Singular matrices and pairwise-tangent circles

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1. *Introduction*

The idea of using the *generalised inverse* of a singular matrix A to solve the matrix equation $A\mathbf{x} = \mathbf{b}$ has been discussed in the earlier papers [1, 2, 3, 4] in the *Gazette*. Here we discuss three simple geometric questions which are of interest in their own right, and which illustrate the use of the generalised inverse of a matrix. The three questions are about polygons and circles in the Euclidean plane. We need not assume that a polygon is a simple closed curve, nor that it is convex: indeed, abstractly, a polygon is just a finite sequence $(v_1, \dots, v_n)$ of its distinct, consecutive, vertices. It is convenient to let $v_{n+1} = v_1$ and (later) $C_{n+1} = C_1$.

Question 1: Given a polygon P with vertex sequence $(v_1, \dots, v_n)$, is it possible to construct circles C_j centred at v_j, such that each C_j is externally tangent to C_{j-1} and C_{j+1}?

Question 2: Given positive numbers $\ell_1, \dots \ell_n$, is it possible to construct a polygon P whose vertex sequence $(v_1, \dots, v_n)$ has sides $[v_j, v_{j+1}]$ of length ℓ_j, and circles C_j centred at v_j, such that each C_j is externally tangent to C_{j-1} and C_{j+1}?

Question 3: Given positive numbers $r_1, \dots, r_n$, is it possible to construct a polygon with vertex sequence $(v_1, \dots, v_n)$, and circles C_j of radius r_j and centred at v_j, such that each C_j is externally tangent to C_{j-1} and C_{j+1}?

In Question I we are given the polygon P; in Question 2 we are given the lengths ℓ_j of the sides of P, but not its vertices. In Question 3 we are given the desired radii r_j of the circles, and place no constraints on P: in this case the reader can experiment by sliding plates of different sizes (turned upside down) around on a table top. We shall consider the (easy) case $n = 3$, and the more interesting case $n = 4$, and leave the cases $n \geqslant 5$ for readers to explore. We shall take an algebraic, and a geometric, point of view but as algebraic arguments are not, in general, sensitive to the geometric constraint that radii and lengths must be positive, we must pay particular attention to this aspect.

2. *Triangles*: *the case* $n = 3$

In Questions 1 and 2 we are given the ℓ_j, and it is clear from a diagram that we can solve these problems by solving the linear equation

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$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix}. \tag{1}$$

As this matrix is non-singular, its inverse exists and we find that (1) is equivalent to the system

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} = 2\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}. \tag{2}$$

In order to solve the geometric questions, we must ensure that the r_j, and the ℓ_j, are positive, and that the ℓ_j satisfy the triangle inequalities

$$\ell_i < \ell_j + \ell_k. \qquad \{i, j, k\} = \{1, 2, 3\}. \tag{3}$$

Now it is clear from (2) that since the ℓ_j are positive and satisfy (3) in Questions I and II, then the corresponding r_j are also positive. Conversely, it is clear from (1) that since the given r_j are positive in Question 3, then so are the ℓ_j and, moreover, that the ℓ_j do satisfy (3). In conclusion (and this is intuitively obvious), providing that the condition (3) is assumed in Question 2, then, when $n = 3$, the answer to all three questions is ‘yes’. Moreover, in all of these cases (ℓ_1, ℓ_2, ℓ_3) determines, and is determined by, (r_1, r_2, r_3). As we shall see, this is *not* the case when $n = 4$.

3. *The generalised triangle inequality*

Before we study the case $n = 4$, we comment on an extension of the triangle inequality (3). First, the real numbers ℓ_1, ℓ_2 and ℓ_3 are the lengths of the sides of some triangle in the plane if, and only if, they are positive and satisfy (3). Now consider a polygon with n sides of lengths $\ell_1, \dots, \ell_n$ arranged in this order around the polygon. Then, obviously, the ℓ_j satisfy the *generalised triangle inequality*

$$\ell_k < \sum_{i=1, i\neq k}^{n} \ell_j, \qquad k = 1, \dots, n. \tag{4}$$

In fact, *the converse is also true*, and we can even insist that the vertices of the polygon are concyclic. This result (which seems plausible after sliding plates around a table, but which does not seem to be as well known as the case $n = 3$), can be stated as follows.

Theorem 1: Let $\ell_1, \dots, \ell_n$, $n \geqslant 3$, be positive numbers which satisfy (4). Then there exists a Euclidean n-gon with consecutive sides of lengths ℓ_j, and whose vertices lie on a circle.

Theorem 1 occurs as [5, Theorem 6.2], and then later as [6, Theorem 1] and [7, Theorem 1.1], and is perhaps a little more subtle than one might

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expect at first sight. Therefore, in order not to disrupt our main line of enquiry, we defer our proof of it until the last section (Section 8) of the paper.

