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DIMENSION OF NULL SPACES WITH APPLICATIONS TO GROUP RINGS

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1. Introduction. In this paper we investigate methods for estimating the dimension of the null space of operators in a finite W^* algebra, the dimension being measured by the trace τ. For the most part we are concerned with operators A which are a finite linear combination of orthogonal unitaries. We give various results which show how certain information about the unitaries and the coefficients can be utilized to derive an upper bound for $\tau(N_A)$, where N_A is the null space of A.

This problem was motivated by and has application to the well known zero divisor problem for group rings. Suppose that G is a discrete group and let $\mathfrak{A}_0(G)$ be the algebraic group ring over the complex numbers C. That is $\mathfrak{A}_0(G)$ consists of all functions from G to C with finite support where the multiplication is by convolution. Then $\mathfrak{A}_0(G)$ acts by convolution on the Hilbert space $l_2(G)$ and its closure in the weak operator topology is a finite W^*-algebra with trace τ defined by $\tau(A) = (Ae, e)$. (Here (,) is the inner product in $l_2(G)$ and e is the group identity. We follow the convention throughout of identifying a group element with its characteristic function.) We denote this W^*-algebra by $\mathfrak{A}(G)$.

The zero divisor conjecture is that if G is torsion free then $\mathfrak{A}_0(G)$ has no proper zero divisors. See [**6**, Section 26] or [**7**, Section III] for more details. In fact this conjecture is stated for the case of an arbitrary field F, but the same argument as in [**7**, Theorem 22.7] shows that it would be sufficient to prove the result for $F = C$ in order to establish it for all fields of characteristic 0.

Now $\mathfrak{A}(G)$, being a W^*-algebra, has several zero divisors. So, if the conjecture is true one feels that there should be some operator theoretic condition on the elements of $\mathfrak{A}_0(G)$ which would render this so. One possibility is that the null space of such an element is suitably restricted. For example, suppose we could show for a given torsion free group G that $\tau(N_A) < \frac{1}{2}$ for all non-zero A in $\mathfrak{A}_0(G)$. This implies immediately that $\mathfrak{A}_0(G)$ has no proper zero divisors. If $AB = 0$ for some non-zero A and B in $\mathfrak{A}_0(G)$ we would have that $R_B \subseteq N_A$ (R denoting the closed range) and hence that $\tau(1 - N_{B^*}) \leqq \tau(N_A)$, which is impossible.

Of course the condition that $\tau(N_A) < \frac{1}{2}$ is stronger than the non-existence of zero divisors. On the other hand we feel that it is plausible that all non-zero elements of $\mathfrak{A}_0(G)$ for a torsion free group G have in fact zero null space. At any rate it is a problem of interest to determine those groups for which this is so.

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This condition is equivalent to showing that all non-zero self-adjoint elements of $\mathfrak{A}_0(G)$ with zero trace have no eigenvalues, since if $Ax = 0$, then $[A^*A - \tau(A^*A)]x = \tau(A^*A)x$. In § 4 we investigate the question of existence of eigenvalues of such elements. This leads to a consideration of certain operator equations involving weighted shifts.

The results of § 4 also have significance towards an auxilliary problem, that of computing or estimating operator norms. In the group ring case this question has been studied extensively by Akemann and Ostrand [**7**]. We discuss this in § 5.

It is easy to see that when G is the infinite cyclic group then all non-zero elements of $\mathfrak{A}_0(G)$ have zero null space. For example this is immediate from the proof of Corollary 6.5 below. Gardner [**4**] has shown that this property is in fact true for all torsion free abelian groups. We have, however, been unable to find any non-commutative examples. The natural case to consider next would be free non-abelian groups. It is of course well known that in this case $\mathfrak{A}_0(G)$ has no proper zero divisors, see [**7**]. We give some results, [Example 3.3] and [Theorem 4.5], which indicate the role of freeness in reducing the size of null spaces.

2. Notation and terminology. In the rest of this paper $\mathfrak{A}$ will denote a W^*-algebra and τ will be a finite faithful normal trace on $\mathfrak{A}$, normalized so that $\tau(1) = 1$. Then τ induces a positive bilinear form on $\mathfrak{A}$, $(A, B) = \tau(B^*A)$, and hence a norm, $||A||_2 = \tau(A^*A)^{1/2}$. This is to be distinguished from the operator norm which we denote by $||A||$. Two elements A and B of $\mathfrak{A}$ are *orthogonal* if $\tau(B^*A) = 0$.

If H is the Hilbert space completion of $\mathfrak{A}$ with respect to $|| \quad ||_2$ then $\mathfrak{A}$ acts in a natural way on H, the so called *standard representation*, with the identity 1 of $\mathfrak{A}$ as a cyclic and separating vector. (The action of $\mathfrak{A}(G)$ on $l_2(G)$ is such an example). We can therefore identify elements of $\mathfrak{A}$ as vectors in H, and we will follow this notation in § 4. See [**3**, Chapitre 1, §§ 4, 5, 6] for more details concerning the above.

