Can. J. Math. Vol. **43** (6), 1991 pp. 1243–1262

# POLYNOMIAL INVARIANT THEORY AND TAYLOR SERIES

*To Tim Rooney on his 65th birthday.*

JOHN E. GILBERT

1. **Introduction.** For any group $K$ and finite-dimensional (right) $K$-module $V$ let

$$\big(\pi(k)f\big)(v) = f(v\cdot k) \qquad (v \in V,\ k \in K), \tag{1.1}$$

be the right regular representation of $K$ on the algebra $\mathcal{P}(V)$ of polynomial functions on $V$. An *Isotypic Component* $\mathcal{V}_\tau$ of $\mathcal{P}(V)$ is the sum of all $K$-submodules of $\mathcal{P}(V)$ on which $\pi$ restricts to an irreducible representation $\tau \in \hat{K}$; each $f$ in $\mathcal{P}(V)$ can then be written as $f = \sum_\tau f_\tau$ with $f_\tau$ in $\mathcal{V}_\tau$. When $K$ is compact this decomposition can be achieved by integral methods. Indeed, if $\{\chi_\tau : \tau \in \hat{K}\}$ is the set of characters of the irreducible unitary representations of $K$, normalized so that $\chi_\tau * \chi_\tau = \chi_\tau$, then

$$f_\tau = \int_K \overline{\chi_\tau(k)}\pi(k)f\,dk \qquad \big(f \in \mathcal{P}(V)\big). \tag{1.2}$$

On the other hand, without restriction on $K$, Taylor series expansions often exhibit such decompositions using infinitesimal methods. For instance, when $V$ is regarded as a $GL(V)$-module, the usual Taylor series expansion

$$f(z) = \sum_{m=0}^{\infty} \frac{1}{m!}\left(z \,\Big|\, \frac{\partial}{\partial\zeta}\right)^m f\Big|_{\zeta=0} \qquad \big(f \in \mathcal{P}(V)\big), \tag{1.3}$$

is a precise expression of the fact that the isotypic components of the right regular representation of $GL(V)$ on $\mathcal{P}(V)$ are the spaces $\mathcal{P}_m(V)$ of polynomials homogeneous of degree $m$ on which $GL(V)$ acts irreducibly. Thus

$$\mathcal{P}(V) = \oplus\sum_{m=0}^{\infty} \mathcal{P}_m(V) \tag{1.4}$$

*identifies* the isotypic components, while the $m^{th}$-order homogeneous Taylor polynomial mapping

$$f \longrightarrow \frac{1}{m!}\left(z \,\Big|\, \frac{\partial}{\partial\zeta}\right)^m f\Big|_{\zeta=0} \qquad (z,\zeta \in V), \tag{1.5}$$

*defines* the $GL(V)$-equivariant projection of $\mathcal{P}(V)$ onto the isotypic component $\mathcal{P}_m(V)$.

Research supported in part by NSF Grants DMS 8502352, DMS 8812414.
Received by the editors October 16, 1990; revised April 15, 1991 .




https://doi.org/10.4153/CJM-1991-071-6 Published online by Cambridge University Press

There are many other well-known examples where the isotypic components of $\mathcal{P}(V)$ can be identified (cf., for instance [15]; [16]; [20]). By a *partition* we shall mean a finite or infinite sequence $\tau = (m_1, m_2, \ldots)$ of non-negative integers such that

$$\text{(1.6)} \qquad \begin{aligned} &\text{(i)} \quad m_1 \geq m_2 \geq \cdots \\ &\text{(ii)} \quad m_j = 0,\ j \text{ sufficiently large;} \end{aligned}$$

we then set

$$\text{(1.7)} \qquad \begin{aligned} &\text{(iii)} \quad \ell(\tau) = \max\{\, j : m_j \neq 0 \,\}, \\ &\text{(iv)} \quad |\tau| = \sum_j m_j. \end{aligned}$$

The Cartan-Schur-Weyl theory establishes a 1–1 correspondence between partitions $\tau$, $\ell(\tau) \leq n$, and irreducible polynomial representations of $GL_n$, *i.e.*, irreducible representations $GL_n \longrightarrow GL(W)$ which extend to polynomial mappings $M_n \longrightarrow \mathrm{Hom}(W)$. If $E, F$ are finite-dimensional vector spaces, let $\mathcal{V}_\tau(E)$, $\mathcal{V}_\tau(F)$ be the respective irreducible $GL(E)$- and $GL(F)$-modules corresponding to a partition $\tau$ with $\ell(\tau) \leq \min(\dim E, \dim F)$. (Note that here, as well as elsewhere, we use $\tau$ simultaneously as a partition (or highest weight), as a label for a representation space, and as a homomorphism.) The following very well-known theorem has many proofs valid in varying degrees of generality (cf. [9]; [16]; [22], ...).

THEOREM 1.7. *When $E \otimes F$ is regarded as a $GL(E) \times GL(F)$-module, then*

$$\text{(1.8)} \qquad \mathcal{P}(E \otimes F) \cong \oplus \sum_\tau \mathcal{V}_\tau(E) \otimes \mathcal{V}_\tau(F),$$

*the sum being taken over all partitions $\tau$, $\ell(\tau) \leq \min(\dim E, \dim F)$.*

Theorem (1.7) thus identifies the isotypic components of $\mathcal{P}(E \otimes F)$, just as (1.4) did in the special case when $E$ is one-dimensional. At an abstract level it is easy to set up differential operators which project $\mathcal{P}(E \otimes F)$ onto these isotypic components, but in practice it is important to have explicit expressions for them. Now, the symbol of the differential operator

$$\text{(1.9)} \qquad f(\zeta) \longrightarrow \Big(z \,\Big|\, \frac{\partial}{\partial \zeta}\Big)^m f(\zeta) \qquad \big(f \in \mathcal{P}(V)\big),$$

used to define the Taylor series (1.3) is the $m$-fold power of the dual pairing $(\cdot|\cdot)$ on $V \times V'$, and by the simplest case of the First Fundamental Theorem (FFT) of Invariant Theory, the set $\{\, (\cdot|\cdot)^m : m \geq 0\}$ of all such powers is a *linear* basis for the $GL(V)$-invariants in $\mathcal{P}(V \times V')$. On the other hand, $\{z \longrightarrow z^m : m \geq 0\}$ is the only set of characters of $GL_1(\mathbb{C})$ which extend to all of $\mathbb{C}$. Consequently, in the more general case the operators in (1.9) will be replaced by differential operators whose symbol is a $GL(E) \times GL(F)$-invariant in $\mathcal{P}(E \otimes F, E' \otimes F')$, and the series for $f$ in $\mathcal{P}(E \otimes F)$ follows taking a linear basis for such invariants derived from the characters of the polynomial representations of $GL(E)$. Such series expansions are valid whether the scalar field $\mathbb{F} = \mathbb{R}$ or

$\mathbb{C}$. As the action of $GL(E) \times GL(F)$ on $\mathcal{P}(E \otimes F)$ is multiplicity-free this isotypic decomposition is actually the irreducible decomposition. Actually, both Theorem 1.7 and the corresponding Taylor series version, Theorem 3.6, follow fairly quickly from well-known identities ("character identities") for symmetric functions. Invariant theory does become important, however, in the use of Capelli operators instead of Euler's operator in deriving variants of the $GL(E) \times GL(F)$-module theory for some $GL(F)$-modules of polynomial functions and $GL(E) \times O(F)$- or $O(F)$-modules of harmonic polynomial functions. These variants will be fundamental in the on-going study of first-order systems of over-determined elliptic differential operators $\mathfrak{d}$ (cf. [6], [7], [8]). The prototypical example of $\mathfrak{d}$ is the Cauchy-Riemann $\bar{\partial}$-operator, but examples including Hodge-deRham $(d, d^*)$- and $(\bar{\partial}, \bar{\partial}^*)$-systems, where $\mathfrak{d}$ commutes with the action of a Lie group $K$, *i.e.*, is an invariant differential operator, occur throughout harmonic analysis and representation theory. For these invariant operators the kernel of $\mathfrak{d}$ is contained in an eigenspace of an invariant second order elliptic differential operator, just as any analytic function is automatically harmonic. One fundamental question—the $K$-type analysis of ker$\mathfrak{d}$—is whether the kernel of $\mathfrak{d}$ can be distinguished within the associated eigenspace of the second order operator, and more generally among all smooth functions, by the occurrence or absence of particular $K$-types. Such is the case for $\bar{\partial}$, since every function

$$f(z) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \qquad (z = re^{i\theta} \in \mathbb{C}), \tag{1.10}$$

is harmonic, whereas $\bar{\partial} f = 0$ if and only if $a_n = 0, n < 0$; Thus the various more general Taylor series expansions are designed to replace (1.10), and the basic problem then comes in deciding which (and how) $\mathfrak{d}$ arises from the finite order invariant differential operators implementing the Taylor series expansions. These applications will be made elsewhere, however.

The author wishes to thank Ray Kunze for introducing him to the fascinating world of invariant theory, and to thank also Roger Richardson as well as the referee for comments and explanations which significantly determined the final form of this paper.

2. **Invariant operators, general theory.** Let $V$ be a finite-dimensional vector space over $\mathbb{F}$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, that is simultaneously a left $H$-module and right $K$-module with respect to groups $H, K$ (the two module actions being assumed to be associative). Then $\mu(h,k)\colon v \longrightarrow hvk^{-1}$ is a representation of $H \times K$ on $V$ and

$$g_1 g_2 = (h_1, k_1; v_1)(h_2, k_2; v_2) = (h_1h_2, k_1k_2; h_2^{-1} v_1 k_2 + v_1) \tag{2.1}$$

defines group multiplication on the semi-direct product $G = (H \times K)$ Ⓢ$V$. When $H \times K$ is identified with the subgroup $\{(h,k;0) : h \in H,\ k \in K\}$ of $G$ and $V$ with the space $(H \times K) \setminus G$ of right cosets, the canonical action of $G$ on this coset-space corresponds to the action of $G$ as a motion group of transformations

$$g = (h,k;w)\colon v \longrightarrow v.g = h^{-1}vk + w \qquad (g \in G,\ v \in V), \tag{2.2}$$

on $V$. The example of $V = \mathbb{F}^{r\times n}$ with $H = GL_r$ and $K = GL_n$ acting by matrix multiplication will be basic to the present paper.