4. *Quadrilaterals*: *the case* $n = 4$

The case $n = 4$ is much more interesting than the case $n = 3$, and we shall prove the following results.

Theorem 2: Let P be a quadrilateral with vertex sequence (v_1, v_2, v_3, v_4) and sides $[v_j, v_{j+1}]$ of length ℓ_j. Then it is possible to construct circles C_j centred at v_j, with each C_j externally tangent to C_{j-1} and to C_{j+1} if, and only if, P has an inscribed circle and if, and only if, $\ell_1 + \ell_3 = \ell_2 + \ell_4$.

Theorem 3: Let r_1, r_2, r_3 and r_4 be any positive numbers. Then there is a cyclic quadrilateral P with vertices v_j, and circles C_j of radius r_j and centre v_j, such that each C_j is externally tangent to C_{j-1} and to C_{j+1}.

Figure 1 provides a 'proof without words' of Theorem 2, and we omit the details. In fact, Theorem 2 (and its proof) are closely related to Pitot's theorem, namely that *a quadrilateral with side lengths* ℓ_1, ℓ_2, ℓ_3 *and* ℓ_4 (*in this order*) *has an inscribed circle if, and only if*, $\ell_1 + \ell_3 = \ell_2 + \ell_4$. For a discussion of Pitot's theorem, see [8].

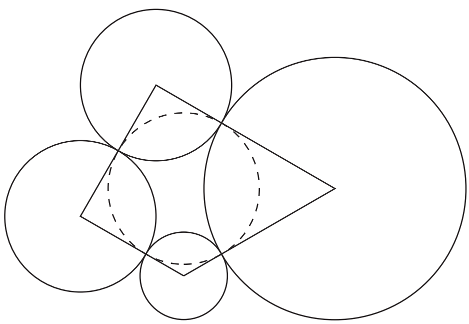

FIGURE 1: A quadrilateral with an inscribed circle

Let us now consider the case $n = 4$ from the perspective of linear algebra. First, given a quadrilateral P with side lengths ℓ_j, we can solve Questions 1 and 2 if there is a *positive* solution (r_1, r_2, r_3, r_4) of the linear system

$$A\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \end{pmatrix}. \tag{5}$$

In contrast to the case $n = 3$, the matrix A is singular so, disregarding (for

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the moment) the signs of the r_j, we see that given a quadrilateral P, *either* (i) there is no solution, *or* (ii) there are infinitely many solutions. In any event, there is definitely not a unique solution (r_j). Now for any solution r_j we have

$$\ell_1 - \ell_2 + \ell_3 - \ell_4 = (r_1 + r_2) - (r_2 + r_3) + (r_3 + r_4) - (r_4 + r_1) = 0,$$

so that

$$\ell_1 + \ell_3 = \ell_2 + \ell_4 \tag{6}$$

is a *necessary condition* for the existence of some *real* solution (r_j).

We shall now show that the condition (6) is also sufficient. Given that (6) holds, the general (real) solution to the equation (5) is given, for any real parameter t, by

$$r_1 = t;$$

$$r_2 = \ell_1 - t;$$

$$r_3 = \ell_2 - \ell_1 + t;$$

$$r_4 = \ell_4 - t = \ell_3 - \ell_2 + \ell_1 - t.$$

Now we shall leave the case $\ell_1 = \ell_2 = \ell_3 = \ell_4$ (when P is a square) to the reader. In all other cases we may assume that we have relabelled the polygon so that $\ell_2 > \ell_1$, and it follows from this that if $t > 0$ then $r_1 > 0$ and $r_3 > 0$. Further if $0 < t < \max\{\ell_1, \ell_4\}$, then $r_2 > 0$ and $r_4 > 0$, so if (6) holds, and if t is positive and sufficiently small, then we do have a solution (r_1, r_2, r_3, r_4) with each r_j positive.

Let us now consider the non-uniqueness of the solution. Geometrically, the non-uniqueness is obvious from Figure 1, for it is clear that given any solution, we can increase the radii of two opposite circles, and decrease the radii of the other two by the same amount. From the perspective of linear algebra, this happens because the kernel K of the transformation A is the set of real vectors $(t, -t, t, -t)^t$ (where $\mathbf{x}^t$ denotes the transpose of the row vector $\mathbf{x}$). We conclude that if $(r_1, r_2, r_3, r_4)^t$ is any solution to our question then, at least for sufficiently small $|t|$, the vector $(r_1 + t, r_2 - t, r_3 + t, r_4 - t)^t$ is also a solution.