For two projections E and F in $\mathfrak{A}$ we have [**3**, p. 226, Exercise 2(b)],

$$(2.1) \quad \tau(E) + \tau(F) = \tau(E \cup F) - \tau(E \cap F).$$

For any $A \in \mathfrak{A}$, N_A will denote the null space of A considered as a projection in $\mathfrak{A}$.

3. A formula for $\tau(N_A)$. In this section we derive a general formula for $\tau(N_A)$ and give some applications.

THEOREM 3.1. *Suppose that A is an element of $\mathfrak{A}$ with $\tau(A) \neq 0$. Let $\mathcal{J}$ be the weakly closed left ideal generated by A and let $\mathcal{N} = \{B \in \mathcal{J} : \tau(B) = 0\}$.*

Then

$$(3.1)\quad \tau(N_A) = 1 - \left(\frac{|\tau(A)|}{d(A,\mathscr{N})}\right)^2$$

where d is the metric associated with the norm $\|\ \|_2$.

Proof. Let $F = 1 - N_A$. Then $\mathscr{I} = \{B \in \mathfrak{A} : BF = B\}$, so that $F \in \mathscr{I}$ and is orthogonal to the hyperplane $\mathscr{N}$ in I. Hence for all B in $\mathscr{I}$, $d(B,\mathscr{N})$ is just the length of the projection of B onto the one dimensional subspace spanned by F. That is,

$$d(B,\mathscr{N})^2 = \frac{|(B,F)|^2}{\|F\|_2^2} = \frac{|\tau(B)|^2}{\tau(F)}$$

and taking $B = A$ yields (3.1).

We apply Theorem 3.1 to conclude that in a certain extreme case, finite linear combinations of unitaries always have a null space of dimension less than $\frac{1}{2}$.

THEOREM 3.2. *Let $(U_1, U_2, \ldots, U_n)$ be unitaries in $\mathfrak{A}$ such that for any two distinct ordered pairs (i,j) and (k,p) the unitaries $U_i^*U_j$ and $U_k^*U_p$ are orthogonal, except of course when $i = j$ and $k = p$. Then if $A = \sum_{i=1}^n a_iU_i$, where $a_1 \ldots a_n$ are non-zero scalars, we have $\tau(N_A) < \frac{1}{2}$.*

Proof. Let $B = A^*A$. Then

$$B = \sum_{i=1}^n |a_i|^2 + \sum_{i\neq j} \bar{a}_i a_j U_i^* U_j.$$

Now using the orthogonality condition,

$$\tau(B^*B) - |\tau(B)|^2 = \sum_{i\neq j} |a_i|^2|a_j|^2 < \left(\sum_{j=1}^n |a_j|^2\right)^2 = |\tau(B)|^2,$$

so that

$$1 - \frac{|\tau(B)|^2}{\|B\|_2^2} < \tfrac{1}{2}.$$

Now from Theorem 3.1 applied to B we have (since of course $0 \in \mathscr{N}$) that $\tau(N_A) = \tau(N_B) < \frac{1}{2}$.

In specific examples we can make more refined use of Theorem 3.1. Consider any A with $\tau(A) = 1$ and let $A' = A^*A - \|A\|_2^2A$. Then A' is an element of $\mathscr{N}$. We define $T(A) = A - cA'$ where $c = \tau(A^*A')\,\|A\|_2^{-2}$ if $A' \neq 0$ and $c = 0$ if $A' = 0$. It is easily seen that $A' = 0$ if and only if $A = \tau(E)^{-1}E$, for some non-zero projection E. The coefficient c is of course chosen in the usual way to minimize the distance to $\mathscr{N}$. Now $T(A)$ is an element in $\mathscr{I}$ of trace 1 and $\|T(A)\|_2^2 = \|A\|_2^2 - |c|^2\|A'\|_2^2$. So T is a mapping from the set of all

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elements of trace 1 to itself which strictly reduces the norm except for scalar multiples of projections which it leaves fixed. From Theorem 3.1 we can conclude that

$$(3.2)\quad \tau(N_A) \leqq 1 - \frac{1}{||T^n(A)||_2^2}$$

for all positive integers n and A such that $\tau(A) = 1$.

The problem in using this result is the difficulty in computing or estimating $||T^n(A)||_2^2$. We give one simple example.

Example 3.3. Let $U_1, U_2, \ldots, U_k$ be unitaries in $\mathfrak{A}$ and consider the self adjoint element

$$A = 1 + \sum_{i=1}^{k} (U_i + U_i^*).$$

We will estimate $\tau(N_A)$ under a certain freedom condition on the given unitaries.