If $(\mathcal{V}_\tau, \tau)$ is a finite-dimensional representation of $H \times K$, unitary or not, and $C^\infty(V, \mathcal{V}_\tau)$ the space of smooth $\mathcal{V}_\tau$-valued functions on $V$, let

$$(2.3) \qquad \big(\pi_\tau(g)f\big)(v) = \tau(h,k)f(v.g) \qquad (v \in V,\ g \in G),$$

be the representation of $G$ on $C^\infty(V, \mathcal{V}_\tau)$ *induced* from $(\mathcal{V}_\tau, \tau)$. Given two such representations, there is a canonical construction of finite-order differential operators $\partial\colon C^\infty(V, \mathcal{V}_\tau) \longrightarrow C^\infty(V, \mathcal{V}_\omega)$ which are invariant in the sense that

$$(2.4) \qquad \partial \circ \pi_\tau(g) = \pi_\omega(g) \circ \partial \qquad (g \in G),$$

(cf. [13], chap. II; [23], §5.4). Such operators will be said to be $H \times K$-*invariant* to emphasize their critical dependence on $H \times K$, the contribution of $V$ being merely to guarantee translation-invariance. We recall this construction in the various forms needed here and elsewhere, setting notation at the same time.

Each $X$ in $V$ gives rise by parallel translation to a vector field $\partial_X$,

$$(2.5) \qquad (\partial_X f)(v) = \frac{d}{dt} f(v + tX)\Big|_{t=0} \qquad \big(f \in C^\infty(V, \mathcal{V}_\tau)\big),$$

and $X \longrightarrow \partial_X$ extends to an isomorphism from the symmetric algebra $\mathcal{S}(V)$ onto the algebra $\mathcal{D}(C^\infty(V))$ of finite order, constant-coefficient differential operators on $C^\infty(V, \mathcal{V}_\tau)$. Thus $A \otimes X \longrightarrow A \circ \partial_X$ extends to a linear isomorphism $\lambda : S \longrightarrow \partial_S$ from $\mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega) \otimes \mathcal{S}(V)$ onto the space $\mathcal{D}\big(C^\infty(V, \mathcal{V}_\tau), C^\infty(V, \mathcal{V}_\omega)\big)$ of finite-order, translation-invariant differential operators taking $C^\infty(V, \mathcal{V}_\tau)$ into $C^\infty(V, \mathcal{V}_\omega)$. Those $S$ for which $\partial_S$ is $H \times K$ invariant are easily characterized.

THEOREM 2.6. *The mapping $\lambda : S \longrightarrow \partial_S$ is a linear isomorphism from the $H \times K$-invariants in* $\mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega) \otimes \mathcal{S}(V)$ *onto the $H \times K$-invariants in* $\mathcal{D}\big(C^\infty(V, \mathcal{V}_\tau), C^\infty(V, \mathcal{V}_\omega)\big)$.

There are useful alternative characterizations in the case of first-order operators. Fix a basis $\{e_1, \ldots, e_n\}$ for $V$, and let $\{\partial_1, \ldots, \partial_n\}$ be the corresponding set of vector fields. Then $\lambda : S \longrightarrow \partial_S$ associates to each $S = \sum_j A_j \otimes e_j$ in $\mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega) \otimes V$ the differential operator

$$(2.7) \qquad \partial_S{:}\, f \longrightarrow \sum_j (A_j \circ \partial_j) f \qquad \big(f \in C^\infty(V, \mathcal{V}_\tau)\big).$$

Such an $S$ is an $H \times K$-invariant precisely when the $\{A_j\}_{j=1}^n \subseteq \mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega)$ satisfy

$$(2.8) \qquad \sum_i k_{ij} A_i = \sum_\ell h_{\ell j}\, \omega(h,k) A_\ell\, \tau(h,k)^{-1} \qquad (h \in H,\ k \in K),$$

for each $j$, $1 \le j \le n$, $[h_{ij}]$ and $[k_{ij}]$ being the matrix representations

$$he_i = \sum_j h_{ij} e_j, \qquad e_i k = \sum_j k_{ij} e_j \qquad (h \in H,\ k \in K)$$

https://doi.org/10.4153/CJM-1991-071-6 Published online by Cambridge University Press

in $\mathbb{F}^{n\times n}$. Thus the first-order $H \times K$-invariant differential operators are all given by (2.7) for any choice of $A_1, \dots, A_n$ satisfying (2.8). In fact, if $\{e'_1, \dots, e'_n\}$ is the dual basis in $V'$, every $A$ in $\mathrm{Hom}(\mathcal{V}_\tau \otimes V', \mathcal{V}_\omega)$ arises as $T_S$ with $S = \sum_j (A \circ I_j) \otimes e_j$ and

$$I_j \colon \mathcal{V}_\tau \longrightarrow \mathcal{V}_\tau \otimes V', \qquad I_j \colon \xi \longrightarrow \xi \otimes e'_j.$$

Combining isomorphisms we thus obtain a linear isomorphism

$$\lambda : A \longrightarrow \partial_A f = (A \circ \nabla) f = \sum_j (A \circ I_j)\partial_j f \qquad \big(f \in C^\infty(V, \mathcal{V}_\tau)\big). \tag{2.9}$$

To characterize the invariants, let $(V', \mu')$ be the representation of $H \times K$ contragredient to $(V, \mu)$.

THEOREM 2.10. *The mapping $\lambda : A \longrightarrow \partial_A = A \circ \nabla$ is a linear isomorphism from the $H \times K$-invariants in* $\mathrm{Hom}(\mathcal{V}_\tau \otimes V', \mathcal{V}_\omega)$ *onto the first-order $H \times K$-invariants in* $\mathcal{D}\big(C^\infty(V, \mathcal{V}_\tau), C^\infty(V, \mathcal{V}_\omega)\big)$.

When $H, K$ are completely reducible and $\mathcal{V}_\omega = \mathcal{V}_\tau \otimes V'$, say, these invariants are easily described by applying Schur's lemma.

THEOREM 2.11. *If $H, K$ are completely reducible, the equivariant projections from $\mathcal{V}_\tau \otimes V'$ onto its irreducible $H \times K$-submodules form a linear basis for the $H \times K$-invariants in* $\mathrm{Hom}(\mathcal{V}_\tau \otimes V')$.

Clearly there will be an analogous result for the case $\mathcal{V}_\omega \neq \mathcal{V}_\tau \otimes V'$. All of the previous discussion is independent of the choice of basis $\{e_1, \dots, e_n\}$ for $V$, as will be subsequent discussion also.

Now let $\mathcal{P}(V', \mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega))$ be the space of $\mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega)$-valued polynomial functions on $V'$. With respect to the coordinate system $v' = (\xi_1, \dots, \xi_n)$ for $V'$ determined by $\{e'_1, \dots, e'_n\}$, each $P$ in $\mathcal{P}(V', \mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega))$ can be regarded as a polynomial $P = P(\xi_1, \dots, \xi_n)$ in $\xi_1, \dots, \xi_n$ with coefficients from $\mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega)$. Consequently,

$$\lambda : P \longrightarrow \partial_P = P(\partial_1, \dots, \partial_n) \tag{2.12}$$

defines a linear isomorphism from $\mathcal{P}\big(V', \mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega)\big)$ onto $\mathcal{D}\big(C^\infty(V, \mathcal{V}_\tau), C^\infty(V, \mathcal{V}_\omega)\big)$; in addition, $\partial_P = \partial_S$ when $P$ corresponds to $S$ under the standard identification of $\mathcal{P}\big(V', \mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega)\big)$ with $\mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega) \otimes \mathcal{S}(V)$.

THEOREM 2.13. *The mapping $\lambda : P \longrightarrow \partial_P = P(\partial_1, \dots, \partial_n)$ is a linear isomorphism from the $H \times K$-invariants in* $\mathcal{P}\big(V', \mathrm{Hom}(\mathcal{V}_\tau, \mathcal{V}_\omega)\big)$ *onto the $H \times K$-invariants in* $\mathcal{D}\big(C^\infty(V, \mathcal{V}_\tau), C^\infty(V, \mathcal{V}_\omega)\big)$.

Roughly speaking, therefore, invariant polynomials arise here as the symbol of invariant differential operators. The invariance property ensures that both the null-space and range-space of these operators are $H \times K$-modules when the operators act on $H \times K$-modules.

To incorporate ideas from classical invariant theory, let $P$ be an $H \times K$-invariant in $\mathcal{P}(V' \times V)$. Then by (2.13), $\lambda : P \longrightarrow \partial_P$ associates to each such $P$ an $H \times K$ invariant differential operator

$$f(\zeta) \longrightarrow (\partial_P f)(z, \zeta) = P\Big(\frac{\partial}{\partial\zeta}, z\Big) f(\zeta) \qquad (\zeta, z \in V), \tag{2.14}$$

from $C^\infty(V)$ into $C^\infty\big(V, \mathcal{P}(V)\big)$. One can think of these $\partial_P = P(\frac{\partial}{\partial\zeta}, z)$ as finite-order differential operators in $\zeta$ having polynomial functions of $z$ as coefficients. When $\zeta'$ denotes the transpose of any matrix $\zeta$, the pull-back $P \circ \gamma$,

$$\gamma : \mathbb{F}^{r\times n} \times \mathbb{F}^{r\times n} \longrightarrow \mathbb{F}^{r\times r}, \qquad \gamma : (\zeta, z) \rightarrow z\zeta', \tag{2.15}$$

of any $P$ in $\mathcal{P}(\mathbb{F}^{r\times r})$ clearly defines a $GL_n$-invariant in $\mathcal{P}(\mathbb{F}^{r\times n} \times \mathbb{F}^{r\times n})$, while $P \circ \gamma$ is a $GL_r \times GL_n$-invariant if and only if $P$ is $Ad\, GL_r$-invariant, *i.e.*,

$$\big(Ad(h)P\big)(x) = P(h^{-1}xh) = P(x) \qquad (h \in GL_r,\ A \in \mathbb{F}^{r\times r}). \tag{2.16}$$

The First Fundamental Theorem of Invariant Theory (FFT) (cf. [19]; [21], chap. XI; [24], chap. 1A) together with Theorem 2.13 thus gives

THEOREM 2.17. *The mapping*

$$\lambda : P \longrightarrow \partial_P = P\Big(z\frac{\partial'}{\partial\zeta}\Big) \qquad (z, \zeta \in \mathbb{F}^{r\times n}),$$

*is a linear isomorphism from* $\mathcal{P}(\mathbb{F}^{r\times r})$ *onto the* $GL_n$*-invariants in* $\mathcal{D}(C^\infty(\mathbb{F}^{r\times n}), C^\infty(\mathbb{F}^{r\times n} \times \mathbb{F}^{r\times n}))$.

