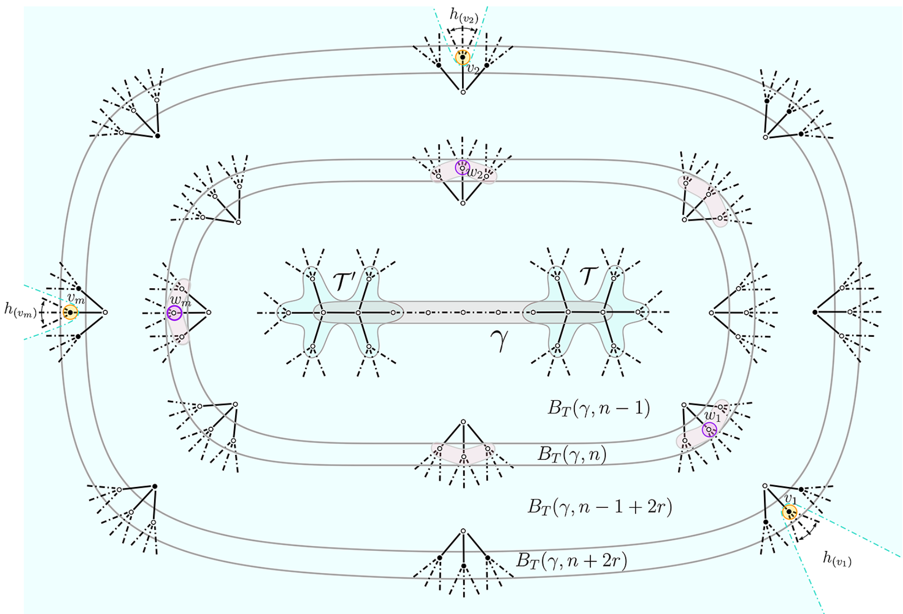

FIGURE 8. Step II of the proof of Lemma 4.9. Colour available online.

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

$\tilde{w}_j$ would be too far inside $\mathcal{T}$ for $\mathrm{Sgn}_{(i)}(h_n, S_{Y_0}(\tilde{w}_j)) = -1$ as h_n fixes pointwise a large enough ball around $\tilde{w}_j$ and if $\tilde{v}_j$ was an interior vertex of $\mathcal{T}'$, $\tilde{w}_j$ would be too far inside $\mathcal{T}'$ for $\mathrm{Sgn}_{(i)}(h_n, S_{Y_0}(\tilde{w}_j)) = -1$ as h_n agrees with α on a large enough ball around $\tilde{w}_j$. This proves as desired that the automorphisms $h_{(\tilde{v}_0)}, \ldots, h_{(\tilde{v}_k)}$ fix $\mathcal{T} \cup \mathcal{T}'$ pointwise. In particular, $\tilde{h}_n = h_n \circ h_{(\tilde{v}_0)} \circ \cdots \circ h_{(\tilde{v}_k)}$ satisfies:

- $\tilde{h}_n|_{B_T(\gamma, n-1+\max(Y_1))} = h_n|_{B_T(\gamma, n-1+\max(Y_1))}$;
- $\mathrm{Sgn}_{(i)}(h_n, S_{Y_0}(w)) = 1$ for all $w \in B_T(\gamma, n+2r) \cap V_{t_0}$;
- $\mathrm{Sgn}_{(i)}(h_n, S_{Y_1}(w)) = 1$ for all $w \in B_T(\gamma, n-1) \cap V_{t_1}$.

Now, let $\{w_0, \ldots, w_m\}$ be the set of vertices $w \in B_T(\gamma, n) \cap V_{t_1}$ such that $\mathrm{Sgn}_{(i)}(\tilde{h}_n, S_{Y_1}(w)) = -1$. For each $j = 0, 1, \ldots, m$, choose $v_j \in \bigcap_{w \in B_T(\gamma,n)-\{w_j\}} T(w_j, w)$ such that $d(v_j, w_j) = \max(Y_1)$. Since $\max(Y_0) + 2r \leq \max(Y_1)$, notice from Remark 3.4 that $v_j \notin S_{Y_0}(w)$ for every $w \in B_T(\gamma, n+2r) \cap V_{t_0}$ and $v_j \notin S_{Y_1}(w)$ for every $w \in B_T(\gamma, n) \cap V_{t_1} - \{w_j\}$. Just as before, notice that the automorphisms $h_{(v_0)}, \ldots, h_{(v_m)}$ fix $\mathcal{T} \cup \mathcal{T}'$ pointwise. In particular, $h_{n+1} = \tilde{h}_n \circ h_{(v_0)} \circ \cdots \circ h_{(v_m)}$ satisfies:

- $h_{n+1}|_{B_T(\gamma, n-1+\max(Y_1))} = h_n|_{B_T(\gamma, n-1+\max(Y_1))}$;
- $\mathrm{Sgn}_{(i)}(h_{n+1}, S_{Y_0}(w)) = 1$ for all $w \in B(\gamma, n+2r) \cap V_{t_0}$;
- $\mathrm{Sgn}_{(i)}(h_{n+1}, S_{Y_1}(w)) = 1$ for all $w \in B(\gamma, n) \cap V_{t_1}$.

This proves that Ω_n is not empty. We now show that Ω_n is compact for every integer $n \geq 1$. To this end, notice that Ω_n is a closed subset of $\mathrm{Aut}(T)^+$ and that

$$\Omega_n \subseteq h_n \mathrm{Fix}_{\mathrm{Aut}(T)^+}(B_T(\gamma, n-1+\max(Y_1))).$$

Since the right-hand side is a compact subset of $\mathrm{Aut}(T)^+$, the results follow. □

Finally, we prove the result announced at the beginning of Section 4.3.

THEOREM 4.10. *The generic filtration $\mathcal{S}_0$ of $G^+_{(i)}(Y_0, Y_1)$ factorises$^+$ at every depth $l \geq 1$.*

PROOF. Let $G = G^+_{(i)}(Y_0, Y_1)$. To prove that $\mathcal{S}_0$ factorises$^+$ at depth $l \geq 1$, we successively check the three conditions of Definition 4.3.

First, we must prove that for every U in the conjugacy class of an element of $\mathcal{S}_0[l]$ and every subgroup V in the conjugacy class of an element of $\mathcal{S}_0$ with $V \nsubseteq U$, there exists a W in the conjugacy class of an element of $\mathcal{S}_0[l-1]$ such that

$$U \subseteq W \subseteq VU.$$

Let U and V be as above. By the definition of $\mathcal{S}_0$ and since $\mathfrak{T}_0$ is stable under the action of G, there exist two subtrees $\mathcal{T}, \mathcal{T}' \in \mathfrak{T}_0$ such that $U = \mathrm{Fix}_G(\mathcal{T})$ and $V = \mathrm{Fix}_G(\mathcal{T}')$. In particular, $\mathcal{T}' = B_T(o, r)$, where o is either a vertex or an edge of T and r is a nonnegative integer. Let v be the vertex of $\mathcal{T}$ at maximal distance from o and notice since $V \nsubseteq U$ that $d_T(v, o) > r$. Now let γ be an infinite geodesic ray starting at v and containing o, let $\mathcal{P}$ be the tree centred at $\gamma(d_T(v, o) + l)$ of radius $r + l$ and let $V' = \mathrm{Fix}_G(\mathcal{P})$. By construction, one has that $\mathcal{T}' \subseteq \mathcal{P}$ so that $V' \subseteq V$. Furthermore, V' has depth greater than l and since $v \notin \mathcal{P}$, Lemma 4.6 ensures that $V' \nsubseteq U$. Now,

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

Proposition 4.7 ensures the existence of a subgroup $W \in \mathcal{S}_0[l-1]$ such that

$$U \subseteq W \subseteq V'U \subseteq VU.$$

This proves the first condition.

Next, we must prove that $N_G(U, V) = \{g \in G \mid g^{-1}Vg \subseteq U\}$ is compact for every V in the conjugacy class of an element of $\mathcal{S}_0$. Just as before, notice that there exists a $\mathcal{T}' \in \mathfrak{T}_0$ such that $V = \mathrm{Fix}_G(\mathcal{T}')$. Since G satisfies the hypothesis (H$_0$), notice that

$$\begin{aligned} N_G(U, V) &= \{g \in G \mid g^{-1}Vg \subseteq U\} = \{g \in G \mid g^{-1}\mathrm{Fix}_G(\mathcal{T}')g \subseteq \mathrm{Fix}_G(\mathcal{T})\} \\ &= \{g \in G \mid \mathrm{Fix}_G(g^{-1}\mathcal{T}') \subseteq \mathrm{Fix}_G(\mathcal{T})\} = \{g \in G \mid g\mathcal{T} \subseteq \mathcal{T}'\}. \end{aligned}$$

In particular, since both $\mathcal{T}$ and $\mathcal{T}'$ are finite subtrees of T, $N_G(U, V)$ is a compact subset of G which proves the second condition.