We first need some notation. Let Y denote the set of all finite sequences of integers, $y = y(1), y(2), \ldots, y(rk)$, with length a multiple of k, and which do not contain any string of $k - 1$ consecutive zeros except possibly for the first $k - 1$ or last $k - 1$ entries. (Note that the sequence consisting of k zeros is in Y). Let w be the mapping from Y to $\mathfrak{A}$ defined by

$$w(y) = U_1^{y(1)} U_2^{y(2)} \ldots U_k^{y(k)} U_1^{y(k+1)} U_2^{y(k+2)} \ldots U_k^{y(rk)}.$$

For each non-negative integer n define an element of $\mathfrak{A}$

$$A_n = \sum_y w(y), \quad \text{with } ||y||_1 = n$$

where, as usual $||y||_1$ denotes $\sum_{i=1}^{rk} |y(i)|$. Note that $A_0 = 1$. Let $p = 2k - 1$. By a straightforward calculation, the number of terms in the sum defining A_n, $n \geqq 1$, is $(p + 1)p^{n-1}$, (of course the summands are not necessarily distinct), and

$$(3.3)\quad A_1^2 = (p + 1)A_2, \quad A_1A_n = A_nA_1 = pA_{n-1} + A_{n+1} \quad \text{for } n > 1.$$

From (3.3) we can write A_nA_m as a linear combination of A_j's for all n and m. For example,

$$(3.4)\quad A_2^2 = A_1(A_1A_2) - (p + 1)A_2 = p(p + 1) + (p - 1)A_2 + A_4.$$

Let us now make the following assumption: For any distinct y_1 and y_2 in Y with both $||y_1||_1$ and $||y_2||_1 \leqq 4$ the unitaries $w(y_1)$ and $w(y_2)$ are orthogonal.

For example, if $x_1, x_2, \ldots, x_k$ are elements of a group G which do not satisfy any non-trivial relation of length $\leqq 4$ then this assumption will hold for the unitaries of $\mathfrak{A}_0(G)$, $U_i = \alpha_i x_i$ where α_i is any scalar of absolute value 1.

We now have that for all n and $m \leqq 4$, A_n and A_m are orthogonal and

(3.5) $\quad ||A_n||_2{}^2 = (p+1)p^{n-1} \quad \text{if } n \geqq 1.$

The operator A is just $1 + A_1$ and using (3.3) – (3.5) we calculate, $A' = A^2 - (p+2)A = -pA_1 + A_2$ so that $||A'||_2{}^2 = (p+1)^2 p$ and then $\tau(AA') = -p(p+1)$. This gives $T(A) = 1 + (p+1)^{-1}[A_1 + A_2]$. Then

$$T(A)' = T(A)^2 - 2T(A) = \frac{1}{(p+1)^2}[2pA_1 + pA_2 + 2A_3 + A_4],$$

$$||T(A')||_2{}^2 = \frac{2p^2(4+p)}{(p+1)^3} \quad \text{and} \quad \tau[T(A)T(A)'] = \frac{p(p+2)}{(p+1)^2},$$

so that

$$||T^2(A)||_2{}^2 = 2 - \tau\frac{[T(A)T(A')]^2}{||T(A')||_2{}^2} = \frac{3p^2 + 16p + 12}{2(p+1)(4+p)}.$$

From 3.2 we conclude that

$$\tau(N_A) \leqq \frac{p^2 + 6p + 4}{3p^2 + 16p + 12}$$

which for odd p is easily seen to be $\leqq 11/31$, (the value at $p = 1$).

Of course, if in our assumption we replace the number 4 by a higher power of 2 we will obtain a lower upper bound, but the calculations get progressively more difficult. In the case that $w(y_1)$ and $w(y_2)$ are orthogonal for all distinct y_1 and y_2 we will show later that the null space is trivial [Theorem 4.5].

4. Non-existence of eigenvalues. Let A be a self-adjoint element in $\mathfrak{A}$ with $\tau(A) = 0$. We consider the question of determining whether or not A has eigenvalues. Since the eigenvalues are algebraic invariants of A, determined by the W^*-subalgebra of A generated by A and 1, we may first replace $\mathfrak{A}$ by this subalgebra. Then, by considering the standard representation as outlined in § 2 we can assume that H is the closed linear span of the elements $\{A^n : n = 0, 1, 2, \ldots\}$.

We now apply the Gram-Schmidt procedure to obtain vectors.

$$(4.1) \quad \begin{aligned} f_n &= A^n - \sum_{i=0}^{n-1} (A^n, e_i)e_i \\ e_n &= \frac{f_n}{||f_n||_2} \quad \text{or} \quad e_n = 0 \text{ if } f_n = 0 \end{aligned}$$

where we begin with $e_0 = 1$. So $\{e_n; e_n \neq 0\}$ is an orthonormal basis of K such that $\{e_n : n \leqq k\}$ spans the subspace consisting of polynomials in A of degree $\leqq k$. It follows that $(Ae_k, e_j) = 0$ for $j \geqq k + 2$ and by self-adjointness this holds as well for $j \leqq k - 2$. Now let

(4.2) $\quad \alpha_n = (Ae_n, e_{n+1}) \quad \text{and} \quad \beta_n = (Ae_n, e_n).$

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Each β_n is of course real and $\beta_0 = 0$ by the assumption that $\tau(A) = 0$. Moreover, each α_n is $\geqq 0$. In fact

$$(Af_n, e_{n+1}) = (A^{n+1}, e_{n+1}) = (f_{n+1}, e_{n+1})$$

so that

$$\text{(4.3)} \quad \alpha_n = \frac{||f_{n+1}||_2}{||f_n||_2} \quad \text{if } ||f_n|| \neq 0.$$

We have now that for $n \geqq 1$, $Ae_n = \alpha_{n-1}e_{n-1} + \alpha_n e_{n+1} + \beta_n e_n$.