The expression $z\frac{\partial'}{\partial\zeta}$ is to be interpreted as the $r \times r$-product obtained from the $r \times n$ matrices

$$z = [z_{rs}] \quad , \quad \frac{\partial}{\partial\zeta} = \Big[\frac{\partial}{\partial\zeta_{rs}}\Big]. \tag{2.18}$$

Equivalent formulations of $z\frac{\partial'}{\partial\zeta}$ will be useful. Let

$$z_j = (z_{j1}, \dots, z_{jn}) \quad , \quad \frac{\partial}{\partial\zeta_k} = \Big(\frac{\partial}{\partial\zeta_{k1}}, \dots, \frac{\partial}{\partial_{kn}}\Big) \tag{2.19}$$

be the rows of $z$ and $\frac{\partial}{\partial\zeta}$ respectively, and set

$$\Big(z_j \mid \frac{\partial}{\partial\zeta_k}\Big) = \sum_{\ell=1}^{n} z_{j\ell} \frac{\partial}{\partial\zeta_{k\ell}} \qquad (1 \le j, k \le r). \tag{2.20}$$

Then $z\frac{\partial'}{\partial\zeta}$ can be interpreted as the $r \times r$ matrix

$$D = [D_{jk}] = \Big[\Big(z_j \mid \frac{\partial}{\partial\zeta_k}\Big)\Big] \qquad (1 \le j, k \le r), \tag{2.21}$$

of $GL_n$-invariants $D_{jk} = \lambda(P_{jk})$ associated with the coordinate functions $P_{jk} : \eta \longrightarrow \eta_{jk}$ on $\mathbb{F}^{r\times r}$; classically, the $D_{jk}$ are known as *Polarization operators* (cf. [21], pp. 110, 207; [24], p. 5).

COROLLARY 2.22. *The mapping* $P \longrightarrow \partial_P = P(D)$, $D = z\frac{\partial'}{\partial\zeta}$ *is* $GL_r$*-equivariant, i.e.,*

$$\partial_{Ad(h)P} = (\lambda \otimes \lambda)(h) \circ \partial_P \circ \lambda(h)^{-1} \qquad (h \in GL_r); \tag{2.23}$$

*in particular,* $\partial_P$ *is* $GL_r \times GL_n$*-invariant if and only if* $P$ *is an* $Ad\, GL_r$*-invariant in* $\mathcal{P}(\mathbb{F}^{r\times r})$.

To describe $Ad\, GL_r$-invariants in $\mathcal{P}(\mathbb{F}^{r\times r})$, define $\{c_j\}_{j=1}^r$ by

$$\det(A - \lambda I) = \sum_{j=0}^{r} (-1)^j \lambda^j c_{r-j}(A) \qquad (A \in \mathbb{F}^{r\times r}). \tag{2.24}$$

It is well-known that

$$c_j(A) = \operatorname{tr}(A \otimes \cdots \otimes A)\big|_{\Lambda^j(\mathbb{F}^n)}, \tag{2.25}$$

and so

$$c_0(A) = I,\ c_1(A) = \operatorname{tr}(A), \ldots, c_r(A) = \det A. \tag{2.26}$$

These $c_j$ are certainly $Ad\, GL_r$-invariant. More generally (cf. [12], §2.1),

THEOREM 2.27. *The polynomials in the subalgebra of* $\mathcal{P}(\mathbb{F}^{r\times r})$ *generated by* $c_1, \ldots, c_r$ *are* $Ad\, GL_r$*-invariants, and, when* $\mathbb{F} = \mathbb{C}$, $\{c_j\}_{j=1}^r$ *generates freely the algebra of all* $Ad\, GL_r$*-invariants.*

COROLLARY 2.28. *The mapping* $\lambda : Q \longrightarrow \partial_Q$,

$$(\partial_Q f)(z, \zeta) = Q\left(c_1\left(z\frac{\partial'}{\partial\zeta}\right), \ldots, c_r\left(z\frac{\partial'}{\partial\zeta}\right)\right) f(\zeta), \qquad z, \zeta \in V,$$

*is a linear isomorphism from* $\mathcal{P}(\mathbb{F}^r)$ *into the* $GL_r \times GL_n$*-invariants in* $\mathcal{D}\big(\mathcal{C}^\infty(\mathbb{F}^{r\times n}), \mathcal{C}^\infty(\mathbb{F}^{r\times n} \times \mathbb{F}^{r\times n})\big)$*, which is surjective when* $\mathbb{F} = \mathbb{C}$.

It is instructive to see the connection with classical Cayley determinantal operators (cf. [21], p. 113). Denote by $\Delta^{j_1 \ldots j_s}_{k_1 \ldots k_s}(\cdot)$ the minor

$$\Delta^{j_1 \ldots j_s}_{k_1 \ldots k_s}(z) = \det \begin{vmatrix} z_{j_1 k_1} & \cdots & z_{j_1 k_s} \\ \vdots & & \vdots \\ z_{j_s k_1} & \cdots & z_{j_s k_s} \end{vmatrix} \tag{2.29}$$

formed from the $j_1, \ldots, j_s$ rows and $k_1, \ldots, k_s$ columns of $z \in \mathbb{F}^{r\times n}$. The Cayley operators $\Omega^{j_1 \ldots j_s}_{k_1 \ldots k_s}$ are just the differential operators

$$\Omega^{j_1 \ldots j_s}_{k_1 \ldots k_s}\left(\frac{\partial}{\partial \xi}\right) = \Delta^{j_1 \ldots j_s}_{k_1 \ldots k_s}\left(\frac{\partial}{\partial \xi}\right) \tag{2.30}$$

On the other hand, if $\{e_1, \ldots, e_n\}$ is the usual basis for $\mathbb{F}^n$,

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} \longrightarrow z_{j_1} \wedge \cdots \wedge z_{j_s} = \sum_{k_1 < \cdots < k_s} \Delta^{j_1 \ldots j_s}_{k_1 \ldots k_s}(z) e_{k_1} \wedge \cdots \wedge e_{k_s}$$

maps $\mathbb{F}^{r\times n}$ onto $\Lambda^j(\mathbb{F}^n)$, and the Cauchy-Binet theorem exhibits a dual pairing

$$(2.31)\qquad \begin{aligned}(z_{j_1}\wedge\cdots\wedge z_{j_s}\mid \xi_{j_1}\wedge\cdots\xi_{j_s}) &= \Delta^{j_1\ldots j_s}_{j_1\ldots j_s}(z\xi')\\ &= \sum_{k_1<\cdots<k_s}\Delta^{j_1\ldots j_s}_{k_1\ldots k_s}(z)\Delta^{j_1\ldots j_s}_{k_1\ldots k_s}(\xi)\end{aligned}$$

for $\Lambda^j(\mathbb{F}^n)\times\Lambda^j(\mathbb{F}^n)$ ([21], pp. 79,82). Regarding $\Delta^{j_1\ldots j_s}_{j_1\ldots j_s}$ as a polynomial on $\mathbb{F}^{r\times r}$, we thus obtain a $GL_n$-invariant differential operator

$$(2.32)\qquad \Delta^{j_1\ldots j_s}_{j_1\ldots j_s}\Big(z\frac{\partial'}{\partial\xi}\Big) = \sum_{k_1<\cdots<k_s}\Delta^{j_1\ldots j_s}_{k_1\ldots k_s}(z)\Omega^{j_1\ldots j_s}_{k_1\ldots k_s}\Big(\frac{\partial}{\partial\xi}\Big)$$

In view of (2.23), therefore,

$$(2.33)\qquad c_s\Big(z\frac{\partial'}{\partial\xi}\Big) = \sum_{j_1<\cdots<j_s}\Delta^{j_1\ldots j_s}_{j_1\ldots j_s}\Big(z\frac{\partial'}{\partial\xi}\Big).$$

is the $GL_r\times GL_n$-invariant differential operator arising from all possible choices of $s$ rows from $\mathbb{F}^{r\times n}$.

3. **Group-invariant Taylor series.** For the moment, let $V$ be an $H\times K$-module as in the previous section. By a group-invariant Taylor series we shall mean the representation of every $f$ in $C^\infty(V)$, or a particular $H\times K$ submodule of $C^\infty(V)$, as a sum of polynomials each of which lies in an $H\times K$ isotypic component, together with $H\times K$-invariant differential operators exhibiting this decomposition of $f$. The general relation between polynomial invariants and the present notion of Taylor series is easily seen, however. For if $\partial_P$ is the invariant operator associated in (2.14) with an $H\times K$-invariant $P$ in $P(V'\times V)$,

$$f(\zeta)\longrightarrow\partial_P f\big|_{\zeta=0}(z) = P\Big(\frac{\partial}{\partial\zeta},z\Big)f(\zeta)\big|_{\zeta=0}\qquad \big(f\in C^\infty(V)\big),$$

is an equivariant mapping whose range is an $H\times K$-submodule of $\mathcal{P}(V)$. For suitable $V$, $H$ and $K$ we might hope to recover $f$, formally at least, as

$$(3.1)\qquad f=\sum_P\frac{1}{|P|}\partial_P f\big|_{n=0}\qquad \big(f\in C^\infty(V)\big),$$

where the sum is taken over a *linear basis* for the space of $H\times K$-polynomial invariants, characters,... in $\mathcal{P}(V'\times V)$ and each $|P|$ is a constant. By restricting to polynomial $f$ in (3.1), convergence questions are avoided.

