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A BIVARIANT CHERN CHARACTER, II

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Introduction. In [Con2] Connes introduced cyclic cohomology $\mathrm{HC}^*(A)$ for an associative algebra A. When A is a complex algebra he constructed a Chern character for p-summable Fredholm modules over A taking values in $\mathrm{HC}^*(A)$. As a very special case, when X is a closed C^∞-manifold and $A = C^\infty(X)$, this construction recovers the usual Chern character, which is a rational isomorphism from the K-homology $K_0(X)$ to $H_{\mathrm{ev}}^{\mathrm{deR}}(X)$, the even dimensional deRham homology of X.

For an associative algebra A over any commutative ring k, the cyclic cohomology ring $\mathrm{HC}^*(k)$ (p. 106 [Con1]) is isomorphic to the polynomial ring $k[u]$, $\deg u = 2$, and the cup product makes $\mathrm{HC}^*(A)$ into a $k[u]$-module, such that the multiplication by u corresponds to the periodicity operator S. The K-theory $K_*(A)$ is dual to K-homology $K^*(A)$ over $K_*(\mathbb{C})$. The homology theory dual to $\mathrm{HC}^*(A)$ over $k[u]$ is the negative cyclic homology $\mathrm{HC}_*^-(A)$ introduced by T. Goodwillie [G2] and Hood-Jones [H-J].

J. Jones and C. Kassel [J-K] introduced a bivariant cyclic theory. For unital k-algebras, its relation to cyclic cohomology and negative cyclic homology is exactly analogous to that of Kasparov's KK-bifunctor to K-theory and K-homology.

The goal in [W1] and this paper is to construct the Chern character from a certain class of cycles in the KK-theory group $KK^*(A, A')$ to the bivariant cyclic theory group which extends simultaneously the Chern character on K-theory and K-homology.

Since cyclic theory can be defined in a purely algebraic context, such a construction should have a purely algebraic description. On the other hand, one wishes to work with K-theory in a topological category, for example, the category of locally convex topological algebras, or the subcategory of C^*-algebras. The topological aspect is particularly important for applications. We shall work in a topological category with purely algebraic situations appearing as a special case where all topologies in sight are discrete.

Even for the K-homology group $K^0(A)$, in general we do not even know the largest possible domain on which Chern character can be defined. Clearly, some restriction on cycles is necessary. Among other things, we shall construct the bivariant Chern character for I-summable quasihomorphisms satisfying excision properties.

For two C^*-algebras A and A', following Cuntz, we call a pair of $*$-homomorphisms $\alpha, \bar{\alpha}\colon A \longrightarrow \mathcal{L}(H_{A'})$ an *I-summable quasihomomorphism* if $\mathrm{Im}(\alpha - \bar{\alpha}) \subset I$, where $I \subset$

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$\mathcal{K}(H_{A'})$ is an ideal of the C^*-algebra $\mathcal{L}(H_{A'})$ of bounded linear operators on the Hilbert A'-module $H_{A'}$ [Kasp1]. We also define I-summable quasihomomorphisms in the algebraic category and categories of more general types of toplogical algebras (Definition 1.1).

Of fundamental importance in our construction is M. Wodzicki's work [Wod1], [Wod2] on excisions in cyclic theory. The natural question is: When does an exact sequence of algebras

$$(\varepsilon) \qquad 0 \longrightarrow I \longrightarrow E \xrightarrow{\pi} A \longrightarrow 0$$

induce long exact sequences in Hochschild homology and cyclic homology? Wodzicki proved the following: this is true for every extension (ε) with kernel I if and only if I is H-unital. Let $K_* = \ker \pi_*$ where $\pi_*\colon C_*(E) \longrightarrow C_*(A)$ is the induced chain map on the Hochschild complexes (the discussion for cyclic complexes is similar). Then Wodzicki's criterion ensures that the imbedding $C_*(I) \hookrightarrow K_*$ induces an isomorphism on homology.

A bivariant Chern character for quasihomomorphisms was first described in [W1]. A crucial observation is the following. Let h be any I-summable quasihomomorphism from A into $A' \otimes K$. Then h determines naturally a filtration F^* of the Hochschild chain complex $C_*\big(A' \otimes \mathcal{L}(H)\big)$, with the last subcomplex given by

$$K_n = F_n^0\big(A' \otimes \mathcal{L}(H)\big) = \sum_{i=0}^{n} \big(A' \otimes \mathcal{L}(H)\big) \otimes \cdots \otimes \underbrace{(A' \otimes I)}_{i\text{-th factor}} \otimes \cdots \otimes \big(A' \otimes \mathcal{L}(H)\big).$$

We may consider F^* as a filtration of K_* itself as well.

In [W1] it was shown that a quasihomomorphism h defines a cycle $h^\#$ in $\mathrm{Hom}_S\big(B(A), B(K_*)\big)^0$. An explicit formula for the decomposition $h^\# = (h^{(i)})_{i=0,1,\ldots}$ was given. The H-unitality of I would imply that $\mathrm{Hom}_S\big(B(A), B(K_*)\big)^0$ and $\mathrm{Hom}_S\big(B(A), B(I)\big)^0$ are quasi-isomorphic. Thus we could define a Chern character of h taking its value in

$$\mathrm{HC}^0(A, I) = H^0\Big(\mathrm{Hom}_S\big(B(A), B(I)\big)^*\Big).$$

The results in each section can be briefly summarized as follows.

In § 1 we recall Cuntz's formalism of KK and define I-summable quasihomomorphisms and I-summable extensions. There are the "cycles" for KK-theory (we will not be working just in the category of C^*-algebras). Then we formulate Connes' notion of p-summable Fredholm modules in the bivariant context. It is shown that the notions of p-summable quasihomomorphisms and p-summable Fredholm modules are equivalent.

In § 2, we discuss Jones-Kassel's bivariant cyclic theory for generalized Connes cyclic complexes. These complexes are suitable subcomplexes of cyclic bicomplexes. The framework is that established by Jones-Kassel. Similar results for cyclic k-modules, or d.g.-Λ-modules are to be found in [J-K]. Loday-Quillen [L-Q] constructed a quasi-isomorphism from the bar complex $B(M)$ to the total complex of the cyclic bicomplex $C_{\cdot\,\cdot}(M)$. We show that there is a quasi-inverse (which was given in Lemma 1, [W1]) to the Loday-Quillen map. The weaker fact that $\mathrm{Hom}_S\big(B(M), B(N)\big)$ and $\mathrm{Hom}_S\big(C_{\cdot\,\cdot}(M),$

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$C_{\cdot\cdot}(N)\big)$ are quasi-isomorphic, which can be derived from only the existence of the Loday-Quillen map, had been proved in [J-K].

§ 3 is the main section. It gives the full proof of the formula for the Chern character for 1-summable quasihomomorphisms (Theorem 3.8, Corollary 3.9). An outline of this proof was given in [W1]. Also, a construction is given of a Chern character $\mathrm{ch}^{2m}(h)$ for any I-summable quasihomomorphism from A to $A'\tilde{\otimes}K$, when I is H-unital (cf. Definition 3.10).

The functorial properties of our Chern character follow easily from the naturalness of our construction.

Kassel has already defined the "character of an extension" in [Kass3], when the ideal is H-unital. In § 4, we study the Chern character of I-summable extensions. By replacing A' by its suspension $\Sigma A'$, the problem is reduced to the even case considered in § 3. Our definition of Chern character uses the long exact sequence of [Kass3] and is shown to agree with Kassel's character.

In § 5 we digress to clarify H-unitarity for $L^p(H)$. The study of operator spaces and their Haagerup tensor products [E] is generalized to L^p-operator spaces and Haagerup L^p-products.