Finally, we comment on Question 3. If we start with with positive numbers r_j, we can use (5) to define the ℓ_j, and then these are obviously positive. Moreover, it is clear from (5) that these ℓ_j also satisfy (4) so, by Theorem 1, there does indeed exist a polygon with sides of lengths ℓ_j.

5. *The generalised inverse of a matrix*

A second solution to a problem always enhances our understanding of it, and with this in mind we consider the case $n = 4$ from the perspective of the generalised inverse of a non-singular matrix. Briefly, if a square matrix A is non-singular, then the inverse matrix A^{-1} immediately provides a unique solution of the equation $A\mathbf{x} = \mathbf{b}$. However, if A is singular, or if A is not a square matrix, then no such inverse exists. However, we can *always* find a matrix B (called the *generalised inverse* of A) which satisfies $ABA = B$ and $BAB = B$, and which is such that $A\mathbf{x} = \mathbf{b}$ has a solution if, and only if, $AB\mathbf{b} = \mathbf{b}$ (that is, $\mathbf{b}$ is an eigenvector of AB with eigenvalue 1). In particular, if we are given A, and can compute B, then we have a necessary and sufficient condition on $\mathbf{b}$ for the existence of a solution of $A\mathbf{x} = \mathbf{b}$.

Now we have a singular matrix A in (5) and, according to the results in [2], the generalised inverse of A is the matrix B, where

$$B = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad AB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{pmatrix}.$$

Now, by the result stated above, the matrix equation (5) has a solution if and only if $(\ell_1, \ell_2, \ell_3, \ell_4)^t$ is an eigenvector of AB with eigenvalue 1, and an easy calculation shows that this is so if, and only if, (6) holds.

6. *The cases $n \geqslant 5$*

We now encourage readers to pursue the cases $n = 5$ and $n = 6$ or, better still, show that, in the general case, the relevant matrix is non-singular when n is odd, and singular when n is even. For example, if we consider the case $n = 6$ (a hexagon) we obtain a 6×6 matrix A, where

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now A is singular, and a straightforward application of the ideas in [2] then shows that

$$B = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, AB = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 \end{pmatrix},$$

and we arrive at the (expected) sufficient condition

$$\ell_1 + \ell_3 + \ell_5 = \ell_2 + \ell_4 + \ell_6$$

for the existence of a solution. In some sense, we may regard this as a generalisation of Pitot's theorem, although the notion of an inscribed circle has disappeared and has been replaced by circles at the vertices!

7. *Higher dimensions*

As an alternative to generalising the results on plane triangles to plane quadrilaterals, we can consider generalising the results on plane triangles to tetrahedra in $\mathbb{R}^3$ (and then on to $\mathbb{R}^4$, and so on). In this case, for a given tetrahedron we can construct mutually tangent spheres at the four vertices if and only if we can solve a system of six linear equations (coming from the six edge lengths) in four variables (the radii of the spheres). Obvious questions now arise, and we leave this for the interested readers to pursue.

8. *The proof of Theorem* 1

We end the paper with our proof of Theorem 1 (which is an expanded version of the proof in [5]).

Proof: Without loss of generality we may assume that $\ell_1 = \max\{\ell_1, \ldots, \ell_n\}$. Then, as the inequalities (4) hold, we find that $\ell_1 < \ell_2 + \ldots + \ell_n$ (in fact, this single inequality is obviously equivalent to the collection of inequalities in (4)). Next, we select a (sufficiently large) positive r, and then for each j we construct the Euclidean triangle T_j illustrated in Figure 2.

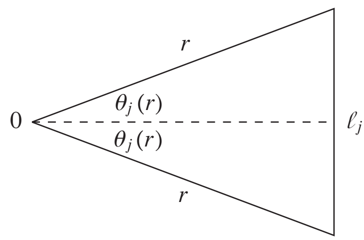

FIGURE 2: The isosceles triangle T_j

The plan is to show that we can choose r so that $\sum_j \theta_j(r) = \pi$, for then it is obvious that we can fit the triangles together, each with its vertex at the origin, and thereby construct a polygon and complete the proof of Theorem 1. Unfortunately, the proof is not this simple because such a polygon would necessarily have the origin in its interior, and this need not be the case. The case we have just described is illustrated (with $n = 4$) in the second circle in Figure 3, but we also have to allow for the possibility that the polygon is as illustrated in the first circle in Figure 3, and in this case we have $\theta_1(r) = \theta_2(r) + \ldots + \theta_n(r)$. We therefore have to prove the existence of some r such that one of the following two equations hold:

$$\theta_2(r) + \ldots + \theta_n(r) = \pi - \theta_1(r), \qquad \theta_2(r) + \ldots + \theta_n(r) = \theta_1(r).$$

As is so often the case, the existence of such an r will follow from an application of the intermediate value theorem.