To exhibit one such basis, let $\tau=(m_1,m_2,\ldots)$ be a partition with $\ell(\tau)=k$, and denote by

$$(3.2)\qquad \tau'=(\mu_1,\mu_2,\ldots)=(\underbrace{k,\ldots,k}_{m_k},\underbrace{k-1,\ldots,k-1}_{m_{k-1}-m_k},\ldots,\underbrace{1,\ldots,1}_{m_1-m_2},0,\ldots)$$

its *conjugate partition* (cf. [17], p. 60; [18], p. 2). Now define $\chi_\tau\in\mathcal{P}(\mathbb{F}^{r\times r})$ by

$$(3.3)\qquad \chi_\tau=\det[c_{\mu_i-i+j}],\qquad \tau'=(\mu_1,\mu_2,\ldots),$$

with the convention that $c_{\mu_i - i + j} \equiv 0$ whenever $\mu_i - i + j < 0$ or $\mu_i - i + j > \ell(\tau)$. For instance, if $\tau = \rho_s = (\underbrace{1, \ldots, 1}_{s}, 0, \ldots)$, then

$$\rho_s' = (s, 0, \ldots), \qquad \chi_{\rho_s} = c_s \qquad (1 \leq s \leq r), \tag{3.4}$$

and so (2.25) ensures that $\chi_{\rho_s}$ is the character of the fundamental representation of $GL_r$ on $\Lambda^s(\mathbb{F}^r)$. More generally, $\chi_\tau$ is the character of the polynomial representation $\mathcal{V}_\tau(\mathbb{F}^r)$ of $GL_r$; on the other hand, an inspection of (3.3) shows that

$$\chi_\tau(A) = c_r(A)^{m_r} \chi_\sigma(A) = \Delta_r(A)^{m_r} \chi_\sigma(A) \qquad (A \in \mathbb{F}^{r \times r}), \tag{3.5}$$

when $\ell(\tau) = r$ and $\sigma = \tau - m_r \rho_r = (m_1 - m_2, \ldots, m_{r-1} - m_r, 0, \ldots)$.

THEOREM 3.6. *Fix $r$, $1 \leq r \leq n$. Then each $f$ in $\mathcal{P}(\mathbb{F}^{r \times n})$ can be written uniquely as*

$$f(z) = \sum_\tau \frac{1}{h(\tau)} \chi_\tau\Big(z \frac{\partial'}{\partial \zeta}\Big) f\Big|_{\zeta = 0} \qquad (z, \zeta \in \mathbb{F}^{r \times n}), \tag{3.7}$$

*the sum being taken over all partitions $\tau$, $\ell(\tau) \leq r$, where $h(\tau)$ is the Hook-Length*

$$h(\tau) = \prod_{j=1}^{\ell(\tau)} \big(m_j + \ell(\tau) - j\big)! \Big/ \prod_{i<j} (m_i - m_j + j - i)$$

*of the partition $\tau$.*

COROLLARY 3.8. *The set*

$$\Big\{ \chi_\tau\Big(z \frac{\partial'}{\partial \zeta}\Big) f\Big|_{\zeta = 0} : f \in \mathcal{P}(\mathbb{F}^{r \times n}) \Big\} \tag{3.9}$$

*is the unique irreducible $GL_r \times GL_n$-submodule of $\mathcal{P}(\mathbb{F}^{r \times n})$ isomorphic to $\mathcal{V}_\tau(\mathbb{F}^n) \otimes \mathcal{V}_\tau(\mathbb{F}^n)$.*

To identify (3.9), denote by $B_s$ the triangular subgroup of $GL_s$ having zero entries below the diagonal, and by $N_s$ its subgroup whose diagonal entries are all 1. Let $\lambda'$, $\pi$ be the representation

$$\big(\lambda'(h) f\big)(z) = f(h' z), \qquad \big(\pi(k) f\big)(z) = f(zk)$$

of $GL_r$ and $GL_n$ respectively on $\mathcal{C}^\infty(\mathbb{F}^{r \times n})$. Then the algebra

$$\mathcal{P}_L(\mathbb{F}^{r \times n}) = \big\{ f \in \mathcal{P}(\mathbb{F}^{r \times n}) : \lambda'(b) f = f,\ b \in N_r \big\}$$

of all $N_r$-invariants is a $GL_n$-module with respect to $\pi$; similarly, the algebra

$$\mathcal{P}_U(\mathbb{F}^{r \times n}) = \big\{ f \in \mathcal{P}(\mathbb{F}^{r \times n}) : \pi(b) f = f,\ b \in N_n \big\}$$

is a $GL_r$-module with respect to either of $\lambda$ and $\lambda'$. But the only $N_r$-invariants in $\mathcal{V}_\tau(\mathbb{F}^r)$ are $\mathbb{F}\,\phi_\tau$ with $\phi_\tau$ a highest weight vector, so

$$\mathcal{P}_L(\mathbb{F}^{r \times n}) \cong \oplus \sum_\tau \mathcal{V}_\tau(\mathbb{F}^n) \qquad \big(\ell(\tau) \leq r\big), \tag{3.10}$$

as a $GL_n$-module, while

$$\mathcal{P}_U(\mathbb{F}^{r\times n}) \cong \oplus \sum_{\tau} \mathcal{V}_\tau(\mathbb{F}^r) \qquad \big(\ell(\tau) \leq r\big), \tag{3.11}$$

is a $GL_r$-module. Now set

$$\Phi_\tau(x) = \Delta_1(x)^{m_1-m_2}\cdots\Delta_{r-1}(x)^{m_{r-1}-m_r}\Delta_r(x)^{m_r} \qquad (x \in \mathbb{F}^{r\times r}), \tag{3.12}$$

where $\tau = (m_1, m_2, \ldots)$ and $\Delta_s$ is the principal minor $\Delta^{1\ldots s}_{1\ldots s}$. Thus we can and shall regard $\Phi_\tau$ as a polynomial on $\mathbb{F}^{r\times r}$ and $\mathbb{F}^{r\times n}$, $r \leq n$. In fact, $\Phi_\tau$ is a character of $B_r$, just as $\chi_\tau$ was a character of $GL_r$, since

$$\lambda'(b)\Phi_\tau = \Phi_\tau(b)\Phi_\tau, \qquad \pi(b)\Phi_\tau = \Phi_\tau(b)\Phi_\tau. \tag{3.13}$$

On the other hand, the pull-back $\Phi_\tau \circ \gamma$ is a $GL_n$-invariant $P$ in $\mathcal{P}(\mathbb{F}^{r\times n} \times \mathbb{F}^{r\times n})$ having the additional semi-invariance property

$$P(b_1' z, b_2' \zeta) = \Phi_\tau(b_1 b_2) P(z, \zeta) \qquad (b_1, b_2 \in B_r); \tag{3.14}$$

in fact, by the bitriangular decomposition for $\mathbb{F}^{r\times r}$ ([21], p. 369), the $\Phi_\tau$ are a linear basis for such invariants. This leads us to the second Taylor series expansion.

THEOREM 3.15. *Fix $r$, $1 \leq r \leq n$. Then each $f$ in $\mathcal{P}_L(\mathbb{F}^{r\times n})$ can be written uniquely as*

$$f(z) = \sum_\tau \frac{1}{h(\tau)} \Phi_\tau\left(z\,\frac{\partial'}{\partial\zeta}\right) f\Big|_{\zeta=0} \qquad (z, \zeta \in \mathbb{F}^{r\times n}), \tag{3.16}$$

*the sum being taken over all partitions $\tau$, $\ell(\tau) \leq r$.*

COROLLARY 3.17. *The unique irreducible $GL_n$-submodule of $\mathcal{P}_L(\mathbb{F}^{r\times n})$ isomorphic to $\mathcal{V}_\tau(\mathbb{F}^n)$ is characterized by either of*

$$\text{(a)}\quad \left\{ \Phi_\tau\left(z\,\frac{\partial'}{\partial\zeta}\right) f\Big|_{\zeta=0} : f \in \mathcal{P}_L(\mathbb{F}^{r\times n}) \right\},$$

$$\text{(b)}\quad \mathcal{P}_\tau(\mathbb{F}^{r\times n}) = \left\{ f \in \mathcal{P}(\mathbb{F}^{r\times n}) : \lambda'(b) f = \Phi_\tau(b) f,\ b \in B_r \right\}. \tag{3.18}$$

Both (3.6) and (3.15) reduce to the classical Taylor series (1.3) when $r = 1$ since

$$\chi_\tau\left(z\frac{\partial}{\partial\zeta}\right) = \left(\sum_j z_j \frac{\partial}{\partial\zeta_j}\right)^m = \Phi_\tau\left(z\frac{\partial}{\partial\zeta}\right) \qquad \big(\tau = (m, 0, \ldots)\big)$$

and $\Phi_\tau(\zeta) = \zeta_1^m$ for $z, \zeta$ in $\mathbb{F}^n$. In particular, by Euler's theorem or direct calculation,

$$\left(\sum_j z_j \frac{\partial}{\partial\zeta_j}\right)^\ell \zeta_1^m\Big|_{\zeta=0} = z_1^\ell \left(\frac{\partial}{\partial\zeta_1}\right)^\ell \zeta_1^m\Big|_{\zeta=0} = m!\, z_1^m$$

when $\ell = m$, while

$$\left(\sum_j z_j \frac{\partial}{\partial\zeta_j}\right)^\ell \zeta_1^m\Big|_{\zeta=0} = z_1^\ell \left(\frac{\partial}{\partial\zeta_1}\right)^\ell \zeta_1^m\Big|_{\zeta=0} = 0$$

when $\ell \neq m$. The extension of these last results to arbitrary $r$, $1 \leq r \leq n$, will play a key role in establishing the generalization of (1.3) to (3.6) and (3.15) and of (1.5) to the corollaries (3.8) and (3.17).