§ 6 spells out the details of Example 1 of [W1]. Also it describes how the definition of the Chern character of an algebraic bivariant theory can be viewed as a special case of 1-summable quasihomomorphisms.

We wish to point out that the condition that $(I, \otimes)$ is H-unital is not at all a necessary condition for a Chern character $\mathrm{ch}^0(h) \in \mathrm{HC}^0(A, I)$ to exist, given an I-summable quasihomomorphism h. To get a Chern character in $\mathrm{HC}^*(A, A')$, we need an excision-free construction, such as that of the Chern character for 1-summable quasihomomorphisms in Theorem 1, [W1].

Our construction of the Chern character using differential homological algebra may perhaps be viewed as part of an algebraicization of geometry. This algebraicization of geometry and analysis is directly in line with Cuntz's program of algebraicization of KK, and first appeared in Connes' quantization of the calculus of differential forms in his non-commutative differential geometry ([Con1]), [Con2], [Con-Cun]). These two approaches appear to be closely related. The simple fact that the universal algebra in Cuntz's picture QA ($\equiv A * A$) and the universal differential graded algebra ΩA are linearly isomorphic, although superficial, may indicate the tip of an iceberg.

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0. **Conventions and notation.** We shall denote by k any commutative ring of characteristic 0. In a (nondiscrete) topological category $k = \mathbb{C}$ or $\mathbb{R}$. All k-algebras are assumed to be projective as k-modules. This of course is automatically true when k is a field of characteristic 0. By a topological algebra, we mean a locally multiplicative-convex (m-convex) topological algebra over $\mathbb{C}$ or $\mathbb{R}$ with continuous multiplication. If A is a k-algebra, then $\tilde{A}$ is the algebra with a (new) unit adjoined to A. By an ideal, we mean a (closed) two-sided ideal. A Λ-module over any ring Λ is meant to be a left Λ-module. We abbreviate "differential graded complex" as "d.g.-complex".

The assumption that algebras be k-projective can be removed, if we work instead with "allowable" categories in the context of MacLane's relative homological algebra (Chapter IX, [Mac]). Allowable objects, e.g., short exact sequences of k-algebras, complexes of A-bimodules (in particular, resolutions), are those which admit k-linear splittings.

As usual H will be a separable infinite dimensional Hilbert space. $\mathcal{L}(H)$ and $\mathcal{K}(H)$ are respectively the C^*-algebra of all bounded linear operators and the C^*-algebra of compact operators. For $1 \leq p \leq \infty$, $L^p(H)$ is the Schatten ideal [Sim]. In particular $L^1(H)$ is the ideal of trace class operators and $L^\infty(H) = \mathcal{K}(H)$. All topological vector spaces, in particular all topological algebras, will be locally convex and Hausdorff.

We will denote by K a generic algebra in other categories which plays the role of $\mathcal{K}(H)$ in the category of C^*-algebras. For instance, in the discrete, or pure algebraic, category of k-algebras, we take $K = \varinjlim_n M_n(k)$.

We shall denote by $M(A)$ the *multiplier algebra* of a k-algebra A. Recall that (e.g. [Bus]) a pair of maps $(T, S): A \longrightarrow A$ is a *multiplier* if for any $a, b \in A$, we have $a \cdot T(b) = S(a) \cdot b$. Such maps are automatically k-linear.

It is easy to check [F-W], when $K = \varinjlim_n M_n(k)$, that the multiplier algebra

$$M(A \otimes K) = \Big\{ (a_{ij})_{i,j \geq 0} \mid a_{ij} \in M(A), \text{ and only finitely many } a_{ij} \neq 0 \text{ for each fixed } i, \text{ and for each fixed } j. \Big\}$$

Let p be a semi-norm of a locally convex topological algebra A. Then there is a semi-norm on $M(A)$ given by

$$P_a(T, S) = p\big(T(a)\big) + p\big(S(a)\big)$$

for each $a \in A$. Let p vary in a fundamental defining family $\mathcal{P}$ of A. The topology on $M(A)$ defined by all these semi-norms is called the *strong topology*. With this topology $M(A)$ may not be locally *multiplicative* convex, and thus the multiplication in $M(A)$ is only separately continuous. In this paper we shall restrict ourselves to multiplier algebras $M(A)$ with a locally convex topology such that

(i) the multiplication is jointly continuous;
(ii) the restriction to A coincides with the topology on A. This assumes implicity;

(iii) the multiplication in A induces an embedding of A into $M(A)$.

We introduce another canonical topology on $M(A)$ for a class of topological algebras so broad that it includes essentially all the complete locally m-convex algebras we encounter.

Recall that a topological algebra is complete if the underlying topological vector space is complete, i.e. every Cauchy net is convergent. Let A be a complete topological algebra with a fundamental family $\mathcal{P}$ of submultiplicative seminorms. Let $p \in \mathcal{P}$; then $\ker p = \{a \in A \mid p(a) = 0\}$ is a closed ideal of A, and $A/\ker p$ is a Banach algebra. We say that A *has no annihilators*, if the Banach algebra $A/\ker p$ satisfies the condition (iii) above, for every $p \in \mathcal{P}$.

PROPOSITION 0.1. *Let A be a topological algebra which has no annihilators. For each seminorm $p \in \mathcal{P}$, let $B_p(A) = \{a \in A \mid p(a) \leq 1\}$. Let $(T,S) \in M(A)$, and set $\|(T,S)\|_p = |T|_p + |S|_p$, where*

$$|T|_p = \sup_{a \in B_p(A)} p\big(T(a)\big), \quad |S|_p = \sup_{a \in B_p(A)} p\big(S(a)\big),$$

Then $\|(T,S)\|_p < \infty$.

PROOF. We note that (T,S) are not *a priori* assumed to be continuous. Both S and T map the closed ideal $\ker p$ into itself. In fact if $p(b) = 0$, and $p\big(T(b)\big) \neq 0$, then by the assumption on A there is some $a \in A$ with $p\big(aT(b)\big) \neq 0$. But $p\big(aT(b)\big) = p\big(S(a)b\big) \leq p\big(S(a)\big)p(b) = 0$, a contradiction. Similarly, we have $S(\ker p) \subset \ker p$. Thus, (T,S) induce k-linear maps $(\dot{T},\dot{S})$ on the quotient $A_p = A/\ker p$, which is a Banach algebra with the quotient norm $\dot{p}$. Both maps $(\dot{T},\dot{S})$ can be shown to be continuous and

$$\sup_{x \in B_{\dot{p}}(A_p)} \dot{p}\big(\dot{T}(x)\big) + \sup_{x \in B_{\dot{p}}(A_p)} \dot{p}\big(\dot{S}(x)\big) < \infty;$$

see the proofs of Proposition 2.5 and Lemma 2.6 of [Bus]. The conclusion follows from the equalities

$$|T|_p = \sup_{x \in B_{\dot{p}}(A_p)} \dot{p}\big(\dot{T}(x)\big), \quad |S|_p = \sup_{x \in B_{\dot{p}}(A_p)} \dot{p}\big(\dot{S}(x)\big). \qquad \blacksquare$$

We shall call the locally convex topology on $M(A)$ defined by the family $\{\|\ \|_p; p \in \mathcal{P}\}$ the *fine topology*, which is finer than the strong topology discussed above.

PROPOSITION 0.2. *Let A be an algebra without annihilators. With the fine topology, the multiplier algebra $M(A)$ is a complete locally convex algebra with jointly continuous multiplication. Furthermore, if A is a Fréchet algebra, $M(A)$ is also a Fréchet algebra.*

PROOF. Let (T,S) and (T',S') be two multipliers of A. Since for $x \in B_{\dot{p}}(A_p)$,

$$\dot{p}\big(\dot{T}\dot{T}'(x)\big) \leq |\dot{T}|_p \dot{p}\big(\dot{T}'(x)\big) \leq |\dot{T}|_p |\dot{T}'|_p,$$

the seminorms $\|\ \|_p$ are submultiplicative. Thus the multiplication in $M(A)$ is jointly continuous.