The effect of this reduction then is that we are led to consider operators of the form $V + V^* + D$ where V is a unilateral weighted shift with non-negative weights (α_n) and D is a self-adjoint diagonal operator with diagonal (β_n), $\beta_0 = 0$, with respect to an orthonormal basis $\{e_n : n = 0, 1, 2, \ldots\}$. (Of course, the space may be finite dimensional but the above still holds with the appropriate interpretations.)

The problem is then to characterize those pairs of sequences (α_n) and (β_n) for which this operator has no eigenvalues. We first investigate conditions on (α_n) and (β_n) which will ensure a zero null space for $V + V^* + D$. Of course, a complete solution to this latter problem would yield a complete solution to the former. The problem in general appears to be quite difficult and we give here only a few partial results.

For convenience in notation we use $-D$ in place of D.

THEOREM 4.1. *Let (α_n) be a sequence of non-zero real numbers converging to* 1 *and let (β_n) be a sequence of real numbers converging to β, $|\beta| < 2$. Suppose moreover that both of the series*

$$\sum_{n=0}^{\infty} \left| \frac{\beta_n}{\alpha_n} - \beta \right| \quad \textit{and} \quad \sum_{n=0}^{\infty} \left| \frac{\alpha_n}{\alpha_{n+1}} - 1 \right| \quad \textit{are} < \infty.$$

Then $V + V^* - D$ has zero null space, where V is the unilateral shift with weights (α_n) and D has diagonal (β_n).

Proof. Suppose that $x = \sum_{n=1}^{\infty} \lambda_n e_n$ is a non-zero vector in the null space. By a direct calculation we must have

$$\text{(4.4)} \quad \lambda_1 = \frac{\beta_0\lambda_0}{\alpha_0} \quad \text{and} \quad \lambda_{n+1} = \frac{\beta_n\lambda_n - \alpha_{n-1}\lambda_{n-1}}{\alpha_n} \quad \text{for } n \geqq 1.$$

If $\lambda_0 = 0$ we would have that $x = 0$. We can therefore assume that $\lambda_0 = 1$ and hence that λ_n is real for all n. Note also that if $\lambda_n \neq 0$ and $\lambda_{n+1} = 0$ then $\lambda_{n+2} \neq 0$, which shows that the sequence (λ_n) cannot have two consecutive zeros.

Let w be a complex number of absolute value 1 such that Re $(w) = \frac{1}{2}\beta$ and let $r = \frac{1}{4}|\text{Im}(w)|$. By assumption $r \neq 0$. Let

$$\sigma_n = \lambda_{n+1} - \frac{\bar{w}\alpha_n\lambda_n}{\alpha_{n+1}} \quad \text{and} \quad \delta_n = \left(\frac{\beta_n}{\alpha_n} - \beta\right) + \bar{w}\left(1 - \frac{\alpha_n}{\alpha_{n+1}}\right).$$

Since $w + \bar{w} = \beta$ and $w\bar{w} = 1$ we see from (4.4) that

(4.5) $\sigma_{n+1} = w\sigma_n + \delta_{n+1}\lambda_{n+1}.$

Choose a positive integer N such that $\sum_{n=N}^{\infty} |\delta_n| < r$ and $|\alpha_n/\alpha_{n+1}| \geqq \frac{1}{2}$ for $n \geqq N$. Let $k \geqq N$ be such that $|\lambda_k| \geqq |\lambda_n|$ for $n \geqq N$. Since (λ_n) is an l_2 sequence such a k exists and by our preliminary remarks $\lambda_k \neq 0$. By recursion on (4.5)

$$\sigma_{n+k} = w^n\sigma_k + \sum_{i=1}^{n} w^{n-i}\delta_{k+i}\lambda_{k+i}.$$

Now $|w^n\sigma_k| \geqq |\mathrm{Im}\sigma_k| \geqq 2r\,|\lambda_k|$ and the absolute value of the second term is $\leqq |\lambda_k| \sum_{i=0}^{n} |\delta_{k+i}| \leqq r\,|\lambda_k|$. Hence $|\sigma_{n+k}| \geqq r\,|\lambda_k|$ for all $n \geqq 0$. This shows that (σ_n) does not converge to 0 and so neither does (λ_n), a contradiction.

In the case where the sequences (α_n) and (β_n) are eventually constant the above proof obviously works for $|\beta| = 2$. However in general if $|\beta| \geqq 2$ or if the convergence to β is not as rapid as indicated then the conclusion of Theorem 4.1 may fail to hold as shown by the following examples.