THEOREM 3.19. *Fix r and let $\gamma, \tau$ be arbitrary partitions of length at most r. Then, for any $z, \zeta$ in $\mathbb{F}^{r\times n}$, $n \geq r$,*

$$\chi_\gamma\Big(z\,\frac{\partial'}{\partial\zeta}\Big)\Phi_\tau\Big|_{\zeta=0} = \Phi_\gamma\Big(z\,\frac{\partial'}{\partial\zeta}\Big)\Phi_\tau\Big|_{\zeta=0} = h(\tau)\Phi_\tau(z) \tag{3.20}$$

*when $\gamma = \tau$, while*

$$\chi_\gamma\Big(z\,\frac{\partial'}{\partial\zeta}\Big)\Phi_\tau\Big|_{\zeta=0} = 0 = \Phi_\gamma\Big(z\,\frac{\partial'}{\partial\zeta}\Big)\Phi_\tau\Big|_{\zeta=0} \tag{3.21}$$

*when $\gamma \neq \tau$.*

Perhaps not surprisingly in view of the earlier comments on the classical case, the proof of (3.19) hinges on the extension of Euler's operator $E = \sum_j \zeta_j \frac{\partial}{\partial\zeta_j}$ to arbitrary $r$. This role will be played by the Capelli operator $H_r$ to which we turn next before beginning the proofs of the series expansions and their corollaries.

4. **The Capelli Identities.** Although the Capelli operator $H_r$ and Capelli identities ([1]; [2]; [3]) have long been known to be a cornerstone of polynomial invariant theory (cf., for instance, [19]; [21]; [24]), their group-theoretic meaning has emerged only recently ([5]; [11]; [15]). It was Howe's work that made clear the role of Capelli operators as generalized Euler operators on polynomials of matrix argument.

Let $E = [E_{jk}] = \zeta\,\frac{\partial'}{\partial\zeta}$ by the $r \times r$-matrix of operators on $\mathcal{C}^\infty(\mathbb{F}^{r\times n})$ defined by

$$E_{jk}f = \Big(\zeta_j \mid \frac{\partial}{\partial\zeta_k}\Big)f(\zeta) \qquad (1 \leq j,k \leq r). \tag{4.1}$$

Then

$$d\lambda' : A = [a_{jk}] \longrightarrow \operatorname{tr}(AE') = \sum_{j,k} a_{jk}E_{jk} \qquad (A \in \mathbb{F}^{r\times r}), \tag{4.2}$$

is a faithful representation of the Lie algebra of $GL_r$, and

$$\Big\{ P\Big(\zeta\,\frac{\partial'}{\partial\zeta}\Big) : P \in \mathcal{P}(\mathbb{F}^{r\times r}) \Big\} \tag{4.3}$$

is a faithful realization of the universal enveloping algebra $U(GL_r)$ of $GL_r$ as the algebra of differential operators on $\mathcal{C}^\infty(\mathbb{F}^{r\times n})$. In addition, since

$$P\Big(\zeta\,\frac{\partial'}{\partial\zeta}\Big)f = (\partial_P f)(\zeta, z)\Big|_{z=\zeta} \qquad (z, \zeta \in \mathbb{F}^{r\times n}), \tag{4.5}$$

contact with Howe's theory is made through the following result.

THEOREM 4.6 (À LA HARISH-CHANDRA). *The mapping $P \longrightarrow P(E)$ is a linear bijection from $\mathcal{P}(\mathbb{F}^{r\times r})$ onto a realization of $U(GL_r)$ as differential operators that satisfy*

$$(i) \qquad Ad(h)\colon P(E) \longrightarrow \lambda(h)^{-1} \circ P(E) \circ \lambda(h) \qquad (h \in GL_r),$$

$$(ii) \qquad \pi(k) \circ P(E) = P(E) \circ \pi(k) \qquad (k \in GL_n),$$

*on the $GL_r \times GL_n$-module $C^\infty(\mathbb{F}^{r\times n})$.*

PROOF. Property (i) follows from Corollary 2.22 while property (ii) is an immediate consequence of Theorem 2.13 and 4.5. ∎

The center of this realization of $U(GL_r)$ consists precisely of all operators

$$P(E) = P\big([E_{jk}]\big), \qquad E_{jk} = \Big(\zeta_j \mid \frac{\partial}{\partial \zeta_k}\Big),$$

with $P$ an $Ad\,GL_r$-invariant in $\mathcal{P}(\mathbb{F}^{r\times r})$, just as the $GL_r \times GL_n$-invariant differential operators were given by

$$P(D) = P\big([D_{jk}]\big), \qquad D_{jk} = \Big(z_j \mid \frac{\partial}{\partial \zeta_k}\Big),$$

for such $P$, the crucial difference being that the $D_{jk}$ all commute whereas the $E_{jk}$ do not. Ingenious modifications introducted by Capelli enabled him to derive the analogue of corollary of (2.28) for the non-commuting $E_{jk}$ (cf. [2], p. 19; [4]; [21], p. 116). With the convention that the determinant $\det[A_{pq}]$ of an $\ell \times \ell$-matrix of (possibly) non-commuting variables is given by

$$\det[A_{pq}] = \sum_{\sigma \in S_\ell} \operatorname{sgn} \sigma A_{\sigma(1)1} \dots A_{\sigma(r)r},$$

define operators $H_r$ on $C^\infty(\mathbb{F}^{r\times n})$ by

$$(4.7) \qquad H_r = \det \begin{bmatrix} E_{11} + r - 1 & E_{12} & \cdots & E_{1r} \\ E_{21} & E_{22} + r - 2 & \cdots & E_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ E_{r1} & E_{r2} & \cdots & E_{rr} \end{bmatrix}$$

(cf. [19]; [21], p. 117; [24], chap. II, § 4). The $GL_r \times GL_n$-invariance of

$$\Big(z_1 \wedge \cdots \wedge z_r \,\Big|\, \frac{\partial}{\partial \zeta_1} \wedge \cdots \wedge \frac{\partial}{\partial \zeta_r}\Big) = \sum_{1 \le k_1 < \cdots < k_r \le n} \Delta^{1\dots r}_{k_1\dots k_r}(z) \Omega^{1\dots r}_{k_1\dots k_r}\Big(\frac{\partial}{\partial \zeta}\Big)$$

(cf. (2.32)) ensures that

$$H_r = \Big(\zeta_1 \wedge \cdots \wedge \zeta_r \,\Big|\, \frac{\partial}{\partial \zeta_1} \wedge \cdots \wedge \frac{\partial}{\partial \zeta_r}\Big) = \sum_{1 \le k_1 < \cdots < k_r \le n} \Delta^{1\dots r}_{k_1\dots k_r}(\zeta) \Omega^{1\dots r}_{k_1\dots k_r}\Big(\frac{\partial}{\partial \zeta}\Big).$$

These operators $E_{jk}$ and $H_r$ are particularly well-adapted to use on $\mathcal{P}_L(\mathbb{F}^{r\times n})$ since $\{E_{jk} : 1 \le j < k \le r\}$ and $\{E_{jk} : 1 \le j \le k \le r\}$ are bases of faithful realizations of $N_r$ and $B_r$ respectively. For by (4.2)

$$(4.8)(\mathrm{i}) \qquad \mathcal{P}_L(\mathbb{F}^{r\times n}) = \big\{ f \in \mathcal{P}(\mathbb{F}^{r\times n}) : E_{jk} f = 0,\ 1 \le j < k \le r \big\},$$

and, if $\tau = (m_1, m_2, \ldots)$,

$$(4.8)(ii) \qquad \mathcal{P}_\tau(\mathbb{F}^{r\times n}) = \left\{ f \in \mathcal{P}_L(\mathbb{F}^{r\times n}) : E_{jj}f = m_j f,\ 1 \le j \le r \right\}.$$

For $f$ in $\mathcal{P}_L(\mathbb{F}^{r\times n})$, therefore, the only non-zero term in $H_r f$ is the diagonal term $\left(\prod_{j=1}^r (E_{jj} + r - 1)\right) f$. This proves (cf. also [21], p. 256):

$$(4.8)(iii) \qquad H_r f = \left( \prod_{j=1}^r (m_j + r - j) \right) f \qquad (f \in \mathcal{P}_\tau),$$

which is a complete analogue of Euler's result (the case $r = 1$).

As a first application of these ideas we give

PROOF OF THEOREM 3.19. It is convenient to begin with the second equality in (3.20). As

$$\Psi(z) = \Phi_\tau\left(z \frac{\partial'}{\partial \zeta}\right) \Phi_\tau \Big|_{\zeta=0} \qquad (z \in \mathbb{F}^{r\times n})$$

is an $N_r \times N_n$-invariant such that $\lambda'(b)\Psi = \Phi_\tau(b)\Psi$, $b \in B_r$, there is a constant $\theta$ in $\mathbb{F}$ such that $\Psi = \theta \Phi_\tau$. To calculate $\theta$, assume first that $m_r \neq 0$. Now, given any $f$ in $\mathcal{P}(\mathbb{F}^{r\times n})$,

$$(4.9) \qquad \begin{aligned} \Delta_r\left(z \frac{\partial'}{\partial \zeta}\right) f &= \left( z_1 \wedge \cdots \wedge z_r \,\Big|\, \frac{\partial}{\partial \zeta_1} \wedge \cdots \wedge \frac{\partial}{\partial \zeta_r} \right) f(\zeta) \\ &= \sum_{1 \le k_1 < \cdots < k_r \le n} \Delta^{1\ldots r}_{k_1\ldots k_r}(z) \Omega^{1\ldots r}_{k_1\ldots k_r}\left(\frac{\partial}{\partial \zeta}\right) f(\zeta) \\ &= \sum_{1 \le k_1 < \cdots < k_r \le n} \Delta^{1\ldots r}_{k_1\ldots k_r}(z) F_{k_1\ldots k_r}(\zeta), \text{ say.} \end{aligned}$$

Consequently, when $f = \Phi_\tau$,

$$\left( z_1 \wedge \cdots \wedge z_r \,\Big|\, \frac{\partial}{\partial \zeta_1} \wedge \cdots \wedge \frac{\partial}{\partial \zeta_r} \right) \Phi_\tau = \Delta_r(z) F(\zeta), \text{ say.}$$

But then by (4.8)(iii),

$$H_r \Phi_\tau = \left( \prod_{j=1}^r (m_j + r - j) \right) \Phi_\tau(\zeta) = \left( \prod_{j=1}^r (m_j + r - j) \right) \Delta_r(\zeta) \Phi_{\tau-\rho_r}(\zeta)$$

where $\tau - \rho_r = (m_1 - 1, \ldots, m_r - 1, 0, \ldots)$. This identifies $F$, and so after $m_r$ such differentiations,

$$(4.10) \qquad \left( \Delta_r\left(z \frac{\partial'}{\partial \zeta}\right) \right)^{m_r} \Phi_\tau(\zeta) = \left\{ \prod_{j=1}^r \frac{(m_j + r - j)!}{(m_j - m_r + r - j)!} \right\} \Delta_r(z)^{m_r} \Phi_\sigma(\zeta)$$

with $\sigma = \tau - m_r \rho_r$ and $\Phi_\sigma$ in $\mathcal{P}_\sigma(\mathbb{F}^{(r-1)\times n})$. Repeating this proof successively in $\mathcal{P}(\mathbb{F}^{(r-1)\times n}), \ldots, \mathcal{P}(\mathbb{F}^n)$, we finally obtain

$$\Phi_\tau\left(z \frac{\partial'}{\partial \zeta}\right) \Phi_\tau(\zeta) = h(\tau) \Phi_\tau(z).$$

Had $m_r$ been 0, the proof would have started at one of these later stages.