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Recall that A is metrizable if and only if it has a countable fundamental defining family of seminorms. Thus A is metrizable if and only if $M(A)$ is. It remains to show tht $M(A)$ is complete.

Let (T_λ, S_λ) be a Cauchy net of multipliers. Then $T_\lambda(a)$ and $S_\lambda(a)$ are Cauchy nets in A for any $a \in A$. Let

$$T(a) = \lim T_\lambda(a), \quad S(a) = \lim S_\lambda(a).$$

Since

$$aT(b) = \lim aT_\lambda(b) = \lim S_\lambda(a)b = S(a)b,$$

the pair (T, S) is also a multiplier. ■

A homomorphism between two involutive algebras will always be assumed to be a $*$-homomorphism.

The discussion in the following sections will be within a certain full subcategory $\mathcal{C}$ of the category of all complete locally m-convex k-spaces and topological homomorphisms, with a topological tensor product $\tilde{\otimes}$: $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$; namely, for any E, $F \in \mathcal{C}$, there is a locally m-convex topology on the algebraic tensor $E \otimes F$ with its completion $E\tilde{\otimes}F \in \mathcal{C}$ such that

(i) The canonical bilinear map $E \times F \longrightarrow E\tilde{\otimes}F$ is jointly continuous.

(ii) There are three canonical isomorphisms

$$\alpha : E\tilde{\otimes}(F\tilde{\otimes}G) \xrightarrow{\simeq} (E\tilde{\otimes}F)\tilde{\otimes}G$$
$$\lambda : k\tilde{\otimes}E \xrightarrow{\simeq} E, \qquad \rho : E\tilde{\otimes}k \xrightarrow{\simeq} E$$

such that $(\mathcal{C}, \tilde{\otimes}, \alpha, \lambda, \rho)$ form a *monoidal category* ([Mac2], p. 158).

Whenever $\tilde{\otimes}$ is used in chain complexes over $\mathcal{C}$, we assume that the Hochschild acyclic differential has a continuous extension. If the flip map on the algebraic tensor product extends to a canonical isomorphism $\gamma : E\tilde{\otimes}F \longrightarrow F\tilde{\otimes}E$, making $\mathcal{C}$ a *symmetric monoidal category* ([Mac2]), p. 180), we say $\tilde{\otimes}$ is *admissible*. Trivally, the algebraic tensor $\otimes$ is admissible for the category of all k-algebras with discrete topology.

"$\otimes$" will denote the algebraic tensor product, "$\hat{\otimes}$" and "$\check{\otimes}$" are respectively the complete projective and injective tensor products as topological vector spaces. They coincide for locally convex nuclear spaces. When A or A' is a nuclear C^*-algebra, the completed C^*-tensor product $\otimes_{\min}$ coincides with the maximum C^*-tensor product $\otimes_{\max}$, and we simply write $A \otimes A'$. In particular, this is the case when $A = \mathcal{K}$.

The question arises whether we should choose some fixed topological tensor product for continuous cyclic homology once and for all to avoid possible confusion. The answer is "No". Which one to choose depends on the kind of topological algebras we are dealing with. As a matter of fact the crucial concept of "H-unital algebra" depends on which tensor product we are referring to.

1. **Kasparov's KK-theory and Cuntz's quasihomomorphisms.** Let A and A' be two separable C^*-algebras. J. Cuntz has shown that Kasparov's group $KK(A, A') \simeq [qA, A' \otimes \mathcal{K}]$, the abelian group of homotopy classes of homomorphisms from qA to $A' \otimes \mathcal{K}$ ([Cun1]), [Cun2]). Recall that Cuntz's universal algebra qA is the closed ideal in the C^*-algebra free product $A * A$ spanned by the elements of the form $a^0 qa^1 \cdots qa^n$ and $qa^1 \cdots qa^n$ with $n \geq 1$. Here $qa = i(a) - \bar{\imath}(a)$ is the difference of the two universal embeddings $i, \bar{\imath}$ of A into $A * A$. One has the identity, $q(ab) = qa \cdot b + aqb - qaqb$, with the convention $i(a) = a$.

We may replace the C^*-algebras A and A' by arbitrary topological algebras, even just abstract algebras, and such quasihomomorphisms will still generate a certain "KK-theory" in the new category (see [W3] for details). From the point of view of Connes' noncommutative differential geometry, a particularly interesting category is that of "smooth subalgebras" of C^*-algebras, i.e. Fréchet algebras which are closed under the holomorphic functional calculus (p. 92, [Con2]).

Cuntz's quasihomomorphisms from A to $A' \otimes \mathcal{K}$ are in one-to-one correspondence with homomorphisms from qA to $A' \otimes \mathcal{K}$. The following definition (a) is more general and at the same time more specific.

DEFINITION 1.1. (a) (Even case). Let A and A' be two topological algebras over $\mathbb{C}$. Let I be an ideal of $A'\tilde{\otimes}K$. An *I-summable quasihomomorphism h from A to $A'\tilde{\otimes}K$* is a pair of homomorphisms $\alpha, \bar{\alpha}: A \longrightarrow M(A\tilde{\otimes}K)$ such that $\mathrm{Im}(\alpha - \bar{\alpha}) \subset I$. In Cuntz's notation $h = (\alpha, \bar{\alpha}): A \rightrightarrows M(A'\tilde{\otimes}K) \triangleright I$.
(b) (Odd case). An *I-summable extension* of A by A' is a short exact sequence

$$0 \longrightarrow I \longrightarrow E \longrightarrow A \longrightarrow 0$$

of algebras. Equivalently, it is a homomorphism $\phi: A \longrightarrow 0(I) := M(I)/I$.

REMARK. In the category of C^*-algebras, if we identify the Hilbert A'-module $H_{A'}$ as $\ell^2(A')$, then $A' \otimes \mathcal{K}$ and $M(A' \otimes \mathcal{K})$ are identified with $\mathcal{K}(H_{A'})$ and $\mathcal{L}(H_{A'})$, the C^*-algebra of compact opertors and the C^*-algebra of bounded operators on H_A, [Kasp1]. The ideal I will be an ideal of $\mathcal{K}(H_{A'})$. In this case I is complete with respect to a certain locally convex topology, but not closed in the operator norm topology on $\mathcal{L}(H_{A'})$.

When A' is a subalgebra of a C^*-algebra, we replace $H_{A'}$ by $H_{\bar{A}'}$, where $\bar{A}'$ is the C^*-closure of A'. In particular, when $I = L^p(H)\tilde{\otimes}A'$, where $L^p(H)$ is the pth Schatten ideal of $\mathcal{K}(H)$ [Sim], $1 \leq p \leq \infty$, an $L^p\tilde{\otimes}A'$-summable quasihomorphism or extension is just a p-summable quasi-homomorphism or extension ([Con-Cun], [W1]). This concept was originally introduced by Connes (p. 265 [Con2]), in the form of p-summable Fredholm modules.

Notice that an I-summable quasihomomorphism h from A to $A'\tilde{\otimes}K$ induces an algebra homomorphism from the algebraic analogue of qA to I. When A and A' are C^*-algebras, h usually defines a C^*-algebraic homomorphism from the C^*-algebraic completion of qA to the closure of I in $A'\tilde{\otimes}\mathcal{K}$, instead of just I.