Example 4.2. For each of the following sequences (β_n) there is a non-zero vector $\sum_{n=0}^{\infty} \lambda_n e_n$ in the null space of $V + V^* - D$ where $(\alpha_n) = 1$ for all n. The verifications are straightforward.

(a) For any $\beta > 2$, choose λ such that $0 < \lambda < 1$ and $\lambda + \lambda^{-1} = \beta$. Let $\beta_0 = \lambda$ and $\beta_n = \beta$ for $n > 0$. Then let $\lambda_n = \lambda^n$.

(b) Let $\beta_0 = \frac{1}{2}$, $\beta_n = [1 + (n^2 + 2n)^{-1}]$ for $n > 0$. So $(\beta_n - 2)$ is an l_1 sequence. Let $\lambda_n = (n + 1)^{-1}$.

(c) Let $\beta_0 = \frac{1}{2}$, $\beta_n = (-1)^{n+1}(2n + 2)/(n^2 + 2n)$. So (β_n) is a l_2 sequence. Let $\lambda_n = \pm(n + 1)^{-1}$ where the sign is positive if $n \equiv 0$ or 1 mod 4 and negative if $n \equiv 2$ or 3 mod 4.

THEOREM 4.3. *Suppose that* $\alpha_0^2 \leqq |\beta_0|$ *and for all* $n \geqq 1$, $|\alpha_n| \leqq 1$ *and* $|\beta_n| \geqq 2$. *Then* $V + V^* - D$ *has zero null space.*

Proof. Using notation as above suppose that x is a non-zero vector in the null space and let $r_n = \lambda_{n+1}/\lambda_n$. So r_n is defined for those n with $\lambda_n \neq 0$ and for such n we have from (4.4) that

$$r_{n+1} = \frac{1}{\alpha_n}\left[\beta_{n+1} - \frac{\alpha_{n-1}}{r_n}\right]$$

showing that $|r_n| \geqq 1$ implies that $|r_{n+1}| \geqq 1$. Now $r_0 = \beta_0/\alpha_0$ so that

$$r_1 = \frac{1}{\alpha_1}\left[\beta_1 - \frac{\alpha_0^2}{\beta_0}\right].$$

Hence $|r_1| \geqq 1$ so that $|r_n| \geqq 1$ for all $n \geqq 1$ contradicting the fact that (λ_n) is an l_2 sequence.

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It would be pleasant if we could reduce our question to one involving the case where $(\alpha_n) = 1$ for all n, i.e. V is the usual unilateral shift which we denote by S. The conditions on (α_n) in both Theorems 4.1 and 4.3 are of course trivially satisfied in this case. This can often be done for questions involving V itself since under some fairly simple conditions V is similar via an invertible diagonal operator to cS for some constant c [**5**, Problem 76]. Unfortunately however, $(V + V^*)$ is not similar to $c(S + S^*)$ via a diagonal operator unless (α_n) is constant to begin with. This can be seen by a simple calculation. Nonetheless, we may in some cases reduce to considering $S + S^*$. For a given nonzero sequence (α_n) define a sequence k_n by $k_0 = 1$, and $k_n = \alpha_{n-1} k_n^{-1}$ for $n \geqq 1$. We have then

$$k_{2n} = \frac{\alpha_1\alpha_3 \dots \alpha_{2n-1}}{\alpha_0\alpha_2 \dots \alpha_{2n-2}}, \quad k_{2n+1} = \frac{\alpha_0\alpha_2 \dots \alpha_{2n}}{\alpha_1\alpha_3 \dots \alpha_{2n-1}}$$

for $n \geqq 1$.

THEOREM 4.4. *Suppose that (β_n) is an l_1 sequence and $|k_n|$ is bounded away from* 0 *and* ∞. *Then $V + V^* - D$ has zero null space.*

Proof. Suppose that $x = \sum_{n=0}^{\infty} \lambda_n e_n$ is any vector in the null space. Let $\lambda_n' = k_n \lambda_n$, $\beta_n' = k_n^{-2} \beta_n$. After a routine calculation we see from (4.4) that

$$\lambda_{n+1}' = \beta_n' \lambda_n' - \lambda_{n-1}'$$

Therefore $x' = \sum_{n=0}^{\infty} \lambda_n' e_n$ is in the null space of $S + S - D'$, where D has diagonal (β_n'). Now (β_n') is also an l_1 sequence so Theorem 4.1 shows that each λ_n' and hence each $\lambda_n = 0$.

Note that the conditions on (k_n) will be satisfied whenever $|\alpha_n|$ converges monotonically to 1. For example if α_n increases to 1 and $|\alpha_N| \geqq \frac{1}{2}$, then for $2n > N$ we have $1 \leqq |k_{2n}| \leqq |\alpha_0|^{-1}$ and $|\alpha_0| \leqq |k_{2n+1}| \leqq 1$.