Finally, we must prove for every W in the conjugacy class of an element of $\mathcal{S}_0[l-1]$ with $U \subseteq W$ that

$$W \subseteq N_G(U, U) = \{g \in G \mid g^{-1}Ug \subseteq U\}.$$

For the same reasons as before, there exists $\mathcal{R} \in \mathfrak{T}_0$ such that $W = \mathrm{Fix}_G(\mathcal{R})$. Furthermore, since $U \subseteq W$ and since G satisfies the hypothesis (H$_0$), notice that $\mathcal{R} \subseteq \mathcal{T}$. Moreover, since $\mathrm{Fix}_G(\mathcal{R})$ has depth $l-1$, notice that $\mathcal{R}$ contains every interior vertex of $\mathcal{T}$. Since G is unimodular and satisfies hypothesis (H$_0$), this implies that

$$\begin{aligned} \mathrm{Fix}_G(\mathcal{R}) &\subseteq \{h \in G \mid h\mathcal{T} \subseteq \mathcal{T}\} = \{h \in G \mid \mathrm{Fix}_G(\mathcal{T}) \subseteq \mathrm{Fix}_G(h\mathcal{T})\} \\ &= \{h \in G \mid h^{-1}\mathrm{Fix}_G(\mathcal{T})h \subseteq \mathrm{Fix}_G(\mathcal{T})\} = N_G(U, U), \end{aligned}$$

which proves the third condition. □

4.4. Description of cuspidal representations. The purpose of this section is to give a description of the cuspidal representations of $G^{+}_{(i)}(Y_0, Y_1)$. This is done by Theorem 4.12 below but requires some preliminaries. We refer to [Sem23] for proofs and details of the formalism.

Let G be a nondiscrete unimodular totally disconnected locally compact group G and let $\mathcal{S}$ be a generic filtration of G factorising$^+$ at depth l. Then, for every irreducible representation π of G at depth l, [Sem23, Theorem A] ensures the existence of a unique conjugacy class $C_\pi \in \mathcal{F}_{\mathcal{S}} = \{C(U) \mid U \in \mathcal{S}\}$ at height l such that π admits nonzero U-invariant vectors for any $U \in C_\pi$. The conjugacy class C_π is called the **seed** of π. We define the **group of automorphisms** $\mathrm{Aut}_G(C)$ **of the seed** C as the quotient $N_G(U)/U$ corresponding to any $U \in C$. This group $\mathrm{Aut}_G(C)$ is finite and does not depend on our choice of $U \in C$ up to isomorphism. Now, let $p_U : N_G(U) \mapsto N_G(U)/U$ denote the quotient map, let

$$\tilde{\mathfrak{H}}_{\mathcal{S}}(U) = \{W \mid \text{there exists } g \in G \text{ such that } gWg^{-1} \in \mathcal{S}[l-1] \text{ and } U \subseteq W\}$$

and set

$$\mathfrak{H}_{\mathcal{S}}(C) = \{p_U(W) \mid W \in \tilde{\mathfrak{H}}_{\mathcal{S}}(U)\}.$$

Notice that $\mathfrak{H}_{\mathcal{S}}(C)$ does not depend on our choice of representative $U \in C$.

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

DEFINITION 4.11. An irreducible representation ω of $\mathrm{Aut}_G(C)$ is called **$\mathcal{S}$-standard** if it has no nonzero H-invariant vector for any $H \in \mathfrak{H}_{\mathcal{S}}(C)$.

The importance of this notion is given by [Sem23, Theorem A], which ensures that the irreducible representations of G at depth l with seed C are obtained from the $\mathcal{S}$-standard representations of $\mathrm{Aut}_G(C)$ when $\mathcal{S}$ factorises$^+$ at depth l. More precisely, we recall that every irreducible representation ω of $\mathrm{Aut}_G(C) \cong N_G(U)/U$ can be lifted to an irreducible representation $\omega \circ p_U$ of $N_G(U)$ with U acting trivially and with representation space $\mathcal{H}_\omega$. The lifted representation can then be induced to G. The resulting representation

$$T(U, \omega) = \mathrm{Ind}_{N_G(U)}^{G}(\omega \circ p_U)$$

is an irreducible representation of G with seed $C(U)$. Conversely, if π is an irreducible representation of G with seed C, notice that $\mathcal{H}_\pi^U$ is a nonzero $N_G(U)$-invariant subspace of $\mathcal{H}_\pi$ for every $U \in C$. In particular, the restriction $(\upharpoonright \pi N_G(U), \mathcal{H}_\pi^U)$ is a representation of $N_G(U)$ whose restriction to U is trivial. This representation passes to the quotient group $N_G(U)/U$ and defines an $\mathcal{S}$-standard representation ω_π of $\mathrm{Aut}_G(C)$.

We now come back to the case in which we are interested. Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 4$, let $V(T) = V_0 \sqcup V_1$ be the associated bipartition and let

$$\mathfrak{T}_0 = \{B_T(v, r) \mid v \in V(T), r \geq 1\} \sqcup \{B_T(e, r) \mid e \in E(T), r \geq 0\}.$$

Let i be a legal colouring of T, let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets and consider the group $G_{(i)}^+(Y_0, Y_1)$. Let $G = G_{(i)}^+(Y_0, Y_1)$. We have shown in Sections 4.2 and 4.3 that

$$\mathcal{S}_0 = \{\mathrm{Fix}_G(\mathcal{T}) \mid \mathcal{T} \in \mathfrak{T}_0\}$$

is a generic filtration of $G_{(i)}^+(Y_0, Y_1)$ that factorises$^+$ at all depths $l \geq 1$. In particular, [Sem23, Theorem A] provides a bijective correspondence between irreducible representations of $G_{(i)}^+(Y_0, Y_1)$ at depth l with seed $C \in \mathcal{F}_{\mathcal{S}_0}$ and the $\mathcal{S}_0$-standard representations of $\mathrm{Aut}_G(C)$. We start by identifying these seeds and show that the cuspidal representations of G are precisely the irreducible representations of G at depth $l \geq 1$ with respect to $\mathcal{S}_0$. In the light of Lemma 4.5, we consider the partition $\mathfrak{T}_0 = \bigsqcup_{l \in \mathbb{N}} \mathfrak{T}_0[l]$ where:

- $\mathfrak{T}_0[l] = \{B_T(e, \frac{l}{2}) \mid e \in E(T)\}$ if l is even;
- $\mathfrak{T}_0[l] = \{B_T(v, ((l+1)/2)) \mid v \in V(T)\}$ if l is odd.

Notice that for all $l \in \mathbb{N}$, $\mathfrak{T}_0[l]$ is stable under the action of G. Furthermore, notice that if l is even, or if l is odd and G is transitive on the vertices, the set $\mathfrak{T}_0[l]$ consists of a single G-orbit. However, if l is odd and G has two orbits of vertices, notice that the set $\mathfrak{T}_0[l]$ consists of two G-orbits, namely $\{B_T(v, ((l+1)/2)) \mid v \in V_0\}$ and $\{B_T(v, ((l+1)/2)) \mid v \in V_1\}$. In particular, in light of Lemma 4.6, there are either one or two elements of $\mathcal{F}_{\mathcal{S}_0} = \{C(U) \mid U \in \mathcal{S}_0\}$ at height l and each such element is of the form

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

$$C = \{\mathrm{Fix}_G(\mathcal{T}) \mid \mathcal{T} \in \mathcal{O}\},$$

where $\mathcal{O}$ is a G-orbit of $\mathfrak{T}_0[l]$. We deduce easily that the irreducible representations of $G_{(i)}^{+}(Y_0, Y_1)$ at depth $l \geq 1$ with respect to $\mathcal{S}_0$ are the cuspidal representations of G. Indeed, π is an irreducible representation at depth $l \geq 1$ with respect to $\mathcal{S}_0$ if and only if π does not admit a nonzero V-invariant vector for any V in a conjugacy class C at depth 0, that is, for any $V \in \{\mathrm{Fix}_G(e) \mid e \in E(T)\}$. Now, let π be a cuspidal representation of G, let $C_\pi \in \mathcal{F}_{\mathcal{S}_0}$ be the seed of π, let $U \in C_\pi$ and let $\mathcal{T} \in \mathfrak{T}_0$ be such that $U = \mathrm{Fix}_G(\mathcal{T})$. Since $\mathcal{S}_0$ factorises$^+$ at all depths $l \geq 1$, [Sem23, Theorem A] ensures that π is induced from an irreducible representation of $N_G(U)$ that passes to the quotient $\mathrm{Aut}_G(C_\pi) \cong N_G(U)/U$. Furthermore, since G satisfies hypothesis (H$_0$), notice that

$$\begin{aligned} N_G(U) = \{g \in G \mid gUg^{-1} = U\} &= \{g \in G \mid g\mathrm{Fix}_G(\mathcal{T})g^{-1} = \mathrm{Fix}_G(\mathcal{T})\} \\ &= \{g \in G \mid \mathrm{Fix}_G(g\mathcal{T}) = \mathrm{Fix}_G(\mathcal{T})\} \\ &= \{g \in G \mid g\mathcal{T} = \mathcal{T}\} = \mathrm{Stab}_G(\mathcal{T}) \end{aligned}$$

is exactly the stabiliser of $\mathcal{T}$ in $G_{(i)}^{+}(Y_0, Y_1)$. In particular, $\mathrm{Aut}_G(C_\pi)$ can be identified with the automorphism group of $\mathcal{T}$ obtained by restricting the action of $\mathrm{Stab}_G(\mathcal{T})$ to $\mathcal{T}$. Moreover, since G satisfies hypothesis (H$_0$), notice that