DEFINITION 1.2. Let A and A' be C^*-algebras. A *p-summable Fredholm A'-module* over A is a triple (H, F, ϕ), where

(i) $H_{A'} = H^+_{A'} \oplus H^-_{A'}$ is a $\mathbb{Z}_2$-graded Hilbert A'-module [Kasp1], with the grading operator $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$;

(ii) ϕ is a trivially $\mathbb{Z}_2$-graded homomorphism of A into the C^*-algebra $\mathcal{L}(H_{A'})$, so $\phi(a) = \begin{bmatrix} a_+ & 0 \\ 0 & a_- \end{bmatrix}$ for $a \in A$;

(iii) $F \in \mathcal{L}(H_{A'})$, $F^2 = I$, $F\varepsilon = -\varepsilon F$, and $[F, \phi(a)] \in L^p(H)\tilde{\otimes}A'$; for all $a \in A$. Writing $F = \begin{bmatrix} 0 & P \\ P^{-1} & 0 \end{bmatrix}$ and $\alpha(a) = a_+$ and $\bar{\alpha}(a) = P^{-1}a_-P$ for any $a \in A$, we have that both α and $\bar{\alpha}$ are $*$-homomorphisms of A into $\mathcal{L}(H) \otimes_{\max} A'$. The following is easy to check.

PROPOSITION 1.1. *The relation $h(a^0qa^1) = \alpha(a^0)\big(\alpha(a^0) - \bar{\alpha}(a^1)\big)$ defines a $*$-algebra homomorphism h from Cuntz's algebra qa into $L^p(H)\tilde{\otimes}A'$.*

As an example, consider the case (p. 272, [Con2]) that $A' = \mathbb{C}$ and the p-summable A'-module is the Fredholm module given in Definition 1, p. 265, of [Con2]. Let $m \geq p-1$ and $n = 2m$. For any $a \in A$, let $da = i[F, \phi(a)]$. Consider the supertrace $\mathrm{Tr}_s(T) = \mathrm{Tr}(\varepsilon F[F, T])$, defined for any $T \in \mathcal{L}(H)$. The character τ of the cycle associated to (H, F) is defined (see Proposition 5, p. 275, [Con2]) by $\tau(a^0, a^1, \ldots, a^n) = (2i\pi)^m \frac{m!}{2} \mathrm{Tr}_s(a^0 da^1 \cdots da^n)$, for $a^i \in A$.

PROPOSITION 1.2.

$$\begin{aligned}\tau(a^0, \ldots, a^n) &= m!\,(2i\pi)^m \,\mathrm{Tr}\big(h(qa^0 \cdots qa^n)\big) \\ &= 2m!\,(2i\pi)^m \,\mathrm{Tr}(a^0 hqa^1 \cdots hqa^n).\end{aligned}$$

PROOF. We have

$$\tau(a^0, a^1, \ldots, a^{2m}) = m!\,(2i\pi)^m \,\mathrm{Tr}\big(\varepsilon F[F, \phi(a^0)][F, \phi(a^1)] \cdots [F, \phi(a^{2m})]\big),$$

since

$$\varepsilon F[F, \phi(a^0)] = \begin{bmatrix} a^0_+ - Pa^0_- P^{-1} & 0 \\ 0 & P^{-1}(a^0_+ - Pa^0_- P^{-1})P \end{bmatrix}$$

and

$$[F, \phi(a^{2i-1})][F, \phi(a^{2i})] =$$
$$\begin{bmatrix} -(a^{2i-1}_+ - Pa^{2i-1}_- P^{-1})(a^{2i}_+ - Pa^{2i}_- P^{-1}) & 0 \\ 0 & P^{-1}(a^{2i-1}_+ - Pa^{2i-1}P^{-1})(a^{2i}_+ - Pa^{2i-1}_- P^{-1})P \end{bmatrix}$$

A straightforward computation yields the results. ∎

Recall that $[\tau_{2m}] \in \mathrm{HC}^{2m}(A)$ is the $2m$-th Chern character of the Fredholm module (H, ϕ, F). It is our goal to merge Cuntz's algebraic approach to KK-theory with Jones-Kassel's differential homological algebraic approach to the bivariant cyclic theory. For this purpose, it is convenient to extend cyclic theory to the category of generalized cyclic complexes.

2. **Jones-Kassel's bivariant cyclic theory and generalized Connes cyclic complexes.** In this section k will be any commutative ring of characteristic 0. In later sections if the discussion is in the continuous, i.e., topological, category, we shall assume the conventions of § 0 and replace $\otimes$ by $\tilde{\otimes}$ accordingly.

Connes constructed a bivariant cyclic theory in [Con3] using cyclic k-modules. Let $\mathcal{C}$ be the Connes cyclic category. Recall that the objects in $\mathcal{C}$ are $\mathbf{n} = \{0, 1, \ldots, n\}$ and the morphisms are generated by the face maps $d_i: \mathbf{n+1} \longrightarrow \mathbf{n}$, the degenerating maps $s_i: \mathbf{n} \longrightarrow \mathbf{n+1}$, for $i = 0, 1, \ldots, n$, and the cyclic permutation $\tau_n: \mathbf{n} \longrightarrow \mathbf{n}$. A *cyclic k-module* is a contravariant functor M from $\mathcal{C}$ to the category of k-linear spaces. For simplicity we still denote the k-linear maps $M(d_i)$, $M(s_i)$, and $M(\tau_n)$ by d_i, s_i, and τ_n. If we denote the k-module $M(n)$ by M_n, then (M_*, b) and (M_*, b') are two chain complexes, where the boundary operators b and $b': M_{n+1} \longrightarrow M_n$ are given by

$$b = \sum_{i=0}^{n}(-1)^i d_i, \quad b' = \sum_{i=0}^{n-1}(-1)^i d_i$$

for $n = 1, 2, \ldots$. Since $b's_0 + s_0 b' = \mathrm{id}$, the complex (M_*, b') is acyclic. A unital k-algebra A defines in an obvious way a functor $A^\natural$ from Connes' cyclic categroy $\mathcal{C}$ to the category of k-linear spaces, with $A^\natural(n) = A^{\otimes(n+1)}$. Clearly $\big((A^\natural)_*, b\big)$ and $\big((A^\natural)_*, b'\big)$ are just the Hochschild complex and the acyclic Hochschild complex over A. Connes defined a bivariant cyclic theory $\mathrm{HC}^n_\lambda(A, A') = \mathrm{Ext}^n(A^\natural, A'^\natural)$, where Ext is the left derived functor of Hom on the category of cyclic k-linear spaces [Con3]. In particular, $\mathrm{Ext}^n(A^\natural, \mathbb{C}^\natural) = \mathrm{HC}^n(A)$, the n-th cyclic cohomology group of A.

We shall use the standard notation for the operators $N = 1 + t + \cdots + t^n$ at degree n and $B = (1 - t)s_0 N$, where t is the signed permutation $(-1)^n\tau$ at degree n. Recall that $b^2 = B^2 = Bb + bB = 0$. Together with the degree +1 differential B, (M_*, b, B) was called an *algebraic S^1-module* in [Bur], and later a *mixed complex* in [Kass1], and a d.g. *Λ-module* in [J-K], where $\Lambda = k[\varepsilon] \simeq H_*(S^1)$ is the generalized exterior algebra generated by a generator ε of $H_1(S^1)$, with zero differential. The multiplication by ε on M_* is given by the action of B. Thus, the category of cyclic k-modules is included into the category of d.g. Λ-modules.

We first gather some background material from [J-K], § 1, and § 2.