For a particular application of Theorems 4.1 and 4.3 let $U_1, U_2 \dots U_k$, $k \geqq 2$, be a set of unitaries in $\mathfrak{A}$ such that in the notation of Example 3.3, $w(y_1)$ and $w(y_2)$ are orthogonal for any distinct y_1 and y_2 in Y. Let $T = \sum_{i=1}^{k} U_i$. We can consider two self adjoint operators of trace 0, $T + T^*$ and $T^*T - k$.

THEOREM 4.5. *Both* (a) $T + T^*$ *and* (b) $T^*T - k$ *have no eigenvalues.*

Proof. (a) $T + T^*$ is just the operator A_1 of Example 3.3. From (3.4) and the fact that now A_n and A_m are orthogonal for any $n \neq m$ it is easy to check that $f_n = A_n$ (see 4.1) and $\beta_n = 0$ (see 4.2) for all n. From (3.4) and (4.3) we see that the sequences (α_n) and (β_n) associated with $(2k - 1)^{-1/2} (T + T^*)$ are given by $\alpha_0 = \sqrt{2k/2k - 1}$, $\alpha_n = 1$ for $n \geqq 1$ and $\beta_n = 0$ for all n. Then for $|\mu| < 2$, μ is not an eigenvalue by Theorem 4.1 and for $|\mu| \geqq 2$, μ is not an eigenvalue by Theorem 4.3.

(b) $T^*T - k$ is again the operator A_1 of Example 3.3 but now based on the unitaries $\{U_i^{-1}U_j : 1 \leqq i < j \leqq n\}$. Let us relabel these as $\{V_1, V_2 \ldots V_r\}$ and our notation will now refer to the V's rather than the original U's. Then A_n's no longer satisfy the orthogonality conditions but we replace these by operators B_n defined as follows. For each unitary W in the group generated by the V's, let

$$l(w) = \min \{n : W = w(y) \text{ for some } y \text{ in } Y \text{ with } ||y_1|| = n\}$$

and let $B_n = \sum W$ with $l(W) = n$. Then B_n and B_m are orthogonal for $n \neq m$. Note that $B_0 = 1$ and $B_1 = T^*T - k$.

Now consider a product $V_h^{\epsilon}W$, $\epsilon = \pm 1$, where $l(W) = n$. Let y be the element in Y with $||y||_1 = n$ such that $w(y) = W$. and suppose that $y(m)$ is the first non-zero entry in y. Suppose that $V_h^{\epsilon} = U_i^{-1}U_j$ and $V_m^{\delta} = U_k^{-1}U_p$ where δ is 1 or -1 respectively if $y(m)$ is positive or negative respectively. Then $l(V_hW)$ will be; $n + 1$ if $j \neq k$, n if $j = k$, $i \neq p$, and $n - 1$ if $j = k$, $i = p$. From these remarks we can deduce recursively that

$$B_1^2 = B_2 + (k-2)B_1 + k(k-1)$$

and for $n \geqq 1$

$$B_1B_n = B_{n+1} + (k-2)B_n + (k-1)^2B_{n-1}$$
$$||B_n||_2^2 = k(k-1)^{2n-1}.$$

It follows that vectors f_n are just B_n and the sequences associated with $(k-1)^{-1}[T^*T - k]$ are given by $\alpha_0 = \sqrt{k/k-1}$, $\beta_0 = 0$ while for $n \geqq 1$, $\alpha_n = 1$ and $\beta_n = (k-1)^{-1}(k-2)$. Then Theorem 4.1 rules out eigenvalues μ such that $|\mu + (k-1)^{-1}(k-2)| < 2$ while if $|\mu + (k-1)^{-1}(k-2)| \geqq 2$ we must have $|\mu| \geqq (k-1)^{-1}k$ and this is ruled out by Theorem 4.3.

Remark. Note that for part (a) of Theorem 4.5 we can include the case $k = 1$. In part (b) we can weaken the condition on the U_i's to hold only for elements y of Y with $||y||_1 \geqq 1$ thus enabling us to include cases where one of the U_i's $= 1$.

In both parts of Theorem 4.5 the sequences (α_n) and (β_n) were constant for $n \geqq 1$. We can use Theorems 4.1 and 4.3 to eliminate some eigenvalues in other cases, but it seems difficult to find more general conditions which will guarantee their absence completely. For example consider the case where $\alpha_n = 1$ for all n, $\beta_0 = 1$, $\beta_1 = 1/6$, $\beta_2 = .3$ and $\beta_n = .55$ for $n \geqq 3$. Theorem 4.1 rules out eigenvalues in the interval $[-2.05, 1.45]$ and Theorem 4.3 rules out those which are $\leqq -2.05$ or $\geqq 11/6$. But 1.5 is an eigenvalue of $V + V^* - D$ with the eigenvector given by the sequence $\lambda_0 = 1$, $\lambda_1 = 1.5$ and $\lambda_n = (1.2).8^{n-3}$ for $n \geqq 3$.