$$\tilde{\mathfrak{H}}_{\mathcal{S}_0}(U) = \{\mathrm{Fix}_G(\mathcal{R}) \mid \mathcal{R} \in \mathfrak{T}_0, \mathcal{R} \subsetneq \mathcal{T} \text{ and } \mathcal{R} \text{ is maximal for this property}\},$$

and

$$\mathfrak{H}_{\mathcal{S}_0}(C_\pi) = \{p_U(W) \mid W \in \tilde{\mathfrak{H}}_{\mathcal{S}_0}(U)\}$$

is the set of fixators (in $\mathrm{Aut}_G(C_\pi)$) of subtrees $\mathcal{R} \in \mathcal{T}_0$ satisfying $\mathcal{R} \subsetneq \mathcal{T}$ and that are maximal for this property. In particular, the $\mathcal{S}_0$-**standard** representations of $\mathrm{Aut}_G(C_\pi)$ are the irreducible representations of the group of automorphisms of $\mathcal{T}$ obtained by restricting the action of $\mathrm{Stab}_G(\mathcal{T})$ and that do not admit any nonzero invariant vector for the fixator of any subtree $\mathcal{R}$ of $\mathcal{T}$ that belongs to $\mathfrak{T}_0$ and is maximal for this property. The above discussion together with [Sem23, Theorem A] leads to the following description of the cuspidal representations of $G_{(i)}^{+}(Y_0, Y_1)$.

THEOREM 4.12. *Let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 4$, let i be a legal colouring of T, let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets, let $G = G_{(i)}^{+}(Y_0, Y_1)$, consider the generic filtration $\mathcal{S}_0$ of G (defined in Section 4.2) and let us use the above notation. Then, the cuspidal representations of G are exactly the irreducible representations at depth $l \geq 1$ with respect to $\mathcal{S}_0$. Furthermore, if π is a cuspidal representation at depth l, we have the following.*

- *π has no nonzero $\mathrm{Fix}_G(\mathcal{R})$-invariant vector for any $\mathcal{R} \in \bigsqcup_{r<l} \mathfrak{T}_0[r]$.*
- *There exists a unique conjugacy class $C_\pi \in \mathcal{F}_{\mathcal{S}_0}$ at height l such that π admits a nonzero U-invariant vector for any (hence for all) $U \in C_\pi$. Equivalently, there*

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

exists a unique G-orbit $\mathcal{O}$ of $\mathfrak{T}_0[l]$ such that π admits a nonzero $\mathrm{Fix}_G(\mathcal{T})$-invariant vector for any (hence for all) $\mathcal{T} \in \mathcal{O}$. Furthermore, $\mathcal{O}$ is the only orbit of $\mathfrak{T}_0$ under the action of G such that $C_\pi = \{\mathrm{Fix}_G(\mathcal{T}) \mid \mathcal{T} \in \mathcal{O}\}$.

- *If $\mathcal{O}$ is the unique G-orbit of $\mathfrak{T}_0[l]$ corresponding to π and if $\mathcal{T} \in \mathcal{O}$, π admits a nonzero diagonal matrix coefficient supported in $\mathrm{Stab}_G(\mathcal{T})$. In particular, π is square-integrable and its equivalence class is isolated in the unitary dual $\widehat{G}$ for the Fell topology.*

Furthermore for each $C \in \mathcal{F}_{\mathcal{S}_0}$ at height $l \geq 1$ with corresponding G-orbit $\mathcal{O}$ in $\mathfrak{T}_0[l]$, that is, $C = \{\mathrm{Fix}_G(\mathcal{T}) \mid \mathcal{T} \in \mathcal{O}\}$, there exists a bijective correspondence between the equivalence classes of irreducible representations of G with seed C and the equivalence classes of $\mathcal{S}_0$-standard representations of $\mathrm{Aut}_G(C)$. More precisely, for every $\mathcal{T} \in \mathcal{O}$, the following hold.

(1) *If π is a cuspidal representation of G with seed C, $(\omega_\pi, \mathcal{H}_\pi^{\mathrm{Fix}_G(\mathcal{T})})$ is an $\mathcal{S}_0$-standard representation of $\mathrm{Aut}_G(C)$ such that*

$$\pi \cong T(\mathrm{Fix}_G(\mathcal{T}), \omega_\pi) = \mathrm{Ind}_{\mathrm{Stab}_G(\mathcal{T})}^{G}(\omega_\pi \circ p_{\mathrm{Fix}_G(\mathcal{T})}).$$

(2) *If ω is an $\mathcal{S}_0$-standard representation of $\mathrm{Aut}_G(C)$, the representation $T(\mathrm{Fix}_G(\mathcal{T}), \omega)$ is a cuspidal representation of G with seed in C.*

Furthermore, if ω_1 and ω_2 are $\mathcal{S}_0$-standard representations of $\mathrm{Aut}_G(C)$, we have that $T(\mathrm{Fix}_G(\mathcal{T}), \omega_1) \cong T(\mathrm{Fix}_G(\mathcal{T}), \omega_2)$ if and only if $\omega_1 \cong \omega_2$. In particular, the above two constructions are inverse of one another.

4.5. Existence of cuspidal representations. As in the previous sections, let T be a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 4$, let $V(T) = V_0 \sqcup V_1$ be the associated bipartition and let

$$\mathfrak{T}_0 = \{B_T(v, r) \mid v \in V(T), r \geq 1\} \sqcup \{B_T(e, r) \mid e \in E(T), r \geq 0\}.$$

Let i be a legal colouring of T, let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets and consider the group $G_{(i)}^{+}(Y_0, Y_1)$. Write $G = G_{(i)}^{+}(Y_0, Y_1)$ if this leads to no confusion. We have shown in Sections 4.2 and 4.3 that

$$\mathcal{S}_0 = \{\mathrm{Fix}_G(\mathcal{T}) \mid \mathcal{T} \in \mathfrak{T}_0\}$$

is a generic filtration of $G_{(i)}^{+}(Y_0, Y_1)$ factorising$^+$ at all depths $l \geq 1$. In particular, [Sem23, Theorem A] provides a bijective correspondence between the irreducible representations of $G_{(i)}^{+}(Y_0, Y_1)$ at depth l with seed $C \in \mathcal{F}_{\mathcal{S}_0}$ and the $\mathcal{S}_0$-standard representations of $\mathrm{Aut}_G(C)$. This leads to a description of the cuspidal representations of $G_{(i)}^{+}(Y_0, Y_1)$; see Theorem 4.12. However, none of these results yet ensures the existence of a cuspidal representation of $G_{(i)}^{+}(Y_0, Y_1)$. The purpose of this section is to prove the existence of a cuspidal representation with seed C for each conjugacy class $C \in \mathcal{F}_{\mathcal{S}_0}$ at height $l \geq 1$. From the description of cuspidal representations of $G_{(i)}^{+}(Y_0, Y_1)$ provided by Theorem 4.12, it is equivalent to prove the following theorem.

THEOREM 4.13. *Let* $G = G_{(i)}^{+}(Y_0, Y_1)$ *and let* $C \in \mathcal{F}_{\mathcal{S}_0}$ *be a conjugacy class at height* $l \geq 1$. *Then, there exists an* $\mathcal{S}_0$*-standard representation of* $\mathrm{Aut}_G(C)$.

The proof of this theorem is gathered in the following results. We start by recalling a result from [Sem23].

PROPOSITION 4.14 [Sem23, Proposition 2.29]. *Let* T *be a locally finite tree, let* $G \leq \mathrm{Aut}(T)$ *be a closed subgroup, let* $\mathcal{T}$ *be a finite subtree of* T *and let* $\{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s\}$ *be a set of distinct finite subtrees of* T *contained in* $\mathcal{T}$ *such that* $\mathcal{T}_i \cup \mathcal{T}_j = \mathcal{T}$ *for every* $i \neq j$. *Suppose that* $\mathrm{Stab}_G(\mathcal{T})$ *acts by permutation on the set* $\{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s\}$ *and that* $\mathrm{Fix}_G(\mathcal{T}) \subsetneq \mathrm{Fix}_G(\mathcal{T}_i) \subsetneq \mathrm{Stab}_G(\mathcal{T})$. *Then, there exists an irreducible representation of* $\mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})$ *without a nonzero* $\mathrm{Fix}_G(\mathcal{T}_i)/\mathrm{Fix}_G(\mathcal{T})$*-invariant vector for every* $i = 1, \ldots, s$.

The following proposition ensures the existence of $\mathcal{S}_0$-standard representations of $\mathrm{Aut}_G(C)$ for every $C \in \mathcal{F}_{\mathcal{S}_0}$ with height $l \geq 2$.

PROPOSITION 4.15. *Let* $G = G_{(i)}^{+}(Y_0, Y_1)$ *and let* $C \in \mathcal{F}_{\mathcal{S}_0}$ *be a conjugacy class at height* $l \geq 2$. *Then, there exists an* $\mathcal{S}_0$*-standard representation of* $\mathrm{Aut}_G(C)$.