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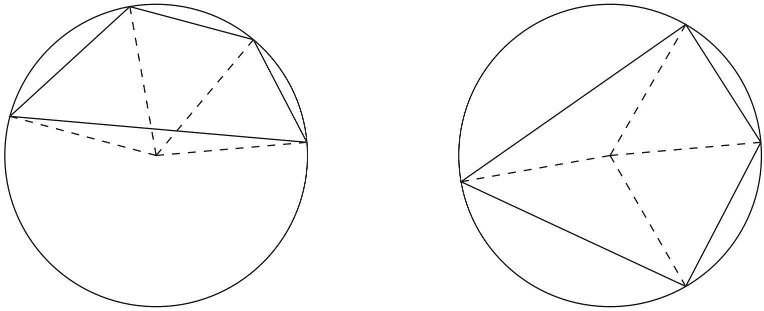

FIGURE 3: The two possibilities

This construction of the triangles T_j is possible if $r \geqslant \ell_1/2$ and, for each j, $\theta_j(r) = \sin^{-1}(\ell_1/2r)$ so that θ_j is a continuous strictly decreasing function on the interval $[\ell_1/2, +\infty)$ with

$$\lim_{r \to \ell_1/2} \theta_j(r) = \theta_j(\ell_1/2) = \sin^{-1}\left(\frac{\ell_j}{\ell_1}\right) \leqslant \tfrac{1}{2}\pi, \theta_j(\ell_1/2) = \tfrac{1}{2}\pi, \lim_{r \to \infty} \theta_j(r) = 0. \quad (7)$$

We now consider each of the following inclusive, but mutually exclusive, possibilities (which correspond to the two cases in Figure 3):

Case 1: $\theta_2(\ell_1/2) + \dots + \theta_n(\ell_1/2) < \pi/2$;

Case 2: $\theta_2(\ell_1/2) + \dots + \theta_n(\ell_1/2) \geqslant \pi/2$.

In Case 1 we recall that $\theta_1(\ell_1/2) = \pi/2$, so that

$$\theta_2(\ell_1/2) + \dots + \theta_n(\ell_1/2) < \theta_1(\ell_1/2). \quad (8)$$

Now as $r \to +\infty$ we see that $\theta_j(r) = \sin^{-1}(\ell_1/2r) \sim \ell_1/2r$. Thus

$$\lim_{r \to +\infty} \frac{\theta_1(r)}{\theta_2(r) + \dots + \theta_n(r)} = \frac{\ell_1}{\ell_2 + \dots + \ell_n} < 1, \quad (9)$$

and this shows that, for some sufficiently large R, we have

$$\theta_1(R) < \theta_2(R) + \dots + \theta_n(R). \quad (10)$$

The inequalities (8) and (10), combined with the intermediate value theorem, now show that for some R_1 with $R_1 > \ell_1/2$, we have

$$\theta_2(R_1) + \dots + \theta_n(R_1) = \theta_1(R_1). \quad (11)$$

Again, as $\theta_1(\ell_1/2) = \pi/2$, in Case 2 we have

$$\theta_2(\ell_1/2) + \dots + \theta_n(\ell_1/2) \geqslant \pi - \theta_1(\ell_1/2). \quad (12)$$

Now (7) and (9) show that

$$\lim_{r \to +\infty} \frac{\pi - \theta_1(r)}{\theta_2(r) + \dots + \theta_n(r)} = +\infty, \quad (13)$$

and this with (12) shows that, for some sufficiently large R_2, we have

$$\theta_2(R_2) + \dots + \theta_n(R_2) = \pi - \theta_1(R_2). \quad (14)$$

The proof is now complete.

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The answers to the *Nemo* page from November 2023 on friction were:

1.	Thomas Hardy	Far from the Madding Crowd	Chapter 3
2.	Charles Dickens	Bleak House	Chapter 66
3.	George Eliot	Middlemarch	Chapter 15
4.	Virginia Woolf	The Voyage Out	Chapter 1
5.	HG Wells	The New Accelerator	
6.	Ambrose Bierce	The Devil's Dictionary	

Congratulations to Bryan Thwaites and Martin Lukarevski on tracking all of these down. This issue, Nemo gathers momentum. Quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 23rd May 2024.

1. The old dog got off his haunches, and his tail, close-curled over his back, began a feeble, excited fluttering; he came waddling forward, gathered momentum, and disappeared over the edge of the fernery.
2. They might have been moving a good deal by a momentum that had begun far back, but they were still brave and personable, still warranted for continuance as long again, and they gave her, in especial collectivity, a sense of pleasant voices, pleasanter than those of actors, of friendly empty words and kind lingerings.

Continued on page 26.