Note that whenever A is such that $\tau(A^n) = 0$ for all odd n, (as in Theorem 4.5(a)), we do indeed get $\beta_n = 0$ for all n. From the definitions (4.1) it follows inductively that e_n is an even or odd polynomial in A respectively when n is even or odd respectively so that β_n is the trace of an odd polynomial.

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5. Operator norms. The remarks at the beginning of § 4 can also be applied to show that the operator norm of a self-adjoint element in $\mathfrak{A}$ of trace 0 is equal to the norm of the operator $V + V^* - D$ acting on the subspace spanned by $\{A^n : n = 0, 1, 2, \ldots\}$. In the group ring case this is a smaller subspace that used in [**1**, Theorem IIB], so it is possible that this may lead to some simplifications.

For any T in $\mathfrak{A}$ let $B = T^*T - ||T||_2^2$ and consider the sequences (α_n) and (β_n) associated with B by (4.2). Let $\alpha = \sup(\alpha_n)$, $\beta = \sup(|\beta_n|)$.

THEOREM 5.1. $||T|| \leqq [2\alpha + \beta + ||T||_2^2]^{1/2}$.

Proof. We have that $||T||^2 = ||T^*T||$ which, from the positivity of T^*T, is equal to $||B|| + ||T||_2^2$. The result now follows from the fact that the weighted shift with weights (α_n) has norm α [**5**, Problem 77] and the operator D with diagonal (β_n) has norm β.

Example 5.2. Consider the T of Theorem 4.5 We have $||T||_2^2 = k$, and from the proof of part (b), $\alpha = \sqrt{k(k-1)} \leqq k$ and $\beta = k - 2$. We see from Theorem 5.1 that

$$||T|| \leqq 2\sqrt{k - \tfrac{1}{2}} \leqq 2||T||_2.$$

Since T is a free operator in the sense of [**1**], the exact answer as computed in [**1**, Theorem IV K] is $||T|| = 2\sqrt{k-1}$. Note that our estimate provides a fairly simple way of deriving the fact that for such operators, $||T|| \leqq c||T||_2$ for some constant c, which is the important result for many applications such as in [**2**].

Actually it should be possible to obtain this exact answer from our data. It would be immediate if α_n were $= k - 1$ for all n but the presence of the larger initial weight seems to cause difficulties.

6. Complete orthogonality. Motivated by the group ring situation we define two unitaries U and V, in $\mathfrak{A}$ to be *completely orthogonal* if $\tau[(V^*U)^n] = 0$ for all positive integers n. If G is a torsion free group then every element of $\mathfrak{A}_0(G)$ is a finite linear combination of pairwise completely orthogonal unitaries.

THEOREM 6.1. *Let U be a unitary in $\mathfrak{A}$ such that* Re $\tau(U^n) \leqq d$ *for all positive integers n. Then*

$$\tau[N_{U-1}] \leqq d.$$

Proof. Let E denote the null space of $U - 1$ and let $r = \tau(E)$. Then if $W = U - E$, we have $U^n = E + W^n$ and hence that

$$\text{(6.1)} \quad \operatorname{Re}\tau(W^n) \leqq d - r \quad \text{and} \quad \operatorname{Re}\tau(W^{*n}) \leqq d - r$$

for all positive integers n. Moreover, W is a partial isometry with both initial and range projections equal to $1 - E$. Hence for all positive integers n and $k \leqq n$, the product $W^k(W^*)^{2n-k}$ is equal to a power of either W or W^* unless

$k = n$ in which case it equals $1 - E$. Now $(W + W^*)^{2n}$ is a positive operator and from (6.1)

$$0 \leqq \operatorname{Re} \tau[(W + W^*)^{2n}] \leqq (1 - r)\binom{2n}{n} + (d - r)\left[2^{(2n)} - \binom{2n}{n}\right].$$

Dividing by $2^{(2n)}$ we obtain

(6.3) $\quad 0 \leqq [-r + s_n] + d[1 - s_n]$

where

$$s_n = \frac{\binom{2n}{n}}{2^{(2n)}}.$$

Now

$$s_n = \prod_{k=1}^{n} \frac{2k-1}{2k} = \frac{2n-1}{2n} \prod_{k=2}^{n} \frac{2k-3}{2k-2} \leqq \frac{2n-1}{2n} \prod_{k=2}^{n} \frac{2k-2}{2k-1}$$

$$s_n^2 \leqq \frac{2n-1}{2n}\left(\prod_{k=1}^{n} \frac{2k-1}{2k}\right)\left(\prod_{k=2}^{n} \frac{2k-2}{2k-1}\right) = \frac{2n-1}{4n^2}.$$

Hence s_n converges to 0 and from (6.2) we have that $r \leqq d$.

COROLLARY 6.2. *Let U be a unitary which is completely orthogonal to* 1 (i.e. $\tau(U^n) = 0$ *for all positive integers n). Then U has no eigenvalues.*

Proof. For any complex λ with $|\lambda| = 1$, the unitary λU is also completely orthogonal to 1, so it is sufficient to show that 1 is not an eigenvalue. This is immediate from Theorem 6.1.