PROOF. Since $l \geq 2$, Lemma 4.5 ensures the existence of a complete finite subtree $\mathcal{T}$ of T containing a ball of radius 1 around an edge such that $C = C(\mathrm{Fix}_G(B_T(e, r)))$. Furthermore, as observed in Section 4.4, we have that $\mathrm{Aut}_G(C) \cong \mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})$ and

$$\mathfrak{H}_{\mathcal{S}_0}(\mathrm{Fix}_G(\mathcal{T})) = \{\mathrm{Fix}_G(\mathcal{R})/\mathrm{Fix}_G(\mathcal{T}) \mid \mathcal{R} \in \mathfrak{T}_0, \mathcal{R} \subsetneq \mathcal{T} \text{ and } \mathcal{R} \text{ is maximal for this property}\}.$$

Now, let $\{\mathcal{T}_1, \ldots, \mathcal{T}_s\}$ be the set of maximal complete proper subtrees of $\mathcal{T}$. These can all be obtained from $\mathcal{T}$ by removing the leaves adjacent to v, where v is a vertex of $\mathcal{T}$ that is not a leaf but such that all but one of its neighbours are leaves of $\mathcal{T}$. Clearly, $\mathcal{T}_i \cup \mathcal{T}_j = \mathcal{T}$ for every $i \neq j$ and $\mathrm{Stab}_G(\mathcal{T})$ acts by permutation on the set $\{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_s\}$. Furthermore, for each $i \in \{1, \ldots, s\}$, notice from Lemma 4.6 that

$$\mathrm{Fix}_G(\mathcal{T}) \subsetneq \mathrm{Fix}_G(\mathcal{T}_i).$$

Furthermore, since $\mathcal{T}$ belongs to $\mathfrak{T}_0$ and contains at least a ball of radius 1 around an edge, notice that $\mathrm{Fix}_G(\mathcal{T}_i) \subseteq \mathrm{Stab}_G(\mathcal{T})$. However, this inclusion must be strict since $\mathrm{Stab}_G(\mathcal{T})$ contains either the fixator of a vertex of $\mathcal{T}_i$ or the fixator of one of its edges. Now, Proposition 4.14 ensures the existence of an irreducible representation of $\mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})$ without a nonzero $\mathrm{Fix}_G(\mathcal{T}_i)/\mathrm{Fix}_G(\mathcal{T})$-invariant vector for every $i = 1, \ldots, s$. Since every proper subtree $\mathcal{R} \in \mathfrak{T}_0$ of $\mathcal{T}$ is contained in one of these $\mathcal{T}_i$, this representation is an $\mathcal{S}_0$-standard representation of $\mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})$. □

The next lemma treats the remaining case $l = 1$ where Proposition 4.14 does not apply.

LEMMA 4.16. *Let* $G = G_{(i)}^{+}(Y_0, Y_1)$ *and let* $C \in \mathcal{F}_{\mathcal{S}_0}$ *be a conjugacy class at height* 1. *Then, there exists an* $\mathcal{S}_0$*-standard representation of* $\mathrm{Aut}_G(C)$.

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

PROOF. Lemma 4.5 ensures the existence of a vertex $v \in V(T)$ such that $C = C(\mathrm{Fix}_G(B_T(v,1)))$. Let $\mathcal{T} = B_T(v,r)$. We recall as observed in Section 4.4 that $\mathrm{Aut}_G(C) \cong \mathrm{Stab}_G(\mathcal{T})/\mathrm{Fix}_G(\mathcal{T})$, where

$$\mathrm{Stab}_G(\mathcal{T}) = \{g \in G \mid g\mathcal{T} \subseteq \mathcal{T}\} = \{g \in G \mid gv = v\} = \mathrm{Fix}_G(v).$$

In particular, $\mathrm{Aut}_G(C)$ can be realised as a the group of automorphisms of $B_T(v,1)$ obtained by restricting the action of $\mathrm{Fix}_G(v)$. Furthermore,

$$\begin{aligned}\mathfrak{H}_{\mathcal{S}_0}(\mathrm{Fix}_G(\mathcal{T})) = \{&\mathrm{Fix}_G(\mathcal{R})/\mathrm{Fix}_G(\mathcal{T}) \mid \mathcal{R} \in \mathfrak{T}_0, \mathcal{R} \subsetneq \mathcal{T} \\ &\text{and } \mathcal{R} \text{ is maximal for this property}\} \\ = \{&\mathrm{Fix}_G(f)/\mathrm{Fix}_G(B_T(v,1)) \mid f \in E(B_T(v,1))\}.\end{aligned}$$

Let d be the degree of v in T, let $X = E(B_T(v,1))$ and let $e \in X$. Since $\mathrm{Alt}(d) \leq \underline{G}(v)$ and since $d \geq 4$, notice that $\underline{G}(v)$ is 2-transitive. In particular, $\mathrm{Aut}_G(C)$ is 2-transitive on X and [Sem23, Lemma 2.27] ensures the existence of an irreducible representation σ of $\mathrm{Aut}_G(C)$ without a nonzero $\mathrm{Fix}_{\mathrm{Aut}_G(C)}(e)$-invariant vector. Since $\mathrm{Fix}_G(v)$ is transitive on $E(B_T(v,1))$, this representation does not admit a nonzero $\mathrm{Fix}_{\mathrm{Aut}_G(C)}(f)$-invariant vector for any $f \in E(B_T(v,1))$. The lemma follows from the fact that $\mathrm{Fix}_{\mathrm{Aut}_G(C)}(f) = \mathrm{Fix}_G(f)/\mathrm{Fix}_G(B_T(v,1))$. □

5. Simple Radu groups are CCR

Let T be a (d_0,d_1)-semi-regular tree with $d_0, d_1 \geq 4$. Let i be a legal colouring of T and let $Y_0, Y_1 \subseteq \mathbb{N}$ be two finite subsets. The purpose of this section is to exploit the classification of the irreducible representations of $G^+_{(i)}(Y_0,Y_1)$ obtained from Sections 2 and 4.4 to prove that $G^+_{(i)}(Y_0,Y_1)$ is uniformly admissible and hence CCR.

We recall that a totally disconnected locally compact group G is uniformly admissible if for every compact open subgroup K, there exists a positive integer k_K such that $\dim(\mathcal{H}_\pi^K) < k_K$ for every irreducible representation π of G. In particular, uniformly admissible groups are CCR. The following classical result ensures that the spherical representations of any Radu group are uniformly admissible.

THEOREM 5.1. *Let T be a (d_0,d_1)-semi-regular tree with $d_0, d_1 \geq 3$, let $G \leq \mathrm{Aut}(T)$ be a closed noncompact subgroup acting transitively on the boundary of T and let $v \in V(T)$. Then, for every integer $n \geq 1$, there exists a constant $k_n \in \mathbb{N}$ such that $\dim(\mathcal{H}_\pi^{K_n}) < k_n$ for every spherical representation π of G admitting a nonzero $\mathrm{Fix}_G(v)$-invariant vector and where $K_n = \mathrm{Fix}_G(B_T(v,n))$.*

PROOF. Let $K = \mathrm{Fix}_G(v)$ and let μ be the Haar measure of G renormalised in such a way that $\mu(K) = 1$. Theorems 2.2 and 2.3 ensure that (G,K) is a Gelfand pair and we observe that $\dim(\mathcal{H}_\pi^K) = 1$. Now let ξ be a unit vector of $\mathcal{H}_\pi^K$ and let η be a unit vector of $\mathcal{H}_\pi^{K_n}$. Notice that

$$\varphi_{\xi,\eta} : G \longrightarrow \mathbb{C} : g \mapsto \langle \pi(g)\xi, \eta\rangle$$

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

is a K-right invariant and K_n-left invariant continuous function. However, since $\dim(\mathcal{H}_\pi^K) = 1$, notice for all $g, h \in G$ that

$$\begin{aligned}\int_K \varphi_{\xi,\eta}(gkh)\, \mathrm{d}\mu(k) &= \Big\langle \int_K \pi(gkh)\xi, \eta \Big\rangle \\ &= \Big\langle \pi(h)\xi, \int_K \pi(k^{-1})\pi(g^{-1})\eta \Big\rangle \\ &= \langle \pi(h)\xi, \alpha(\eta, g)\xi\rangle = \overline{\alpha(\eta, g)}\varphi_{\xi,\xi}(h)\end{aligned}$$

for some $\alpha(\eta, g) \in \mathbb{C}$. However, $\varphi_{\xi,\xi}(1_G) = 1$ and hence $\overline{\alpha(\eta, g)} = \varphi_{\xi,\eta}(g)$. This implies for all $g, h \in G$ that

$$\int_K \varphi_{\xi,\eta}(gkh)\, d\mu(k) = \varphi_{\xi,\eta}(g)\varphi_{\xi,\xi}(h). \tag{5-1}$$