The bar construction $B(\Lambda)$ on the algebra Λ is a graded Hopf algebra isomorphic to $H^*(BS^1)$; as a k-algebra $B(\Lambda)$ is isomorphic to $K[v]$, $\deg v = -2$; as a coalgebra $B(\Lambda)^*$, it is isomorphic to $K[u]$, $\deg u = 2$. The comultiplication $B(\Lambda) \longrightarrow B(\Lambda) \otimes B(\Lambda)$ is given by

$$\Delta(u^n) = 1 \otimes u^n + u \otimes u^{n-1} + \cdots + u^n \otimes 1.$$

Let M be a d.g. Λ-module. Then $B(M) := B(\Lambda) \otimes_k M$ has a natural d.g. $B(\Lambda)$-comodule structure; the differential is

$$d(u^n \otimes m) = u^n \otimes bm + u^{n-1} \otimes Bm.$$

Dually, $B(M)^- := B(\Lambda)^* \tilde{\otimes}_k M$ is a d.g. $B(\Lambda)^*$-module, where $\tilde{\otimes}_k$ stands for the algebraically complete tensor product with respect to the inverse limit.

Thus, an n-chain in $B(M)_n$ has the form

$$1 \otimes m_n + u \otimes m_{n-2} + \cdots + u^{[n/2]} \otimes m_0 \quad (\text{ or } u^{[n/2]} \otimes m_1)$$

and an n-chain in $B^-(M)_n$ has the form

$$1 \otimes m_n + v \otimes m_{n+2} + v^2 \otimes m_{n+4} + \cdots,$$

where $m_k \in M_k, \forall k$.

The $B(\Lambda)^*$-comodule structure on $B(M)$ and $B(\Lambda)$-module structure on $B(M)^-$ define the periodicity operator S:

$$S(u^n \otimes m) = u^{n-1} \otimes m,$$
$$S(v^n \otimes m) = v^{n+1} \otimes m.$$

The category of d.g. $B(\Lambda)$-comodules and the category of d.g. $B(\Lambda)^-$-modules are equivalent, and are equivalent to the category of cyclic k-modules. Moreover, for any d.g. Λ-modules M and M', the d.g. Hom complex $\mathrm{Hom}_S\big(B(M), B(M')\big)$ of $B(\Lambda)$-comodule maps (i.e., the k-linear maps commuting with S) is isomorphic to the d.g. Hom complex $\mathrm{Hom}_S\big(B(M)^-, B(M')^-\big)$ of $B(\Lambda)^*$-module maps (Lemma 2.1, [J-K]).

The following are two trivial observations.

PROPOSITION 2.1. *Every cochain* $f \in \mathrm{Hom}_S\big(B(A), B(A')\big)^{2n}$ *(resp.* $\mathrm{Hom}_S\big(B(M), B(M')\big)^{2n+1}$*) is given by a family* $(f^{(i)})_{i=n,n-1,\ldots}$*, where* $f^{(i)} \in \mathrm{Hom}(M, M')_{-2i}$ *(resp.* $\mathrm{Hom}(M, M')_{-2i-1}$*) such that*

$$f(u^p \otimes m) = \sum_{k=0}^{p} u^{p-k} \otimes f^{(n-k)}(m). \tag{2.1}$$

PROOF. Recall ([Kass2] p. 230, Lemma 2.1 [J-K]) that

$$f(u^p \otimes m) = \sum_{k \in \mathbf{Z}} u^{p-k} \otimes f^{(n-k)}(m).$$

Since $u_i = 0$ if $i < 0$, we can write

$$f(u^p \otimes m) = \sum_{k \le p} u^{p-k} \otimes f^{(n-k)}(m).$$

We assert that $f^{(n-k)} = 0$ if $k < 0$. If not, say $f^{(n-k)}(m) \neq 0$, then $S\big(u^{-k} \otimes f^{(n-k)}(m)\big) = u^{-k-1} \otimes f^{(n-k)}(m)$ is the image of $Sf(1 \otimes m)$, but $fS(1 \otimes m) = 0$, a contradiction. ■

Note that under the isomorphism from $\mathrm{Hom}_S\big(B(M), B(M')\big)$ to $\mathrm{Hom}_S\big(B(M)^-, B(M')^-\big)$, corresponding to (2.1), the image of f is given ([J-K], Lemma 2.1) by

$$f(v^p \otimes m) = \sum_{k \ge 0} v^{k+p} \otimes f^{(n-k)}(m). \tag{2.2}$$

In comparison with (2.1), there are infinitely many nonzero terms on the right of (2.2) in general.

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PROPOSITION 2.2. *Assume* $f \in \mathrm{Hom}_S\big(B(A), B(A')\big)^n$ *is represented by a family* $(f^{(i)})_{i=n,n-1,\dots}$. *Then* $S^m \circ f \in \mathrm{Hom}_S\big(B(A), B(A')\big)^{n+2m}$ *is given by* $(f^{(i)})_{i=n+m,n+m-1,\dots}$, *with* $f^{(n+m)} = \cdots = f^{(n+1)} = 0$.

PROOF. It is enough to show this for $m = 1$. By (2.1),

$$(Sf)(u^p \otimes m) = \sum_{k=0}^{p-1} j^{p-k-1} \otimes f^{(n-k)}(m)$$
$$= \sum_{k=0}^{p-1} u^{p-k} \otimes f^{(n-k+1)}(m)$$

with $f^{(n+1)} = 0$. ∎

Jones-Kassel defined in [J-K] the bivariant cyclic group

$$\mathrm{HC}^n(M, M') = H^n(\mathrm{Hom}_S(B(M), B(M')^*)).$$

The condition that a cochain $f = (f^{(i)})_{i=n,n-1,\dots}$ is cocycle (p. 204, [Kass1], p. 230 [Kass2]) is

$$\begin{cases} [b, f^{(n)}] = 0, \\ [b, f^{(i)}] + [B, f^{(i+1)}] = 0, \quad i = n-1, n-2, \dots. \end{cases}$$

The condition for a cochain $f = (f^{(i)})_{i=n,n-1,\dots}$ to be a coboundary is (see *loc.cit.*) that there exists some $g = (g^{(i)})_{i=n,n-1,\dots}$, where $\deg g^{(i)} = \deg f^{(i)} + 1$ such that

$$\begin{cases} [b, g^{(n)}] = f^{(n)}, \\ [b, g^{(i)}] + [B, g^{(i+1)}] = f^{(i)}, \quad i = n-1, n-2, \dots. \end{cases}$$

For the discussion in the following sections, it is necessary to extend the (bivariant) cyclic theory slightly beyond the category of cyclic k-modules.

Let $(C_{n,m}(M), d_i)$, $n, m \geq 0$, $i = 1, 2$, be the d.g. bicomplex associated to a cyclic k-module M (p. 955, [Con3]). Here $C_{n,m}(M) = M_m$ for all n, m, the vertical differential d_2 is equal to b if n is even and to $-b'$ if n is odd, the horizontal differential d_1 is equal to $(1-t)$ if n is odd, and to N if n is even.

Note that the periodicity operator S acts on $C_{\cdot\,\cdot}(M)$ by deleting the first two columns. We may write $M_{n,m}$ for $C_{n,m}(M)$ if no confusion arises.

DEFINITION 2.3. A *generalized cyclic k-complex* $(N_{n,m}, d_i)$, $i = 1, 2$, $n, m \geq 0$ is a differential graded subcomplex of any d.g. bicomplex $(M_{n,m}, d_i)$, $i = 1, 2$, $n, m \geq 0$, given by a cyclic k-module, such that $N_{n,m}$ depends only on m for all m and is invariant under the action of t.

A generalized cyclic k-complex is *H-unital* if there is a homotopy operator $s\colon N_m \longrightarrow N_{m+1}$, $m \geq 0$, such that s satisfies all the relations s_0 has with other operators, in particular, $sb' + b's = 1$.

For the reason for this definition, see Definitions 3.11 and 3.12.