The proof of the equality

$$\chi_\tau\Big(z\,\frac{\partial'}{\partial\zeta}\Big)\Phi_\tau = h(\tau)\Phi_\tau(z) \qquad (z \in \mathbb{F}^{r\times n}), \tag{4.11}$$

is much the same. Assume first that $m_r \neq 0$. Then in view of (3.5) and (4.10)

$$\begin{aligned}\chi_\tau\Big(z\,\frac{\partial'}{\partial\zeta}\Big)\Phi_\tau &= \chi_\sigma\Big(z\,\frac{\partial'}{\partial\zeta}\Big)\Big(\Delta_r\Big(z\,\frac{\partial'}{\partial\zeta}\Big)\Big)^{m_r}\Phi_\tau \\ &= \Big\{\prod_{j=1}^{r}\frac{(m_j+r-j)!}{(m_j-m_r+r-j)!}\Big\}\Delta_r(z)^{m_r}\chi_\sigma\Big(z\,\frac{\partial'}{\partial\zeta}\Big)\Phi_\sigma.\end{aligned}$$

Since $\Phi_\sigma$ can be regarded equally as a polynomial on $\mathbb{F}^{(r-1)\times n}$, or one on $\mathbb{F}^{r\times n}$ that is independent of the last row of variables, the value of $\chi_\sigma(D)\Phi_\sigma$ will be the same whether $\chi_\sigma$, $\sigma = (m_1 - m_r, \ldots, m_{r-1} - m_r, 0, \ldots)$, is constructed via 3.3 as a polynomial on $\mathbb{F}^{(r-1)\times(r-1)}$ or on $\mathbb{F}^{r\times r}$. But in the first of these cases,

$$\chi_\sigma(x) = \Delta_{r-1}(x)^{m_{r-1}-m_r}\chi_\delta(x) \qquad (x \in \mathbb{F}^{(r-1)\times(r-1)}),$$

where $\delta = (m_1 - m_{r-1}, \ldots, m_{r-2} - m_{r-1}, 0, \ldots)$. Thus the same induction argument proceeds as before, yielding (4.11). Had $m_r$ been 0, the proof would again have started at one of the later stages of the induction proof. ∎

PROOF OF COROLLARY 3.8. By $N_r \times B_n$-invariance, each of

$$\chi_\gamma\Big(z\,\frac{\partial'}{\partial\zeta}\Big)\Phi_\tau\Big|_{\zeta=0}, \qquad \Phi_\gamma\Big(z\,\frac{\partial'}{\partial\zeta}\Big)\Phi_\tau\Big|_{\zeta=0}$$

is of the form $\theta_\gamma\Phi_\tau(z)$ for some $\theta_\gamma$ in $\mathbb{F}$. Thus for any $b = \operatorname{diag}(t_1, \ldots, t_r)$ in $B_r$,

$$\Big(\frac{\partial}{\partial t_1}\Big)^{m_1}\cdots\Big(\frac{\partial}{\partial t_r}\Big)^{m_r}\chi_\gamma(bD)\Phi_\tau\Big|_{\zeta=0} = m_1!\ldots m_r!\theta_\gamma\Phi_\tau(z),\ D = z\,\frac{\partial'}{\partial\zeta}, \tag{4.12}$$

and

$$\Big(\frac{\partial}{\partial t_1}\Big)^{m_1}\cdots\Big(\frac{\partial}{\partial t_r}\Big)^{m_r}\Phi_\gamma(bD)\Phi_\tau\Big|_{\zeta=0} = m_1!\ldots m_r!\theta_\gamma\Phi_\tau(z). \tag{4.13}$$

Now suppose $\gamma \neq \tau$. Then by (3.13),

$$\Big(\frac{\partial}{\partial t_1}\Big)^{m_1}\cdots\Big(\frac{\partial}{\partial t_r}\Big)^{m_r}\Phi_\gamma(bA)\Big|_{t_1=t_2=\cdots=0} = 0 \qquad (A \in \mathbb{F}^{r\times r}),$$

while by 3.3 and 2.33,

$$\Big(\frac{\partial}{\partial t_1}\Big)^{m_1}\cdots\Big(\frac{\partial}{\partial t_r}\Big)^{m_r}\chi_\gamma(bA)\Big|_{t_1=t_2=\cdots=0} = 0 \qquad (A \in \mathbb{F}^{r\times r}),$$

Hence the term $\theta_\gamma$ in both of 4.12 and 4.13 must be zero when $\gamma \neq \tau$. This establishes the corollary. ∎

We now have everything needed to establish Theorem 3.6 and its corollary.

PROOF OF 3.6 AND 3.8. Set

$$Q_\tau = \left\{ \chi_\tau\left(z\frac{\partial'}{\partial\zeta}\right)f\Big|_{\zeta=0} : f \in \mathcal{P}(\mathbb{F}^{r\times n})\right\}. \tag{4.14}$$

By (3.20) and the invariance of $\chi_\tau(z\frac{\partial'}{\partial\zeta})$, this is a $GL_r \times GL_n$-module containing $\Phi_\tau$. Now with respect to $B_r \times B_n$, $\Phi_\tau$ is a $GL_r \times GL_n$-highest weight vector of weight $\tau$, and so the unique irreducible $GL_r \times GL_n$-module in $\mathcal{P}(\mathbb{F}^{r\times n})$ of weight $\tau$, *i.e.*, the one isomorphic to $\mathcal{V}_\tau(\mathbb{F}^r) \otimes \mathcal{V}_\tau(\mathbb{F}^n)$, has $\Phi_\tau$ as a highest weight vector. Hence, $Q_\tau$ contains this copy of $\mathcal{V}_\tau(\mathbb{F}^r) \otimes \mathcal{V}_\tau(\mathbb{F}^n)$. But then by Corollary 3.11, $Q_\tau$ contains no copy of any other $\mathcal{V}_\gamma(\mathbb{F}^r) \otimes \mathcal{V}_\gamma(\mathbb{F}^n)$, and so $Q_\tau$ is the irreducible $GL_r \times GL_n$-sub-module of $\mathcal{P}(\mathbb{F}^{r\times n})$ isomorphic to $\mathcal{V}_\tau(\mathbb{F}^r) \otimes \mathcal{V}_\tau(\mathbb{F}^n)$. This proves Corollary 3.8. But by invariance,

$$\chi_\tau\left(z\frac{\partial'}{\partial\zeta}\right)\Big|_{\zeta=0} = h(\tau)f(z) \tag{4.15}$$

for any $f$ in $Q_\tau$. Hence

$$f(z) = \sum_\tau \frac{1}{h(\tau)}\chi_\tau\left(z\frac{\partial'}{\partial\zeta}\right)f\Big|_{\zeta=0} \qquad \left(f \in \mathcal{P}(\mathbb{F}^{r\times n})\right),$$

completing the proof of 3.6. ∎

One further result, the analogue of 3.19 and 4.15, is needed before Theorem 3.15 and its corollary can be established.

THEOREM 4.16. *For partitions $\tau$, $\gamma$ of length at most $r$*

$$\Phi_\tau\left(z\frac{\partial'}{\partial\zeta}\right)f\Big|_{\zeta=0} = \begin{cases} h(\tau)f(z), & \tau=\gamma \\ 0, & \tau\neq\gamma \end{cases}$$

*whenever $f$ is in $\mathcal{P}_\gamma(\mathbb{F}^{r\times n})$.*

PROOF. By the same equivariance argument as in the Proof of Corollary 3.6 the left-hand side of 4.16 will be zero for all $f$ in $\mathcal{P}_\gamma(\mathbb{F}^{r\times n})$ unless $\tau = \gamma$. Thus from the outset we consider only $f$ in $\mathcal{P}_\tau(\mathbb{F}^{r\times n})$, $\ell(\tau) = r$. Now by 4.9,

$$\Delta_r\left(z\frac{\partial'}{\partial\zeta}\right)f(\zeta) = \sum_{1\le k_1<\cdots<k_r\le n} \Delta^{1\ldots r}_{k_1\ldots k_r}(z)F_{k_1\ldots k_r}(\zeta) \tag{4.17}$$

and

$$\sum_{1\le k_1<\cdots<k_r\le n} \Delta^{1\ldots r}_{k_1\ldots k_r}(\zeta)F_{k_1\ldots k_r}(\zeta) = (H_rf)(\zeta) \tag{4.18}$$

where

$$F_{k_1\ldots k_r}(\zeta) = \Omega^{1\ldots r}_{k_1\ldots k_r}\left(\frac{\partial}{\partial\zeta}\right)f(\zeta). \tag{4.19}$$