Since $\varphi_{\xi,\eta}$ is K-right invariant and K_n-left invariant, notice that it can be realised as a function $\phi : Gv \to \mathbb{C}$ on the orbit Gv of v in $V(T)$ that is constant on the K_n-orbits of v. However, since K_n is an open subgroup of the compact group K, the index of K_n in K is finite. Since K is transitive on the boundary of the tree, this implies that K_n has finitely many orbits on ∂T. In particular, there exists an integer $N_n \geq \max\{2, n\}$ such that $\partial T(w, v)$ is contained in a single K_n-orbit for all $w \in \partial B_T(v, N_n)$, where $\partial T(w, v)$ is the set of ends of $T(w, v) = \{u \in V(T) \mid d_T(u, w) < d_T(u, v)\}$ that are not vertices. Now notice that the value of ϕ on every vertex $w \in Gv$ can be computed iteratively from the values that ϕ takes on vertices of $B_T(v, N_n)$. More precisely, let w be any vertex in $Gv - B_T(v, N_n)$ and let u be the vertex of Gv at distance 2 from w that is closer to $B_T(v, N_n)$ than any other vertex of Gv at distance 2 from w. Let $t \in G$ be such that $d_T(v, tv) = 2$, $g \in G$ be such that $gv = u$ and note from Equation (5-1) (taking $h = t$) that

$$\int_K \phi(gktv)\, d\mu(k) = \phi(u)\varphi_{\xi,\xi}(t). \tag{5-2}$$

Now note that every vertex in $\{gktv : k \in K\}$ is either in the same K_n-orbit as w or is strictly closer to $B_T(v, N_n)$ than w. In particular, Equation (5-2) expresses $\phi(w)$ as a linear combination of the values that ϕ takes on vertices that are strictly closer to $B_T(v, N_n)$ than w. Now, for each of these vertices that do not already belong to $B_T(v, N_n)$, we can apply the same reasoning and obtain the value that ϕ takes on it in terms of the values that ϕ takes on vertices that are even closer to $B_T(v, N_n)$. Since N_n has been chosen to be at least 2, this process must end at some point and one obtains an expression for $\phi(w)$ as a linear combination of the values that ϕ takes on the vertices of $B_T(v, N_n)$. This implies that the space $\mathcal{L}_n$ of functions $\varphi : G \to \mathbb{C}$ that are K-right invariant, K_n-left invariant and satisfy Equation (5-1) has finite dimension bounded by the cardinality k_n of $B_T(v, N_n)$. Furthermore, since π is irreducible, notice that ξ is cyclic and therefore that the linear map $\Psi_n : \mathcal{H}_\pi^{K_n} \to \mathcal{L}_n : \eta \to \overline{\varphi_{\xi,\eta}}$ is injective. This proves as desired that $\dim(\mathcal{H}_\pi^{K_n}) \leq \dim(\mathcal{L}_n) \leq k_n < +\infty$. □

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

In addition, the following classical result provides a bound for every special and cuspidal representation.

THEOREM 5.2 [HC70, Corollary of Theorem 2]. *Let G be a locally compact group, π be an irreducible square-integrable representation of G and $K \leq G$ be a compact open subgroup. Then, there exists a positive integer $k_{K,\pi}$ depending on K and π such that* $\dim(\mathcal{H}_\pi^K) \leq k_{K,\pi}$.

Putting it all together, we can prove the result announced at the beginning of the section.

THEOREM 5.3. *$G_{(i)}^+(Y_0, Y_1)$ is uniformly admissible, and hence CCR.*

PROOF. Let $G = G_{(i)}^+(Y_0, Y_1)$. Let $K \leq G$ be a compact open subgroup, and let $v \in V$. For each $n \in \mathbb{N}$, let $K_n = \mathrm{Fix}_G(B_T(v, n))$ and notice that Theorem 5.1 ensures the existence of a positive integer k_n such that $\dim(\mathcal{H}_\pi^{K_n}) \leq k_n$ for every spherical representation π of G. Now, for every $n \in \mathbb{N}$, we let Σ_n be the subset of all equivalence classes of nonspherical irreducible representations of G admitting nonzero K_n-invariant vectors. The classification of special and cuspidal representations of G provided by Theorems 2.5 and 4.12 ensures that Σ_n is finite. Furthermore, for every $\sigma \in \Sigma_n$, Theorem 5.2 provides a constant $k_{\sigma,n}$ such that $\dim(\mathcal{H}_\pi^{K_n}) \leq k_{\sigma,n}$. In particular, for each $n \in \mathbb{N}$, the constant $k'_n = \max_{\sigma\in\Sigma_n} k_{\sigma,n}$ is finite and $\dim(\mathcal{H}_\pi^{K_n}) < k'_n$ for each special and cuspidal representation π of G. It follows that $\dim(\mathcal{H}_\pi^{K_n}) \leq \max\{k_n, k'_n\}$ for all $\pi \in \widehat{G}$ and every $n \in \mathbb{N}$. However, since $(K_n)_{n\in\mathbb{N}}$ is a basis of neighbourhoods of the identity, there exists some $N \in \mathbb{N}$ such that $K_N \subseteq K$. It follows that

$$\dim(\mathcal{H}_\pi^K) \leq \max\{k_N, k'_N\} \quad \forall \pi \in \widehat{G}.$$

□

Appendix. Irreducible representations of a group and subgroups of index 2

The purpose of this appendix is to explain the relations between the irreducible unitary representations of a locally compact group G and the irreducible representations of its closed subgroups $H \leq G$ of index 2. Among other things, Theorem A.2 below explains the correspondence between these representations. Furthermore, when G is a totally disconnected locally compact group, we show that G is uniformly admissible if and only if H is uniformly admissible; see Lemma A.9.

The relevance of this appendix is given by Theorem 3.5, which ensures that every Radu group G belongs to a finite chain $H_n \geq \cdots \geq H_0$ with $n \in \{0, 1, 2, 3\}$ such that $H_n = G$, $[H_t : H_{t-1}] = 2$ for all t and where H_0 is conjugate in $\mathrm{Aut}(T)^+$ to $G_{(i)}^+(Y_0, Y_1)$ if T is a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 6$. As a direct consequence, we therefore obtain a description of the irreducible representations of these groups and observe that they are uniformly admissible. More precisely, the spherical and special representations of any Radu group G are classified by Section 2 and a description of the cuspidal representations of these groups can be obtained from the description of cuspidal representations of $G_{(i)}^+(Y_0, Y_1)$ given in Section 4 by applying Theorem A.2 several times.

A.1. Preliminaries. Let G be a locally compact group and let $H \leq G$ be a closed normal subgroup of finite index (in particular, H is open in G). The purpose of this section is to recall three operations that can be applied either to the representations of G or to those of H.

We start with the conjugation of representations. For every $g \in G$, we denote by

$$c(g) : G \to G : h \mapsto ghg^{-1}$$

the conjugation map and for every representation π of H, we define the morphism $\pi^g = \pi \circ c(g)$. Since H is normal in G, notice that π^g is a well-defined representation of H on the Hilbert space $\mathcal{H}_\pi$. This representation is called the **conjugate representation** of π by g. Furthermore, notice that the conjugate representation π^g depends up to equivalence only on the coset gH and that π^g is irreducible if and only if π is irreducible. In particular, the action by conjugation of G on $\widehat{H}$ passes to the quotient G/H.

Now, let G' be another topological group, let $\phi : G \to G'$ be a continuous group homomorphism, let π be a representation of G and let $\mathcal{X}$ be a unitary character of G'. We define the **twisted representation** $\pi^{\mathcal{X}}$ as the representation of G on $\mathcal{H}_\pi$ given by

$$\pi^{\mathcal{X}}(g) = \mathcal{X}(\phi(g))\pi(g) \quad \forall g \in G.$$

Notice that this representation is still continuous and unitary since $\mathcal{X}, \pi$ and ϕ are continuous group homomorphisms, and since $\mathcal{X}$ and π are unitary. Note also that $\pi^{\mathcal{X}} \cong \pi$ if $\mathcal{X}$ is the trivial representation of G'.

LEMMA A.1. *$\pi^{\mathcal{X}}$ is irreducible if and only if π is irreducible.*

PROOF. Since $\mathcal{X}(g)$ is a unitary complex number for every $g \in G$, notice that, for every $\xi \in \mathcal{H}_\pi$, the subspace of $\mathcal{H}_\pi$ spanned by $\{\pi(g)\xi \mid g \in G\}$ is the same as the subspace spanned by $\{\mathcal{X}(g)\pi(g)\xi \mid g \in G\} = \{\pi^{\mathcal{X}}(g) \mid g \in G\}$. The result therefore follows from the fact that a representation is irreducible if and only if every nonzero vector is cyclic. □

Finally, we recall the notion of **induction**. Since most of the complexity vanishes when H is an open subgroup of G (because the quotient space G/H is discrete) and since this is the only setup encountered in these notes, we work under this hypothesis. We refer to [KT13, Chs. 2.1 and 2.2] for details. Let G be a locally compact group, let $H \leq G$ be an open subgroup and let σ be a representation of H. The induced representation $\mathrm{Ind}_H^G(\sigma)$ is a representation of G with representation space given by

$$\mathrm{Ind}_H^G(\mathcal{H}_\sigma) = \left\{\phi : G \to \mathcal{H}_\sigma \,\middle|\, \phi(gh) = \sigma(h^{-1})\phi(g), \sum_{gH \in G/H} \langle \phi(g), \phi(g)\rangle < +\infty\right\}.$$

For $\psi, \phi \in \mathrm{Ind}_H^G(\mathcal{H}_\sigma)$,

$$\langle \psi, \phi\rangle_{\mathrm{Ind}_H^G(\mathcal{H}_\sigma)} = \sum_{gH \in G/H} \langle \psi(g), \phi(g)\rangle.$$

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

Equipped with this inner product, $\mathrm{Ind}_H^G(\mathcal{H}_\sigma)$ is a separable complex Hilbert space. The induced representation $\mathrm{Ind}_H^G(\sigma)$ is the representation of G on $\mathrm{Ind}_H^G(\mathcal{H}_\sigma)$ defined by

$$[\mathrm{Ind}_H^G(\sigma)(h)]\phi(g) = \phi(h^{-1}g) \quad \forall \phi \in \mathrm{Ind}_H^G(\mathcal{H}_\sigma) \quad \text{and} \quad \forall g, h \in G.$$

A.2. The explicit correspondence. We come back to the context in which we are interested. Let G be a locally compact group and let $H \leq G$ be a closed subgroup of index 2 in G. In particular, H is an normal open subgroup of G and the quotient $G' = G/H$ is isomorphic to the cyclic group of order two. Let τ denote the only irreducible nontrivial representation of G/H. Let $t \in G - H$, let $\phi : G \to G/H$ be the canonical projection on the quotient and, for every representation π of G, let π^τ be the twisted representation of G as defined in Section A.1. The purpose of this section is to prove the following theorem.