We note that $(N_{n,m})$ is a bihomogeneous submodule of $(M_{n,m})$ (p. 60 [C-E]), $m \geq 0$. It is not always possible to make N_* a d.g. Λ-module. Only H-unital generalized cyclic

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k-complexes can be made into d.g. Λ-modules. Not every d.g. Λ-module comes from a cyclic k-module. These three categories are all inequivalent, although some of them are contained in others. A typical case is

EXAMPLE 2.4. Let I be an ideal of a unital algebra A. Let $C_{..}(A)$ be the d.g. bicomplex given by the cyclic k-module $A^\natural$ (see e.g. [L-Q], p. 567). Let $\pi: A \longrightarrow A/I$ be the quotient map. Set $K_* = \ker \pi_*$, where π_* is the induced chain map $\pi_*: C_*(A) \longrightarrow C_*(A/I)$. Then for $\ell \geq 0$, K_ℓ is the closure of the linear span of elements $a^0 \otimes a^1 \otimes \cdots \otimes a^\ell$ with at least one $a^i \in I$.

There is a canonical filtration $F_k(A, I)$ (for short, $F_k(A)$ or F_k), $k = -1, 0, 1, \ldots$, of K_* defined as follows: $F_{-1}(A) = 0$, $F_0(A) = C_*(I)$. In general $F_k(A)_{k+\ell}$ is the closure of the linear span of $a^0 \otimes a^1 \otimes \cdots \otimes a^{k+\ell}$, where at least $(\ell + 1)$ {respectively, at least 1} of a^i are in I, for $\ell \geq 0$ {respectively for $-k \leq \ell \leq 0$} $(k = 1, 2, \ldots)$. Clearly $F_{-1}(A) \subset F_0(A) \subset F_1(A) \subset \cdots \subset K_* \subset C_*(A)$ and $\cup_{k=1}^{\infty} F_k(A) = K_*$. Each F_k defines a generalized cyclic complex.

The following is easy to verify

PROPOSITION 2.5. *Both K_* and the filtration $F_k(A)$, $k = -1, 0, 1 \ldots$, are invariant under the cyclic action t. They preserve both the Hochschild boundary b and the acyclic Hochschild boundary b'.*

COROLLARY 2.6. *With $K_{n,m} = K_m$ and $F_k(A)_{n,m} = F_k(A)_m$, for all $n, m \geq 0$, both $(K_{..}, d_i)$ and $(F_k(A), d_i)$, $i = 1, 2$, are generalized cyclic k-complexes.*

Note that the periodicity generator S acts on a generalized cyclic k-complex M by deleting the first two columns, and so also acts on the total complex $\mathrm{Tot}(M)$. Thus $\mathrm{Tot}(M)$ is a d.g. S-module as defined by [J-K].

DEFINITION 2.7. Let M, M' be generalized cyclic k-complexes. The *bivariant cyclic groups* of M and M' are

$$\mathrm{HC}^n(M, M') = H^n\Big(\mathrm{Hom}_S\big(\mathrm{Tot}(M), \mathrm{Tot}(M')\big)^*\Big).$$

LEMMA 2.8. *Let $M_{..}$ and $M'_{..}$ be two positively bigraded modules. Then there is a natural isomorphism*

$$\Gamma: \mathrm{Hom}\big(\mathrm{Tot}(M_{..}), \mathrm{Tot}(M')\big) \longrightarrow \mathrm{Tot}\,\mathrm{Hom}(M_{..}, M'_{..}).$$

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PROOF. *For any $n \in \mathbb{Z}$, we have*

$$\begin{aligned}
\mathrm{Hom}(\mathrm{Tot}\, M, \mathrm{Tot}\, M')^n &= \prod_{m\geq 0}\big((\mathrm{Tot}\, M)_m, (\mathrm{Tot}\, M')_{m-n}\big)\\
&= \prod_{m\geq 0}\Big(\prod_{\substack{k+\ell=m\\ k,\ell\geq 0}} M_{k,\ell}, \prod_{\substack{k'+\ell'=m-n\\ k',\ell'\geq 0}} M'_{k',\ell'}\Big)\\
&= \prod_{m\geq 0}\prod_{\substack{k+\ell=m\\ p+q=n}} \mathrm{Hom}(M_{k,\ell}, M'_{k-p,\ell-q})\\
&= \prod_{\substack{p+q=n\\ p,q\in\mathbb{Z}}} \mathrm{Hom}(M_{\cdot\cdot}, M'_{\cdot\cdot})^{p,q}\\
&= \mathrm{Tot}\,\mathrm{Hom}(M_{\cdot\cdot}, M'_{\cdot\cdot})^n.
\end{aligned}$$

■

LEMMA 2.9. *Let M and M' be two generalized cyclic k-complexes. Suppose that $f \in \mathrm{Hom}(M_{\cdot\cdot}, M'_{\cdot\cdot})$. Write $f = (f^{k,\ell}_{p,q})^{k,\ell\geq 0}_{p\leq k, q\leq \ell}$, where $f^{k,\ell}_{p,q} \in \mathrm{Hom}(M_{k,\ell}, M_{k-p,\ell-q})$. Then $f \in \mathrm{Hom}_S(M_{\cdot\cdot}, M'_{\cdot\cdot})$ if and only if $f^{k,\ell}_{p,q} = f^{k-2,\ell}_{p,q}$ (modulo the identification $M_\ell = M_{k,\ell}$, etc.) for all $k-p\geq 2$, $\ell\geq q$.*

PROOF. This follows from the commuting diagram

$$\begin{array}{ccccccc}
M_\ell & = M_{k,\ell} & \xrightarrow{f^{k,\ell}_{p,q}} & M'_{k-p,\ell-q} & = M_{\ell-q}\\
\| & S\downarrow & & \downarrow S & \|\\
M_\ell & = M_{k-2,\ell} & \xrightarrow{f^{k-2,\ell}_{p,q}} & M'_{k-p-2,\ell-q} & = M_{\ell-q}
\end{array}$$

■

THEOREM 2.10. *Suppose that $f \in \mathrm{Hom}_S(M_{\cdot\cdot}, M'_{\cdot\cdot})^n$. Then f is determined by two families $(f_0^{(q)})$, $q\leq n+1$, $(f_1^{(q)})$, $q\leq n-1$, of maps, where $(f_0^{(q)})$ and $(f_1^{(q)})$: $M_\ell \longrightarrow M_{\ell-2q}$, for all ℓ, are of degree 2. More precisely, if $f = (f^{k,\ell}_{p,q})^{k,\ell\geq 0}_{p\leq k, q\leq\ell}$, $p+q=n$, then*

(i) $f^{2k,\ell}_{n-q,q} = f_0^{(q)}|_{M_\ell}$, for all $2k\geq n-q$, $\ell\geq q$;

(ii) $f^{2k-1,\ell}_{n-q,q} = f_1^{(q)}|_{M_\ell}$, for all $2k-1\geq n-q$, $\ell\geq q$.

PROOF. By Lemma 2.9, the map of complexes $f = (f^{k,\ell}_{p,q})^{k,\ell\geq 0}_{p\leq k, q\leq\ell}$ satisfies $f^{k,\ell}_{p,q} = f^{k-2,\ell}_{p,q}$, for $k-p\geq 2$. So f is determined by $f^{k,\ell}_{p,q}$, for $p\leq k < p+2$, $q\leq\ell$ and $k,\ell\geq 0$. For fixed n, $p = n-q$, there is a unique k_0 such that $n-q\leq 2k_0 < n-q+2$, and $n-q\leq 2k_0-1 < n-q+2$. Let $f_0^{(q)}|_{M_\ell} = f^{2k_0,\ell}_{n-q,q}$ and $f_1^{(q)} = f^{2k_0-1,\ell}_{n-q,q}$. ■

THEOREM 2.11. *Let M and M' be two generalized cyclic k-complexes. There is a natural isomorphism of d.g. S-modules:*

$$\Gamma\colon \mathrm{Hom}_S(\mathrm{Tot}\, M, \mathrm{Tot}\, M') \xrightarrow{\simeq} \mathrm{Tot}\,\mathrm{Hom}_S(M, M').$$

PROOF. This is a consequence of Lemmas 2.8 and 2.9 and Theorem 2.10. The criterion for $Sf = fS$ given in Theorem 2.10 is "invariant" under the isomorphism Γ in Lemma 2.8. ■

Now we have another definition of the bivariant cyclic groups.