But each polynomial $F_{k_1\dots k_r}$ is in $\mathcal{P}_{\tau-\rho_r}(\mathbb{F}^{r\times n})$. Indeed

$$\Omega^{1\dots s}_{k_1\dots k_s}\Big(\frac{\partial}{\partial\zeta}\Big)E_{jk}f = \Omega^{1\dots s}_{k_1\dots k_s}\Big(\frac{\partial}{\partial\zeta}\Big)\Big(\zeta_j \,\Big|\, \frac{\partial}{\partial\zeta_k}\Big)f(\zeta) = E_{jk}\Omega^{1\dots s}_{k_1\dots k_s}\Big(\frac{\partial}{\partial\zeta}\Big)f(\zeta)$$

for any $f$ in $\mathcal{P}(\mathbb{F}^{r\times n})$ and all $1 \le j < k \le s$; on the other hand,

$$\begin{aligned}\Omega^{1\dots s}_{k_1\dots k_s}\Big(\frac{\partial}{\partial\zeta}\Big)E_{jj}f &= \Omega^{1\dots s}_{k_1\dots k_s}\Big(\frac{\partial}{\partial\zeta}\Big)\Big(\zeta_j \,\Big|\, \frac{\partial}{\partial\zeta_j}\Big)f(\zeta)\\ &= \Omega^{1\dots s}_{k_1\dots k_s}\Big(\frac{\partial}{\partial\zeta}\Big)f + E_{jj}\Omega^{1\dots s}_{k_1\dots k_s}\Big(\frac{\partial}{\partial\zeta}\Big)f.\end{aligned}$$

Thus by 4.8, the $F_{k_1,\dots k_r}$ in 4.19 belong to $\mathcal{P}_{\tau-\rho_r}(\mathbb{F}^{r\times n})$. The theorem now follows immediately from 4.10 using virtually the same induction argument as in the proof of 3.19. ∎

PROOF OF 3.15 AND 3.17. Set

$$\mathcal{R}_\tau = \Big\{\Phi_\tau\Big(z\frac{\partial'}{\partial\zeta}\Big)f\Big|_{\zeta=0} : f \in \mathcal{P}_L(\mathbb{F}^{r\times n})\Big\}.$$

By 3.20 and the invariance of $\Phi_\tau(z\frac{\partial'}{\partial\zeta})$, this $\mathcal{R}_\tau$ is a $GL_n$-module containing $\Phi_\tau$. Now with respect to $B_n$, $\Phi_\tau$ is a $GL_n$-highest weight vector of weight $\tau$, and so both $\mathcal{R}_\tau$ and $\mathcal{P}_\tau$ must contain the only copy of $\mathcal{V}_\tau(\mathbb{F}^{r\times n})$ in $\mathcal{P}_L(\mathbb{F}^{r\times n})$. Theorem 4.16 thus ensures that both $\mathcal{R}_\tau$ and $\mathcal{P}_\tau$ must coincide with this copy, since $\mathcal{P}_L(\mathbb{F}^{r\times n}) \cong \sum_\tau \mathcal{V}_\tau(\mathbb{F}^{r\times n})$. This completes the proof of the Corollary 3.17. But then by 4.16 again,

$$f(z) = \sum_\tau \frac{1}{h(\tau)}\Phi_\tau\Big(z\frac{\partial'}{\partial\zeta}\Big)f\Big|_{\zeta=0} \qquad (f \in \mathcal{P}_L(\mathbb{F}^{r\times n})),$$

completing the Proof of 3.15. ∎

To complete this section we note a generalization of the Capelli Identity for $H_r$ posed by Turnbull ([21], p. 119) as an exercise: if the term $\xi_j$ in $\xi_1 \wedge \cdots \wedge \xi_r$ is replaced by $\eta$, then the constant $(r-j-1)$ is omitted from the $j^{th}$ diagonal entry in $H_r$, *i.e.*,

$$(4.20)\qquad \begin{aligned}&\Big(\xi_1\wedge\cdots\wedge\eta\wedge\cdots\wedge\xi_r \,\Big|\, \frac{\partial}{\partial\xi_1}\wedge\cdots\wedge\frac{\partial}{\partial\xi_j}\wedge\cdots\wedge\frac{\partial}{\partial\xi_j}\Big)\\ &= \det\begin{bmatrix} E_{11}+r-1 & \cdots & E_{1j} & \cdots & E_{1r}\\ \vdots & & \vdots & & \vdots\\ \Big(\eta\,\Big|\,\frac{\partial}{\partial\xi_1}\Big) & \cdots & \Big(\eta\,\Big|\,\frac{\partial}{\partial\xi_j}\Big) & \cdots & \Big(\eta\,\Big|\,\frac{\partial}{\partial\xi_r}\Big)\\ \vdots & & \vdots & & \vdots\\ E_{r1} & \cdots & E_{rj} & \cdots & E_{rr}\end{bmatrix}.\end{aligned}$$

This result follows easily from Theorem 4.9 by differentiating the Capelli identity for $H_r$ with respect to $(\eta \mid \frac{\partial}{\partial\xi_j})$. Such generalizations as 4.20 are very useful in practice.

5. **Clebsch-Gordan decompostions and Taylor series.** With the specific realization of $\mathcal{V}_\tau(\mathbb{F}^n)$ as the space $\mathcal{P}_\tau$ of polynomial functions on $\mathbb{F}^{r\times n}$, the Taylor series 3.6 provides a series implementing the $GL_n$-isotypic decomposition of the space $\mathcal{P}(\mathbb{F}^n, \mathcal{V}_\tau(\mathbb{F}^n))$. Indeed, by 3.18, $\mathcal{P}(\mathbb{F}^n, \mathcal{P}_\tau)$ can be identified first with the $GL_n$-submodule of $\mathcal{P}(\mathbb{F}^n \times \mathbb{F}^{r\times n})$ of all $F = F(x, \xi)$ satisfying the Homogeneity condition

$$F(x, b'\xi) = \Phi_\tau(b)F(x, \xi) \qquad (b \in B_r), \tag{5.1}$$

and thence with a $GL_n$-submodule of $\mathcal{P}(\mathbb{F}^{(r+1)\times n})$, after identifying $\mathbb{F}^n \times \mathbb{F}^{r\times n}$ with $\mathbb{F}^{(r+1)\times n}$. The Taylor series 3.6 will be applied to this last sub-module. Independently of these identifications, however, the usual Taylor series expansion applied to $\mathcal{P}(\mathbb{F}^n, \mathcal{P}_\tau) \sim \mathcal{P}(\mathbb{F}^n) \otimes \mathcal{P}_\tau$ gives

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(x \,\Big|\, \frac{\partial}{\partial y}\right)^m f\Big|_{y=0} \qquad (x \in \mathbb{F}^n),$$

where

$$\left\{ \left(x \,\Big|\, \frac{\partial}{\partial y}\right)^m f\Big|_{y=0} : f \in C^\infty(\mathbb{F}^n, \mathcal{P}_\tau) \right\} \cong \mathcal{P}_m(\mathbb{F}^n) \otimes \mathcal{P}_\tau .$$

From this the $GL_n$-isotypic decomposition of $\mathcal{P}(\mathbb{F}^n, \mathcal{P}_\tau)$ follows. For there is a Clebsch-Gordan type decomposition

$$\mathcal{P}_m(\mathbb{F}^n) \otimes \mathcal{V}_\tau(\mathbb{F}^n) \cong \oplus \sum_\mu \mathcal{V}_\mu(\mathbb{F}^n), \tag{5.2}$$

with sum taken over all partitions $\mu = (\mu_1, \mu_2, \ldots)$, $\ell(\mu) \leqq \ell(\tau) + 1$, satisfying

$$\begin{aligned} &\text{(i)} && \mu_1 \geq m_1 \geq \mu_2 \geq \cdots \geq m_r \geq \mu_{r+1}, \qquad r = \ell(\tau), \\ &\text{(ii)} && |\mu| = |\tau| + m \end{aligned} \tag{5.3}$$

(cf. [25]); and so in general,

$$\mathcal{P}(\mathbb{F}^n, \mathcal{P}_\tau) \cong \oplus \sum_\mu \mathcal{V}_\mu(\mathbb{F}^n), \tag{5.4}$$

summing over all $\mu$, $\ell(\mu) \leq \ell(\tau) + 1$, satisfying just 5.3(i). Hence, on restricting the Taylor series 3.7 we obtain

THEOREM 5.5. *Fix a partition $\tau = (m_1, m_2, \ldots)$, $\ell(\tau) = r$. Then each $f = f(z)$ in $\mathcal{P}(\mathbb{F}^n, \mathcal{P}_\tau)$ can be written uniquely as*

$$f(z) = \sum_\mu \frac{1}{h(\mu)} \chi_\mu\left(z \, \frac{\partial'}{\partial \zeta}\right) f\Big|_{\zeta=0}, \qquad z, \zeta \in \mathbb{F}^{(r+1)\times n}, \tag{5.6}$$

*the sum being taken over all partitions $\mu = (\mu_1, \mu_2, \ldots)$, $\ell(\mu) \leq r + 1$, satisfying*

$$\mu_1 \geq m_1 \geq \mu_2 \geq \cdots \geq m_r \geq \mu_{r+1}. \tag{5.7}$$

COROLLARY 5.8. *The set*

$$\left\{ \chi_\mu\left(z\frac{\partial'}{\partial\zeta}\right)f\Big|_{\zeta=0} : f \in \mathcal{P}(\mathbb{F}^n, \mathcal{P}_\tau)\right\} \tag{5.9}$$

*is the unique irreducible $GL_n$-module in $\mathcal{P}(\mathbb{F}^n, \mathcal{P}_\tau)$ isomorphic to $\mathcal{V}_\mu(\mathbb{F}^n)$.*

The $GL_n$-module 5.12 can be specifically identified within $\mathcal{P}(\mathbb{F}^{(r+1)\times n})$ using 'Standard Monomial' Theory (cf. [10]; [14],...).