THEOREM A.2. *For every irreducible representation π of G:*

- $\pi \ncong \pi^\tau$ *if and only if* $\mathrm{Res}_H^G(\pi)$ *is an irreducible representation of H and in that case,* $\mathrm{Res}_H^G(\pi) \cong \mathrm{Res}_H^G(\pi)^t$;
- $\pi \cong \pi^\tau$ *if and only if* $\mathrm{Res}_H^G(\pi) \cong \sigma \oplus \sigma^t$ *for some irreducible representation σ of H and in that case,* $\sigma \ncong \sigma^t$.

For every irreducible representation σ of H:

- $\sigma \ncong \sigma^t$ *if and only if* $\mathrm{Ind}_H^G(\sigma)$ *is an irreducible representation of G and in that case,* $\mathrm{Ind}_H^G(\sigma) \cong \mathrm{Ind}_H^G(\sigma)^\tau$;
- $\sigma \cong \sigma^t$ *if and only if* $\mathrm{Ind}_H^G(\sigma) \cong \pi \oplus \pi^\tau$ *for some irreducible representation π of G and in that case,* $\pi \ncong \pi^\tau$.

Furthermore:

(1) *every irreducible representation π of G satisfies $\pi \leq \mathrm{Ind}_H^G(\sigma)$ for some irreducible representation σ of H;*
(2) *every irreducible representation σ of H satisfies $\sigma \leq \mathrm{Res}_H^G(\pi)$ for some irreducible representation π of G.*

The proof of this theorem is gathered in the following few results. First, let us recall the weak version of Frobenius reciprocity that we use several times throughout the proof.

THEOREM A.3 [Mac76, Corollary 1 of Theorem 3.8]. *Let G be a locally compact group and let $H \leq G$ be a closed subgroup of G. Then, for every representation π of G and every representation σ of H,*

$$\mathrm{I}(\mathrm{Ind}_H^G(\sigma), \pi) \leq \mathrm{I}(\sigma, \mathrm{Res}_H^G(\pi)),$$

where $\mathrm{I}(\pi_1, \pi_2)$ is the dimension of the space of intertwining operators between the two representations π_1 and π_2. Furthermore, if the index of H in G is finite, this relation becomes an equality.

Now, let us make a few observations.

LEMMA A.4. *Let σ be an irreducible representation of H. Then,*

$$\operatorname{Res}_H^G(\operatorname{Ind}_H^G(\sigma)) \cong \sigma \oplus \sigma^t.$$

PROOF. Set

$$\mathcal{L} = \{\varphi \in \operatorname{Ind}_H^G(\mathcal{H}_\sigma) \mid \operatorname{supp}(\varphi) \subseteq H\} \text{ and } \mathcal{L}^t = \{\varphi \in \operatorname{Ind}_H^G(\mathcal{H}_\sigma) \mid \operatorname{supp}(\varphi) \subseteq Ht\}.$$

By definition of $\operatorname{Ind}_H^G(\mathcal{H}_\sigma)$ and since $G = H \sqcup Ht$, it is clear that

$$\operatorname{Ind}_H^G(\mathcal{H}_\sigma) = \mathcal{L} \oplus \mathcal{L}^t.$$

Now, notice that $\mathcal{U} : \mathcal{L} \to \mathcal{H}_\sigma : \varphi \mapsto \varphi(1_G)$ is a unitary operator and that for every $h \in H$ and every $\varphi \in \mathcal{L}$,

$$\sigma(h)\mathcal{U}\varphi = \sigma(h)\varphi(1_G) = \varphi(h^{-1}) = [\operatorname{Ind}_H^G(\sigma)(h)]\varphi(1_G) = \mathcal{U}[\operatorname{Ind}_H^G(\sigma)(h)]\varphi.$$

In particular, this proves that $(\operatorname{Res}_H^G(\operatorname{Ind}_H^G(\sigma)), \mathcal{L}) \cong (\sigma, \mathcal{H}_\sigma)$. Similarly, notice that $\mathcal{U}^t : \mathcal{L}^t \to \mathcal{H}_\sigma : \varphi \mapsto \varphi(t^{-1})$ is a unitary operator and that for every $h \in H$ and every $\varphi \in \mathcal{L}^t$,

$$\begin{aligned}\sigma^t(h)\mathcal{U}^t\varphi &= \sigma(tht^{-1})\varphi(t^{-1}) = \varphi(t^{-1}th^{-1}t^{-1})\\ &= \varphi(h^{-1}t^{-1}) = [\operatorname{Ind}_H^G(\sigma)(h)]\varphi(t^{-1}) = \mathcal{U}^t[\operatorname{Ind}_H^G(\sigma)(h)]\varphi.\end{aligned}$$

This proves that $(\operatorname{Res}_H^G(\operatorname{Ind}_H^G(\sigma)), \mathcal{L}^t) \cong (\sigma^t, \mathcal{H}_{\sigma^t})$ and we obtain as desired that $\operatorname{Res}_H^G(\operatorname{Ind}_H^G(\sigma)) \cong \sigma \oplus \sigma^t$. □

LEMMA A.5. *Let π be an irreducible representation of G. Then exactly one of the following happens:*

- $\operatorname{Res}_H^G(\pi)$ *is an irreducible representation of H and* $\operatorname{Res}_H^G(\pi) \cong \operatorname{Res}_H^G(\pi)^t$;
- $\operatorname{Res}_H^G(\pi) \cong \sigma \oplus \sigma^t$ *for some irreducible representation σ of H.*

PROOF. If $\operatorname{Res}_H^G(\pi)$ is an irreducible representation of H, notice for every $h \in H$ and every $\xi \in \mathcal{H}_\pi$ that

$$\begin{aligned}\pi(t)[\operatorname{Res}_H^G(\pi)(h)]\xi &= \pi(t)\pi(h)\xi = \pi(tht^{-1})\pi(t)\xi\\ &= [\operatorname{Res}_H^G(\pi)(tht^{-1})]\pi(t)\xi = [\operatorname{Res}_H^G(\pi)^t(h)]\pi(t)\xi.\end{aligned}$$

In particular, $\pi(t) : \mathcal{H}_\pi \to \mathcal{H}_\pi$ is an intertwining operator between $\operatorname{Res}_H^G(\pi)$ and $\operatorname{Res}_H^G(\pi)^t$, which settles the first case.

Now, suppose that $\operatorname{Res}_H^G(\pi)$ is not an irreducible representation of G. Since π is irreducible, any nonzero $\xi \in \mathcal{H}_\pi$ is a cyclic vector. Hence, the subspace spanned by $\{\pi(g)\xi \mid g \in G\}$ is dense in $\mathcal{H}_\pi$. However, since $\operatorname{Res}_H^G(\pi)$ is not irreducible, there exists a nonzero vector $\xi \in \mathcal{H}_\pi$ such that the subspace spanned by $\{\pi(h)\xi \mid h \in H\}$ is not dense in $\mathcal{H}_\pi$. Let M denote the closure of this space. First, let us show that $(\operatorname{Res}_H^G(\pi), M)$ is irreducible. To do this, let N be a proper $\pi(H)$-invariant subspace of M and let us show that $N = \{0\}$. To begin with, notice that for every closed $\pi(H)$-invariant subspace L of