THEOREM 2.12. *Let M, M' be two generalized cyclic k-complexes. Then*

$$\mathrm{HC}^n(M,M') = H^n\big(\mathrm{Tot}\,\mathrm{Hom}_S(M,M')\big). \qquad \blacksquare$$

From Theorems 2.11 and 2.12 and Definition 2.17, we may just write $\mathrm{HC}^n(M,M') = H^n\big(\mathrm{Hom}_S(M,M')\big)$.

If M is an H-unital generalized cyclic complex (Definition 2.13), then one can construct Connes' double complex $\big(B.\,.(M),b,B\big)$ as before (e.g., p. 570, [L-Q]), with $B_{m,n}(M) = M_{2m,n-m}$, $b = d_2 \mid M_{2m}$, $B = d, s_0d_1 = (1-t)s_0N$.

We shall denote the total complex of $\big(B.\,.(M),b,B\big)$ simply by $B(M)$. It coincides with the comodule $B(M)$ of Jones-Kassel, if M is a cyclic k-module. The periodicity operator S acts on the double complex $B.\,.(M)$ by deleting the first column and thus acts on $B(M)$.

We will say two d.g. (S-) complexes (M,d) and (N,d') are *quasi-isomorphic* (not to be confused with quasi-homomorphisms!) if there is a sequence of d.g. (S-) complexes (M_i,d_i), $i = 0,1,\dots,n$, such that

1) $(M_0,d_0) = (M,d)$, $(M_n,d_n) = (N,d')$; and
2) there is an (S-) complex map between (M_i,d_i) and (M_{i+1},d_{i+1}) which induces an isomorphism between the homology of the complexes.

Such an (S-) complex map is called a *quasi-isomorphism*.

We note that the obvious "projection" from $\mathrm{Tot}(M)$ to $B(M)$ is not a chain map. Although in general there is no quasi-inverse to a quasi-isomorphism, we show here this is the case. The construction of the quasi-inverse ψ can be read off from (ii) of Lemma 1, [W1].

THEOREM 2.13. *For any generalized H-unital cyclic complex M, there is a map of complexes $\psi\colon \mathrm{Tot}\,M \longrightarrow B(M)$ commuting with S, which is a quasi-inverse to Loday-Quillen's map.*

PROOF. We define ψ by the following formula: $\psi\colon M_{2n,\ell} \longrightarrow M_{2n,\ell}$ is the identity, while $\psi(m) = (1-t)sm$ if $m \in M_{2n+1,\ell}$, for $\ell, n = 0,1,2,\dots$.

It is easy to check that ψ is a chain map. Let d_C and d_B be the differentials in $\mathrm{Tot}\,M$ and $B(M)$ respectively. If $m \in M_{2n,\ell}$, then $\psi(m) = m$, $d_Cm = (Nm,bm)$ and $\psi d_Cm = \big((1-t)sNm,Bm\big) = d_B\psi(m)$. If $m \in M_{2n+1,\ell}$, then $\psi(m) = (1-t)sm \in M_{2n,\ell+1}$ and $d_B\psi(m) = \big(B(1-t)sm, b(1-t)sm\big) \in M_{2(n-1),\ell+2} \oplus M_{2n,\ell}$. On the other hand, $d_Cm = \big((1-t)m, -b'm\big) \in M_{2n,\ell} \oplus M_{2n+1,\ell-1}$, and $\psi d_Cm = \big((1-t)m - (1-t)sb'm\big) \in M_{2n,\ell}$. Since $B(1-t)s = 0$ and $b(1-t) = (1-t)b'$, $b's = 1 - sb'$, we get $d_B\psi = \psi d_C$.

We check that $\psi \circ \phi \simeq 1_B$, so ψ is a quasi-inverse of ϕ.

From our definition, $\psi\phi(m) = m + (1-t)ssNm$. Thus the decomposition of $(\psi\phi - \mathrm{Id}_B) \in \mathrm{Hom}_S\big(B(M),B(M)\big)^0$ is $\big(0,(1-t)ssN,0,\dots\big)$, i.e., the degree 2 map $(\psi\phi - \mathrm{Id}_B)^{(-1)} = (1-t)ssN$, and the rest $(\psi\phi - \mathrm{Id}_B)^{(-i)} = 0$, $i = 0,2,3,\dots$. We define $f \in \mathrm{Hom}_S\big(B(M),B(M)\big)^1$ such that $f^{(0)} = 0 = f^{(-2)} = f^{(-3)} = \cdots$, but

$$f^{(-1)} = (1-t)sssN\colon M_n \longrightarrow M_{n+3}, \forall n.$$

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It is easy to check that

$$\begin{aligned}[b,f^{(-1)}] &= bf^{(-1)} - (-1)^{|f|}f^{(-1)}b \\ &= bf^{(-1)} + f^{(-1)}b \\ &= (1-t)b'sssN + (1-t)sssb'N.\end{aligned}$$

Since $b's = 1 - sb'$, we have

$$b'ss = s - sb's = s - s(1-sb') = ssb',$$

so $[b,f^{(-1)}] = (1-t)ss(b's + sb')N = (1-t)ssN = (\psi\phi - \mathrm{Id}_B)^{(-1)}$.

On the other hand,

$$[B,f^{(-1)}] = Bf^{(-1)} + f^{(-1)}B = B(1-t)sssN + (1-t)sssNB.$$

Recalling that $B = (1-t)sN$ and that $N(1-t) = (1-t)N = 0$, we have $[B,f^{(-1)}] = 0$. It follows that $[\psi\phi] - [1_B] = 0$ in $H^0(\mathrm{Hom}_S(B(M), B(M)))$. Since we already know the Loday-Quillen map ϕ is a quasi-isomorphism the other relation

$$[\phi\psi] = [1_C] \in H^0\big(\mathrm{Hom}_S\big(\mathrm{Tot}(M), \mathrm{Tot}(M)\big)\big)$$

follows automatically. ∎

From Theorem 2.13, it follows in particular that for any generalized H-unital cyclic complex M with $ss = 0$, the bar complex $B(M)$ is a *strong deformation retract* of the total complex $M.$. (By definition, a complex M is a strong deformation retract of a complex N if there are two complex maps ϕ and ψ with $\phi: M \longrightarrow N$ injective such that $\psi\phi = \mathrm{id}_M$ and $\phi\psi = \mathrm{id}_N + df + fd$ for some $f \in \mathrm{Hom}(N,N)^1$.)