Let $\gamma = (\ell_1, \ell_2, \ldots)$ be a partition of length $\sigma$. A *Young Tableau of shape* $\gamma$ consists of $|\gamma|$ positive integers, not necessarily all distinct, arranged in $\sigma$ flush-left rows of successive length $\ell_1, \ell_2, \ldots, \ell_\sigma$, so that, for example,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 4 \\ 3 & 2 & \end{bmatrix} \tag{5.10}$$

are all Young Tableaux of shape $(3, 2, 0, \ldots)$ ([10]). Any such tableau is said to be a *Standard* Young tableau when the entries in each row are non-decreasing, but the entries in each column are strictly increasing as one proceeds from the upper left hand corner; in 5.10, for instance, the first and third tableaux are standard, whereas the second is not. To each pair $[\alpha \mid \beta]$ of standard Young tableaux

$$[\alpha \mid \beta] = \left[\begin{array}{ccc|ccc} j_1 & k_1 & \cdots & u_1 & v_1 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \\ \cdot & k_\delta & & \cdot & v_\delta & \\ j_\sigma & & & u_\sigma & & \end{array}\right] \tag{5.11}$$

having the same shape $\gamma = (\ell_1, \ell_2, \ldots)$, $\ell(\gamma) = \sigma$, there corresponds the '*Standard Monomial*' $\Phi_{(\alpha,\beta)}$ in $\mathcal{P}(\mathbb{F}^{s\times n})$ defined as the product

$$\Phi_{(\alpha,\beta)}(z) = \Delta^{j_1\ldots j_\sigma}_{u_1\ldots u_\sigma}(z)\Delta^{k_1\ldots k_\delta}_{v_1\ldots v_\delta}(z)\ldots \qquad (z \in \mathbb{F}^{s\times n}), \tag{5.12}$$

of minors specified by the $\ell_1$ successive pairs of columns of $\alpha, \beta$. To be well-defined the entries of $\alpha$ must all be taken from $\{1, \ldots, s\}$, while those of $\beta$ must be taken from $\{1, \ldots, n\}$; all such standard Young tableaux of shape $\gamma$ will be denoted by $Y^{(s)}_\gamma$ and $Y^{(n)}_\gamma$, respectively. Thus, if $\alpha \in Y^{(s)}_\gamma$, $\beta \in Y^{(n)}_\gamma$, and

$$\alpha_j = \mathrm{card}\{j \in \alpha\}, \qquad \beta_k = \mathrm{card}\{k \in \beta\},$$

then

$$\lambda'(b)\Phi_{(\alpha,\beta)} = (b_{11}^{\alpha_1}\cdots b_{ss}^{\alpha_s})\Phi_{(\alpha,\beta)}, \qquad \pi(c)\Phi_{(\alpha,\beta)} = (c_{11}^{\beta_1}\cdots c_{nn}^{\beta_n})\Phi_{(\alpha,\beta)}$$

for all *diagonal* matrices $b \in B_s$, $c \in B_n$. The pair $[(\alpha_1, \ldots, \alpha_s) \mid (\beta_1, \ldots, \beta_n)]$ is known as the *Content* of $[\alpha \mid \beta]$ and the *Weight* of $\Phi_{(\alpha,\beta)}$. The $\Phi_\tau$ defined earlier in 3.12 corresponds to $\Phi_{(\alpha,\beta)}$ with $\alpha, \beta$ both being the *Canonical* standard Young tableau

$$\begin{bmatrix} 1 & \cdots & \cdot & \cdot & \cdot & 1 \\ 2 & \cdots & \cdot & \cdot & 2 & \\ \vdots & & & & & \\ r & \cdots & r & & & \end{bmatrix} \tag{5.13}$$

of shape $\tau = (m_1, m_2, \ldots)$, $\ell(\tau) = r$, and content $[(m_1, \ldots, m_r) \mid (m_1, \ldots, m_\sigma)]$ (cf. [9]). The canonical standard Young tableau 5.13 of shape $\tau$ will be denoted by $\tau$. For such $\tau$,

$$(5.14) \qquad \left\{\Phi_{(\alpha,\tau)} : \alpha \in Y_\tau^{(r)}\right\} \subseteq \mathcal{P}_U(\mathbb{F}^{r\times n}) \quad , \quad \left\{\Phi_{(\tau,\beta)} : \beta \in Y_\tau^{(n)}\right\} \subseteq \mathcal{P}_L(\mathbb{F}^{r\times n}).$$

We can now begin the identification of 5.9. A standard Young tableau $\alpha$ with entries taken from $\{1, \ldots, r+1\}$, $1 \le r < n$, is said to *Augment* the canonical standard Young tableau $\tau$, $\ell(\tau) = r$, when $\alpha$ consists of $\tau$ together with additional entries all having the value $r+1$. For instance,

$$\begin{bmatrix} 1 & 1 & 1 & 3 & 3 \\ 2 & 3 & & & \end{bmatrix} \quad , \quad \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 3 & & \\ 3 & & & \end{bmatrix}$$

both augment the canonical standard Young tableau corresponding to $(3, 1, 0, \ldots)$; their respective shapes $(5, 2, 0, \ldots)$, $(4, 2, 1, 0, \ldots)$ satisfy 5.3 with $m = 3$. More generally, there is a 1–1 correspondence between the standard Young tableaux of shape $\mu = (\mu_1, \mu_2, \ldots)$ augmenting the canonical standard Young tableau of shape $\tau = (m_1, m_2, \ldots)$ and the partitions $\mu$ satisfying 5.3(i). Given such $\tau$ and $\mu$, let $\alpha$ be the corresponding standard Young tableau, and set $m = \mathrm{card}\{(r+1) \in \alpha\}$. Then by writing $z = (x, \xi)$ for an element of $\mathbb{F}^n \times \mathbb{F}^{r\times n}$, regarding $x$ as the $(r+1)^{th}$-row, we deduce that

$$\Phi_{(\alpha,\mu)}(\lambda x, b'\xi) = \lambda^m \Phi_\tau(b) \Phi_{(\alpha,\mu)}(x, \xi) \qquad (\lambda \in \mathbb{F},\ b \in B_r),$$

since $\Phi_{(\alpha,\mu)}$ has weight $[(m_1, \ldots, m_r, m) \mid (\mu_1, \mu_2, \ldots)]$. Hence $\Phi_{(\alpha,\mu)} \in \mathcal{P}_m(\mathbb{F}^n) \otimes \mathcal{P}_\tau$; in fact, since $\Phi_{(\alpha,\mu)}$ is a highest weight vector having weight $\mu$ (cf. 5.14), $\Phi_{(\alpha,\mu)}$ must be the essentially unique highest weight vector in the single irreducible $GL_n$-submodule of $\mathcal{P}_m(\mathbb{F}^n) \otimes \mathcal{P}_\tau$ isomorphic to $\mathcal{V}_\mu(\mathbb{F}^n)$. On the other hand, as an element of $\mathcal{P}(\mathbb{F}^{(r+1)\times n})$, $\Phi_{(\alpha,\mu)}$ satisfies

$$\chi_\mu\left(z \frac{\partial'}{\partial \zeta}\right) \Phi_{(\alpha,\mu)} = h(\mu) \Phi_{(\alpha,\mu)}(z).$$

Together these specify (5.19).

## REFERENCES


1. A. Capelli, *Über die Zurückführung der Cayley'schen Operation Ω auf gewöhnliche Polar-Operationen*, Math. Ann. **29**(1887), 331–338.
2. ———, *Sur les opérations dans la théorie des formes algèbriques*, Math. Ann. **37**(1890), 1–37.
3. ———, *Lezioni sulla Teoria delle forme algebriche*. (Pellerano, ed.) Napoli. Libr. Sc. 1902.
4. ———, *Quelques formules relatives aux opérations de polaire*, Proc. Int. Congress. Math. (1894), 35–39.
5. R. W. Carter and G. Lusztig, *On the modular representations of the general linear and symmetric groups*, Math. Z. **136**(1974), 193–242.
6. K. M. Davis, J. E. Gilbert and R. A. Kunze, *Elliptic differential operators in harmonic analysis, I. Generalized Cauchy-Riemann systems*. Amer. J. Math. **113**(1991), 75–116.
7. ———, *Invariant differential operators in harmonic analysis on real hyperbolic space*. Proc. Int. Conf. Harmonic Analysis and Operator Algebras, Aust. Nat. Univ., Canberra, 1987, 58–78.
8. ———, *Elliptic differential operators and harmonic analysis on hyperbolic space*. In preparation.
9. C. DeConcini, D. Eisenbud and C. Procesi, *Young diagrams and determinantal varieties*, Inventiones Math. **56**(1980), 129–165.

**10.** C. DeConcini and C. Procesi, *A characteristic-free approach to invariant theory*, Adv. in Math. **21**(1976), 330–354.

**11.** J. Dixmier, *Algèbres enveloppantes*. Gauthier-Villars, Paris, 1973.

**12.** P. B. Gilkey, *Invariance Theory, The Heat Equation, and the Atiyah-Singer Index Theorem*. Math. Lect. Series, Vol.11, Publish or Perish Inc., Wilmington, Delaware, 1984.

**13.** S. Helgason, *Groups and geometric analysis*. Academic Press, New York, 1984.

**14.** W. V. D. Hodge, *Some enumerative results in the theory of forms*, Proc. Camb. Phil. Soc. **39**(1943), 22–30.

**15.** R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313**(1989), 539–570.

**16.** M. Kashiwara and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Inventiones Mathematicae **44**(1978), 1–47.

**17.** D. E. Littlewood, *The theory of group characters and matrix representations of groups*, $2^{nd}$ edition, Oxford Univ. Press, Oxford 1950.

**18.** I. G. MacDonald, *Symmetric functions and Hall polynomials*. Clarendon Press, Oxford, 1979.

**19.** C. Procesi, *A primer of invariant theory*. Brandeis Lecture Notes, **1**(1982).

**20.** R. M. Thrall, *On symmetrized Kronecker powers and the structure of the free Lie ring*, Amer. J. Math. **64**(1942), 371–388.

**21.** H. W. Turnbull, *The Theory of Determinants, Matrices and Invariants*. Third Edition, Dover Publications, 1960.

**22.** D.-N. Verma, *On a classical stability result on invariants of isotypical modules*, J. of Algebra **63**(1980), 15–40.

**23.** N. R. Wallach, *Harmonic analysis on homogeneous spaces*. Marcel Dekker Inc., New York, 1973.

**24.** H. Weyl, *Classical Groups*. Second Edition, Princeton University Press, Princeton, N.J., 1953.

**25.** D. P. Zelobenko, *Compact Lie Groups and their Representations*,. Transl. Math. Monographs, Amer. Math. Soc., Providence, Rhode Island, 1973.

*The University of Texas at Austin*
*Austin, Texas*
*U.S.A. 78712*