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

$\mathcal{H}_\pi$, the subspace $\pi(t)L$ is also $\pi(H)$-invariant since $\pi(H)\pi(t)L = \pi(Ht)L = \pi(tH)L = \pi(t)L$. Now, since ξ is a cyclic vector for π, notice that $\mathcal{H}_\pi = M + \pi(t)M$ (where the sum is *a priori* not a direct sum). However, since $\mathcal{H}_\pi \neq M$ and since $t^2 \in H$, notice that $M \not\subseteq \pi(t)M$. In particular, replacing N by $N^\perp \cap M$ if necessary, we can assume that $\mathcal{H}_\pi \neq N + \pi(t)M$ and therefore that $\mathcal{H}_\pi \neq N + \pi(t)N$. However, $N + \pi(t)N$ is a closed $\pi(G)$-invariant subspace of π. Since π is irreducible, this implies that $N + \pi(t)N = \{0\}$ and therefore that $N = \{0\}$, which proves that $(\mathrm{Res}_H^G(\pi), M)$ is irreducible. Applying the same reasoning by taking $\xi \in M^\perp$, we also obtain that $(\mathrm{Res}_H^G(\pi), M^\perp)$ is irreducible. In particular, Res_H^G decomposes as a direct sum of two irreducible representations of H. Let σ denote the irreducible representation $(\mathrm{Res}_H^G(\pi), M)$ and notice from Theorem A.3 that

$$0 < I(\sigma, \mathrm{Res}_H^G(\pi)) = I(\mathrm{Ind}_H^G(\sigma), \pi).$$

In particular, $\pi \leq \mathrm{Ind}_H^G(\sigma)$, so that $\mathrm{Res}_H^G(\pi) \leq \mathrm{Res}_H^G(\mathrm{Ind}_H^G(\sigma))$. The result therefore follows from Lemma A.4, which ensures that $\mathrm{Res}_H^G(\mathrm{Ind}_H^G(\sigma)) \cong \sigma \oplus \sigma^t$. □

LEMMA A.6. *Let π be an irreducible representation of G such that $\pi \cong \pi^\tau$. Then,* $\mathrm{Res}_H^G(\pi) \cong \sigma \oplus \sigma^t$ *for some irreducible representation σ of H.*

PROOF. Lemma A.5 ensures that $\mathrm{Res}_H^G(\pi)$ is either irreducible or split as desired. Suppose for a contradiction that $\sigma = \mathrm{Res}_H^G(\pi)$ is irreducible and let $\mathcal{U} : \mathcal{H}_\pi \to \mathcal{H}_\pi$ be the unitary operator intertwining π and π^τ. Notice that π and π^τ have the same representation space $\mathcal{H}_\pi$. Furthermore, for every $h \in H$, we have that $\pi(h) = \pi^\tau(h) = \sigma(h)$. In particular, $\mathcal{U}$ is a unitary operator that intertwines σ with itself. Since σ is irreducible, this implies that $\mathcal{U}$ is a scalar multiple of the identity. However, this is impossible since for every $h \in H$ and every $\xi \in \mathcal{H}_\pi$,

$$\mathcal{U}\pi(th)\xi = \pi^\tau(th)\mathcal{U}\xi = -\pi(th)\mathcal{U}\xi.$$

We obtain as desired that $\mathrm{Res}_H^G(\pi) \cong \sigma \oplus \sigma^t$ for some irreducible representation σ of H when $\pi \cong \pi^\tau$. □

PROPOSITION A.7. *Let σ be an irreducible representation of H. Then, the following hold:*

- $\sigma \ncong \sigma^t$ *if and only if* $\mathrm{Ind}_H^G(\sigma)$ *is an irreducible representation of G and in that case* $\mathrm{Ind}_H^G(\sigma) \cong \mathrm{Ind}_H^G(\sigma)^\tau$*;*
- $\sigma \cong \sigma^t$ *if and only if* $\mathrm{Ind}_H^G(\sigma) \cong \pi \oplus \pi^\tau$ *for some irreducible representation π of G.*

PROOF. Theorem A.3 ensures that

$$\mathrm{I}(\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_H^G(\sigma)) = \mathrm{I}(\sigma, \mathrm{Res}_H^G(\mathrm{Ind}_H^G(\sigma)).$$

In particular, in light of Lemma A.4, $\mathrm{I}(\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_H^G(\sigma)) = 1$ (that is, $\mathrm{Ind}_H^G(\sigma)$ is irreducible) if and only if $\sigma \ncong \sigma^t$. Furthermore, in that case, Theorem A.3 ensures

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

that

$$\begin{aligned}\mathrm{I}(\,\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_H^G(\sigma)^\tau) &= \mathrm{I}(\sigma, \mathrm{Res}_H^G\,(\,\mathrm{Ind}_H^G(\sigma)^\tau)) \\ &= \mathrm{I}(\sigma, \mathrm{Res}_H^G\,(\,\mathrm{Ind}_H^G(\sigma))) \\ &= \mathrm{I}(\,\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_H^G(\sigma)) = 1,\end{aligned}$$

which proves that $\mathrm{Ind}_H^G(\sigma) \cong \mathrm{Ind}_H^G(\sigma)^\tau$ and settles the first case.

Now, suppose that $\sigma \cong \sigma^t$. In that case, $\mathrm{Ind}_H^G(\mathcal{H}_\sigma)$ must split as a sum of two nonzero closed G-invariant subspaces M and M'. However, since $\mathrm{Res}_H^G(\mathrm{Ind}_H^G(\sigma))$ splits as a sum of two irreducible representations of H by Lemma A.4, and since every G-invariant subspace is H-invariant, M and M' do not admit any proper invariant subspaces. This proves that $\mathrm{Ind}_H^G(\sigma) \cong \pi \oplus \pi'$ for some irreducible representations π and π' of G. Furthermore, since $\mathrm{Res}_H^G(\pi) = \mathrm{Res}_H^G(\pi^\tau)$, Theorem A.3 ensures that

$$\mathrm{I}(\mathrm{Ind}_H^G(\sigma), \pi) = \mathrm{I}(\sigma, \mathrm{Res}_H^G(\pi)) = \mathrm{I}(\sigma, \mathrm{Res}_H^G(\pi^\tau)) = \mathrm{I}(\mathrm{Ind}_H^G(\sigma), \pi^\tau)$$

for every irreducible representation π of G. In particular, if $\pi \not\cong \pi^\tau$, we obtain that $\mathrm{Ind}_H^G(\sigma) \cong \pi \oplus \pi^\tau$. However, if $\pi \cong \pi^\tau$, notice from Lemma A.6 that $\mathrm{Res}_H^G(\pi) \cong \sigma \oplus \sigma^t$. Hence, since $\sigma \cong \sigma^t$, Theorem A.3 implies that

$$\mathrm{I}(\mathrm{Ind}_H^G(\sigma), \pi) = \mathrm{I}(\sigma, \mathrm{Res}_H^G(\pi)) = \mathrm{I}(\sigma, \sigma \oplus \sigma^t) > 1,$$

which proves that $\mathrm{Ind}_H^G(\sigma) \cong \pi \oplus \pi \cong \pi \oplus \pi^\tau$. □

The first part of Theorem A.2 follows from Lemmas A.4, A.5, A.6, Proposition A.7, and from the impossibility to have simultaneously that $\pi \cong \pi^\tau$ and that $\sigma \cong \sigma^t$. Indeed, if $\pi \cong \pi^\tau$, Lemma A.6 ensures that $\mathrm{Res}_H^G(\pi) \cong \sigma \oplus \sigma^t$. However, if $\sigma \cong \sigma^t$, Proposition A.7 ensures that $\mathrm{Ind}_H^G(\sigma) \cong \pi \oplus \pi^\tau$. In particular, if these conditions were satisfied simultaneously, one would obtain that

$$\mathrm{Res}_H^G\,(\,\mathrm{Ind}_H^G(\sigma)) \cong \sigma \oplus \sigma^t \oplus \sigma \oplus \sigma^t \cong 4\sigma,$$

which is impossible since Proposition A.7 ensures that

$$\mathrm{Res}_H^G(\mathrm{Ind}_H^G(\sigma)) \cong \sigma \oplus \sigma^t \cong 2\sigma.$$

The following result completes the proof of Theorem A.2.

LEMMA A.8. *Every irreducible representation π of G satisfies $\pi \leq \mathrm{Ind}_H^G(\sigma)$ for some irreducible representation σ of H and every irreducible representation σ of H satisfies $\sigma \leq \mathrm{Res}_H^G(\pi)$ for some irreducible representation π of G.*

PROOF. Let π be an irreducible representation of G and let us show that $\pi \leq \mathrm{Ind}_H^G(\sigma)$ for some irreducible representation σ of H. Notice from Theorem A.3 that

$$\mathrm{I}(\mathrm{Ind}_H^G(\mathrm{Res}_H^G(\pi)), \pi) = \mathrm{I}(\mathrm{Res}_H^G(\pi), \mathrm{Res}_H^G(\pi)) \geq 1,$$

which proves that $\pi \leq \mathrm{Ind}_H^G(\mathrm{Res}_H^G(\pi))$. If $\mathrm{Res}_H^G(\pi)$ is irreducible, the result follows trivially. However, if $\mathrm{Res}_H^G(\pi)$ is not irreducible, Lemma A.5 ensures that

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

$\mathrm{Res}_H^G(\pi) \cong \sigma \oplus \sigma^t$ for some irreducible representation σ of H. In particular, since $\pi \leq \mathrm{Ind}_H^G(\mathrm{Res}_H^G(\pi)) \cong \mathrm{Ind}_H^G(\sigma) \oplus \mathrm{Ind}_H^G(\sigma^t)$, we obtain either that $\pi \leq \mathrm{Ind}_H^G(\sigma)$ or that $\pi \leq \mathrm{Ind}_H^G(\sigma^t)$.