We also could prove this from Theorem 4.5. If $Ux = x$ for some non-zero vector x then $(U^* + U)x = 2x$. But this contradicts part (a) of this theorem taking $k = 1$, (see the remark following the theorem), and $U_1 = U$.

COROLLARY 6.3. *Let $(U_0, U_1, \ldots, U_n)$ be unitaries in $\mathfrak{A}$ such that for some indices j and k, U_j and U_k are completely orthogonal. Then for any non-zero scalars $(a_0, a_1, \ldots, a_n)$ with $\sum_{i=1}^{n} |a_i| \leqq |a_0|$, the operator $A = \sum_{i=1}^{n} a_i U_i$ has zero null space.*

Proof. Multiplying by $a_0^{-1}U^*$ we can assume that $U_0 = 1$, and $a_0 = 1$. If $\sum_{i=1}^{n} |a_i| < 1$ the conclusion is immediate as A is easily seen to be invertible. So suppose that the sum is equal to 1. Then for any unit vector x in N_A we would have

$$\sum_{i=1}^{n} ||a_i U_i x|| = 1 = ||-x|| = \left\| \sum_{i=1}^{n} a_i U_i x \right\|.$$

From the strict convexity of Hilbert space it follows that $U_i x = -x$ for all i so that x is an eigenvector for $U_j^* U_k$, contradicting Theorem 6.2.

Note in particular that if U and V are completely orthogonal then Corollary 6.3 shows that $aU + bV$ has zero null space for any non-zero scalars a and b.

COROLLARY 6.4. *Let $A \in \mathfrak{A}$. Suppose that there is a unitary U in $\mathfrak{A}$ and a trace preserving automorphism Φ of $\mathfrak{A}$ such that $\Phi(A - U) = A - U$, while $\Phi(U)$ is completely orthogonal to U. Then $\tau(N_A) \leqq \frac{1}{2}$.*

Proof. Let E denote N_A and suppose that $\tau(E) > \frac{1}{2}$. Then $\tau(\Phi(E)) > \frac{1}{2}$ and $\Phi(E)$ is the null space of $\Phi(A)$. By (2.2) there is a non-zero vector x in $E \cap \Phi(E)$. So $0 = [A - \Phi(A)]x = [U - \Phi(U)]x$ which contradicts Corollary 6.3.

For a particular example of the use of this corollary let $A = \sum_{i=1}^{n} a_i U_i$ for some unitaries $U_1, \ldots U_n$ and non-zero scalars $a_1 \ldots a_n$. Suppose there is a unitary V which commutes with U_i for $i < n$ while $V^* U_n V$ is completely orthogonal to U_n. Apply Corollary 6.4, taking $U = U_n$ and $\Phi(B) = V^*BV$ to conclude that $\tau(N_A) \leqq \frac{1}{2}$.

COROLLARY 6.5. *Let U be a unitary in $\mathfrak{A}$ which is completely orthogonal to 1 and let p and q be two polynomials such that for some complex z of absolute value 1, $|p(z)| \neq |q(z)|$. Then for any unitary V in $\mathfrak{A}$ which commutes with U, the operator $A = p(U) + V q(U)$ has zero null spare.*

Proof. Let E be the null space of A. Then

$$p(U)E = -V q(U)E. \tag{6.3}$$

Since U commutes with A it commutes with E. Multiplying both sides of (6.3) by their adjoints we see that

$$[p(U)^* p(U) - q(U)^* q(U)]E = 0$$

That is, $f(U)E = 0$ where f is the function defined on $|z| = 1$ by $f(z) = |p(z)|^2 - |q(z)|^2$ which has the form $\sum_{n=-k}^{k} a_n z^n$ for some positive integer k and scalars a_n. By assumption, f is not the zero function so $f(U)$ factors as $cU^{-k} (c \neq 0)$ times a product of linear factors each of which have zero null space by Corollary 6.2. It follows that $E = 0$.

Note that in Corollaries 6.3–6.5 the only use made of complete orthogonality is the absence of eigenvalues so they are obviously true under more general conditions.

REFERENCES

1. C. A. Akemann and P. A. Ostrand, *Computing norms in group C^*-algebras*, Amer. J. Math. *98* (1976), 1015–1047.
2. ——— *On a tensor product C^*-algebra associated with the free groups on two generators*, preprint.
3. J. Dixmier, *Les algebres d'operateurs dans l'espace hilbertien*, second edition (Gauthier Villars, Paris, 1969).
4. L. T. Gardner, *The trivial null space conjecture for torsion free abelian groups*, preprint.
5. P. R. Halmos, *A hilbert space problem book* (Van Nostrand, Princeton, 1967).
6. D. S. Passman, *Infinite-group rings* (Marcel Dekker, New York, 1971).
7. W. Rudin and H. Schneider, *Idempotents in group rings*, Duke Math. J. *31* (1946), 585–602.

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