Now, let σ be an irreducible representation of H and let us show that $\sigma \leq \mathrm{Res}_H^G(\pi)$ for some irreducible representation π of G. Notice from Theorem A.3 that

$$\mathrm{I}(\sigma, \mathrm{Res}_H^G(\mathrm{Ind}_H^G(\sigma))) = \mathrm{I}(\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_H^G(\sigma)) \geq 1,$$

which proves that $\sigma \leq \mathrm{Res}_H^G(\mathrm{Ind}_H^G(\sigma))$. If $\mathrm{Ind}_H^G(\sigma)$ is irreducible, the result follows trivially. However, if $\mathrm{Ind}_H^G(\sigma)$ is not irreducible, Proposition A.7 ensures that $\mathrm{Ind}_H^G(\sigma) \cong \pi \oplus \pi^\tau$ for some irreducible representation π of G. In particular, it follows that $\sigma \leq \mathrm{Res}_H^G(\pi)$ or that $\sigma \leq \mathrm{Res}_H^G(\pi^\tau)$. □

Finally, as a consequence of Theorem 5.3 and of the correspondence provided by Theorem A.2, the following lemma ensures that every Radu group on a (d_0, d_1)-semi-regular tree with $d_0, d_1 \geq 6$ is uniformly admissible.

LEMMA A.9. *Let G be a totally disconnected locally compact group and let H be a closed subgroup of index* 2. *Then, G is uniformly admissible if and only if H is uniformly admissible.*

PROOF. Suppose that G is uniformly admissible and let K be a compact open subgroup of H. Since H has index 2 in G, it is a clopen subgroup of G, which implies that K is a compact open subgroup of G. Since G is uniformly admissible, there exists a constant $k_K \in \mathbb{N}$ such that $\dim(\mathcal{H}_\pi^K) \leq k_K$ for every irreducible representation π of G. Let σ be an irreducible representation of H. Theorem A.2 ensures that $\mathrm{Ind}_H^G(\sigma)$ is either irreducible or splits as a sum of two irreducible representations of G. However, notice from Theorem A.3 that

$$\mathrm{I}(\sigma, \mathrm{Res}_H^G(\mathrm{Ind}_H^G(\sigma))) = \mathrm{I}(\mathrm{Ind}_H^G(\sigma), \mathrm{Ind}_H^G(\sigma)) \geq 1,$$

which implies that $\sigma \leq \mathrm{Res}_H^G(\mathrm{Ind}_H^G(\sigma))$. All together, this proves that

$$\dim(\mathcal{H}_\sigma^K) \leq \dim(\mathcal{H}^K_{\mathrm{Ind}_H^G(\sigma)}) \leq 2k_K$$

and H is uniformly admissible.

Suppose now that H is uniformly admissible and let K be a compact open subgroup of G. Since H has index 2 in G, it is a clopen subgroup of G and $K \cap H$ is a compact open subgroup of H. Since H is uniformly admissible, this implies the existence of a constant $k_{K\cap H}$ such that $\dim(\mathcal{H}_\sigma) \leq k_{K\cap H}$ for every irreducible representation σ of H. Furthermore, Theorem A.2 ensures that $\mathrm{Res}_H^G(\pi)$ is either an irreducible representation of G or splits as a direct sum of 2 irreducible representations of H. This implies that

$$\dim(\mathcal{H}_\pi^K) \leq \dim(\mathcal{H}_\pi^{K\cap H}) = \dim(\mathcal{H}^{K\cap H}_{\mathrm{Res}_H^G(\pi)}) \leq 2k_{K\cap H}.$$

Hence, G is uniformly admissible. □

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press

Acknowledgements

I warmly thank Pierre-Emmanuel Caprace for all the insightful discussions we shared and for all his comments on preliminary versions of this paper. I am also very grateful to the anonymous referee for their suggestions in this paper review and for all their valuable comments.

References

[Ama03] O. E. Amann, 'Groups of tree-automorphisms and their unitary representations', PhD Thesis, ETH Zurich, 2003.

[BdlH20] B. Bekka and P. de la Harpe, *Unitary Representations of Groups, Duals, and Characters*, Mathematical Surveys and Monographs, 250 (American Mathematical Society, Providence, RI, 2020).

[Ber74] I. N. Bernšteĭn, 'All reductive p-adic groups are of type I', *Funktsional. Anal. i Prilozhen.* **8**(2) (1974), 3–6.

[BEW15] C. Banks, M. Elder and G. A. Willis, 'Simple groups of automorphisms of trees determined by their actions on finite subtrees', *J. Group Theory* **18**(2) (2015), 235–261.

[BK88] F. Bouaziz-Kellil, 'Représentations sphériques des groupes agissant transitivement sur un arbre semi-homogène', *Bull. Soc. Math. France* **116**(3) (1988), 255–278.

[BM00] M. Burger and S. Mozes, 'Groups acting on trees: from local to global structure', *Publ. Math. Inst. Hautes Études Sci.* **92** (2000), 113–150.

[CC15] P.-E. Caprace and C. Ciobotaru, 'Gelfand pairs and strong transitivity for Euclidean buildings', *Ergodic Theory Dynam. Systems* **35**(4) (2015), 1056–1078.

[Cho94] F. M. Choucroun, 'Analyse harmonique des groupes d'automorphismes d'arbres de Bruhat–Tits', *Mém. Soc. Math. Fr. (N.S.)* **58** (1994), 170.

[Cio15] C. Ciobotaru, 'A note on type I groups acting on d-regular trees', Preprint, 2015, arXiv:1506.02950.

[CKM23] P.-E. Caprace, M. Kalantar and N. Monod, 'A type I conjecture and boundary representations of hyperbolic groups', *Proc. Lond. Math. Soc.* **127**(2) (2023), 447–486.

[CMoS94] D. I. Cartwright, W. Młotkowski and T. Steger, 'Property (T) and $\tilde{A}_2$ groups', *Ann. Inst. Fourier (Grenoble)* **44**(1) (1994), 213–248.

[FTN91] A. Figà-Talamanca and C. Nebbia, *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees*, London Mathematical Society Lecture Note Series, 162 (Cambridge University Press, Cambridge, 1991).

[HC70] Harish-Chandra, *Harmonic Analysis on Reductive p-adic Groups*, Lecture Notes in Mathematics, 162 (Springer-Verlag, Berlin–New York, 1970); notes by G. van Dijk.

[HR19] C. Houdayer and S. Raum, 'Locally compact groups acting on trees, the type I conjecture and non-amenable von Neumann algebras', *Comment. Math. Helv.* **94**(1) (2019), 185–219.

[IP83] A. Iozzi and M. A. Picardello, 'Spherical functions on symmetric graphs', in *Harmonic Analysis (Cortona, 1982)*, Lecture Notes in Mathematics, 992 (eds. G. Mauceri, F. Ricci and G. Weiss) (Springer, Berlin, 1983), 344–386.

[Kal70] R. R. Kallman, 'Certain topological groups are type I', *Bull. Amer. Math. Soc. (N.S.)* **76** (1970), 404–406.

[KT13] E. Kaniuth and K. F. Taylor, *Induced Representations of Locally Compact Groups*, Cambridge Tracts in Mathematics, 197 (Cambridge University Press, Cambridge, 2013).

[Lan85] S. Lang, *SL2(R)*, Graduate Texts in Mathematics, 197 (Springer-Verlag, New York, 1985); reprint of the 1975 edition.

[Mac76] G. W. Mackey, *The Theory of Unitary Group Representations* (University of Chicago Press, Chicago–London, 1976); based on notes by James M. G. Fell and David B. Lowdenslager of lectures given at the University of Chicago, Chicago, IL, 1955, Chicago Lectures in Mathematics.

[Mat69] H. Matsumoto, 'Fonctions sphériques sur un groupe semi-simple p-adique', *C. R. Math. Acad. Sci. Paris* **269** (1969), 829–832.

[Mat77] H. Matsumoto, *Analyse Harmonique dans les Systèmes de Tits Bornologiques de Type Affine*, Lecture Notes in Mathematics, 590 (Springer-Verlag, Berlin–New York, 1977).

[Neb99] C. Nebbia, 'Groups of isometries of a tree and the CCR property', *Rocky Mountain J. Math.* **29**(1) (1999), 311–316.

[Ol'77] G. I. Ol'shanskii, 'Classification of the irreducible representations of the automorphism groups of Bruhat–Tits trees', *Funktsional. Anal. i Prilozhen.* **11**(1) (1977), 32–42, 96.

[Ol'82] G. I. Ol'shanskii, 'New "large" groups of type I', *J. Math. Sci.* **18** (1982), 22–39. (Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1980), 31–52, 228 (in Russian).

[Rad17] N. Radu, 'A classification theorem for boundary 2-transitive automorphism groups of trees', *Invent. Math.* **209**(1) (2017), 1–60.

[Sem23] L. Semal, 'Unitary representations of totally disconnected locally compact groups satisfying Ol'shanskii's factorisation', *Repr. Theory* **27**(11) (2023), 356–414.

[Smi17] S. M. Smith, 'A product for permutation groups and topological groups', *Duke Math. J.* **166**(15) (2017), 2965–2999.

LANCELOT SEMAL, UCLouvain, 1348 Louvain-la-Neuve, Belgium
e-mail: lancelot.semal@uclouvain.be

https://doi.org/10.1017/S1446788723000381 Published online by Cambridge University Press