COROLLARY 2.14. *Let M and M' be two generalized cyclic bicomplexes. Then the two* Hom *complexes* $\mathrm{Hom}_S(B(M), B(M'))$ *and* $\mathrm{Hom}_S(\mathrm{Tot}\, M, \mathrm{Tot}\, M')$ *are quasi-isomorphic.*

PROOF. The map taking $f \in \mathrm{Hom}_S(B(M), B(M'))$ to $(\phi_{M'} \circ f \circ \psi_M) \in \mathrm{Hom}_S(\mathrm{Tot}\, M, \mathrm{Tot}\, M')$ is an S-complex map. It is easy to check that it has a quasi-inverse sending $g \in \mathrm{Hom}_S(\mathrm{Tot}\, M, \mathrm{Tot}\, M')$ to $\psi_{M'} g \circ \phi_M \in \mathrm{Hom}_S(B(M), B(M'))$. The notation ϕ_M, ψ_M, is self-explanatory. ∎

As another corollary, we have

THEOREM 2.15. *Let M and M' be two cyclic bicomplexes. Then there is a canonical isomorphism $\phi_{M_*} \circ \psi_{M_*}$*

$$\phi_{M_*} \circ \psi_{M_*} : H^n(\mathrm{Hom}_S(B(M), B(M')) \xrightarrow{\simeq} H^n(\mathrm{Hom}_S(\mathrm{Tot}\, M, \mathrm{Tot}\, M')).$$

3. ***I*-summable quasihomomorphisms and the bivariant Chern character.** Let A, A' be topological algebras and I be an ideal of $A'\tilde{\otimes}K$ as described in § 0. Our discussion is simultaneously for both algebraic and topological categories (in the discrete category, replace $\tilde{\otimes}$ by $\otimes$; see the beginning of § 2). Let $h = (\alpha, \bar{\alpha}): A \rightrightarrows M(A'\tilde{\otimes}K)$ be an I-summable quasihomomorphism from A to $A'\tilde{\otimes}K$. Let E be the completion of unitalized algebra generated by the ideal I and the images of homomorphisms α and $\bar{\alpha}$ in $M(A'\tilde{\otimes}K)$. For those who do not mind working with nonseparable algebras, in the following, we may simply replace E by $M(A'\tilde{\otimes}K)$ itself. Thus, associated to h, there is an exact sequence

$$0 \longrightarrow I \longrightarrow E \xrightarrow{\pi} R \longrightarrow 0 \tag{3.1}$$

where by definition R is the quotient algebra E/I. Let $K_* = K_*(E)$ be the kernel $\ker \pi_*$, where π_* is the complex map induced by the quotient π_* on the Hochschild complexes:

$$0 \longrightarrow K_*(E) \longrightarrow C_*(E) \xrightarrow{\pi_*} C_*(R) \longrightarrow 0 \tag{3.2}$$

In example 2.4 we exhibited a filtration $F_k(E)$ of $C_*(E)$, $k = -1, 0, 1, \ldots$, and showed that $K_*(E)$ and $F_k(E)$ for each k give rise to generalized cyclic complexes.

Now we recall from [W1] some of the crucial construction. The algebra homomorphisms α and $\bar{\alpha}: A \longrightarrow E$ define canonically two cocycles $\alpha^{\#}$ and $\bar{\alpha}^{\#}$ in $\mathrm{Hom}_S\big(C..(A), C..(E)\big)^{0,0}$. They are actually two bicomplex maps commuting with the periodicity operator S. Let $h_m = h_{n,m}: C^*_{n,m}(A) \longrightarrow C^*_{n,m}(E)$ be the restriction of $h^{\#} = (\alpha^{\#} - \bar{\alpha}^{\#})$ at the (n, m)-chain. (Recall that $C^*_{n,m}(A) = C^*_m(A)$ and $C^*_{n,m}(E) = C^*_m(E)$.)

LEMMA 3.1. *The linear map h_m, $m \geq 0$, is given by the following recurrence formula: $h_0(a^0) = h(qa)$, and if we suppose h_{m-1} is defined then*

$$\begin{aligned} h_m(a^0, a^1, \ldots, a^m) = \big(h(qa^0), \alpha(a^1), \ldots, \alpha(a^m)\big) + \big(\alpha(a^0), h_{m-1}(\alpha^1, \ldots, a^m)\big) \\ - \big(h(qa^0), h_{m-1}(\alpha^1, \ldots, a^m)\big). \end{aligned} \tag{3.3}$$

PROOF. For $m = 0$, by the definition of quasihomomorphism

$$h_0(a^0) = \alpha(\alpha^0) - \bar{\alpha}(a^0) = h(qa^0).$$

Now let us return to the notation $\tilde{\otimes}$ for the moment:

$$\begin{aligned} & a^{\#}(a^0\tilde{\otimes}a^1\tilde{\otimes}\cdots\tilde{\otimes}a^m) - \bar{\alpha}^{\#}(a^0\tilde{\otimes}a^1\tilde{\otimes}\cdots\tilde{\otimes}a^m) \\ &= \alpha(a^0)\tilde{\otimes}\alpha(a^1)\tilde{\otimes}\cdots\tilde{\otimes}\alpha(a^m) - \bar{\alpha}(a^0)\tilde{\otimes}\bar{\alpha}(a^1)\tilde{\otimes}\cdots\tilde{\otimes}\bar{\alpha}(a^m) \\ &= \big(\alpha(a^0) - \bar{\alpha}(a^0)\big)\tilde{\otimes}\alpha(a^1)\tilde{\otimes}\cdots\tilde{\otimes}\alpha(a^m) \\ &\quad + \bar{\alpha}(a^0)\tilde{\otimes}[\alpha(a^1)\tilde{\otimes}\cdots\tilde{\otimes}\alpha(a^m) - \bar{\alpha}(a^1)\tilde{\otimes}\cdots\tilde{\otimes}\bar{\alpha}(a^m)]. \end{aligned} \tag{3.4}$$

Replace $\bar{\alpha}(a^0)$ by $\alpha(a^0) - hqa^0$ in the second tensor; we obtain (3.3) immediately. ∎

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LEMMA 3.2. *The recurrence formula (3.3) can be replaced by the general formula*

$$
\begin{aligned}
h_m(a^0, a^1, \ldots, a^m) = {} & \big(h(qa^0), \alpha(a^1), \alpha(a^2), \ldots, \alpha(a^m)\big) \\
& + \big(\bar{\alpha}(a^0), h(qa^1), \alpha(a^2), \ldots, \alpha(a^m)\big) + \cdots \\
& + \big(\bar{\alpha}(a^0), \bar{\alpha}(a^1), \ldots, \bar{\alpha}(a^{m-1}), h(qa^m)\big).
\end{aligned} \tag{3.5}
$$

PROOF. Use induction on $m \geq 0$. The formula holds for $m = 0$. Suppose (3.5) holds for $(m-1)$. Then the truth of (3.5) for m follows from (3.3). ∎

COROLLARY 3.3. *The general formula for h_m in terms of only the homomorphism α and the homomorphism $h\colon qA \longrightarrow I$ is given by*

$$
\begin{aligned}
h_m(a^0, a^1, \ldots, a^m) = {} & \big(h(qa^0), \alpha(a^1), \alpha(a^2), \ldots, \alpha(a^m)\big) \\
& + \big(\alpha(a^0) - h(qa^0), h(qa^1), \ldots, \alpha(a^m)\big) + \cdots \\
& + \big(\alpha(a^0) - h(qa^0), \alpha(a^1) - h(qa^1), \ldots, \alpha(a^{m-1}) \\
& \quad - hq(a^{m-1}), hq(a^m)\big).
\end{aligned} \tag{3.6}
$$

We conclude:

THEOREM 3.4. *An I-summable quasihomomorphism $h(\alpha, \bar{\alpha})\colon A \rightrightarrows I$ from A to $A'\hat{\otimes}K$ defines uniquely a cocycle $h^\#$ in*

$$\operatorname{Hom}_S\big(C.\,.(A), K.\,.(E)\big)^{0,0} \subset \operatorname{Hom}_S\big(\operatorname{Tot} C.\,.(A), \operatorname{Tot} K.\,.(E)\big)^0.$$

The chain maps $h_m = h_{n,m}$ are given by any one of the formulae (3.3), (3.5), and (3.6). We shall denote both the class of $h^\#$ in $\mathrm{HC}^0\big(A, K_*(E)\big)$ and the class of $h^\#$ in $\mathrm{HC}^0(A, E)$ by $[h]$. Which group we are referring to is usually clear from the context.

We know that $\operatorname{Hom}_S\big(B(A), B(E)\big) \simeq \operatorname{Hom}_S\big(\operatorname{Tot}(C.\,.(A)), \operatorname{Tot}(C.\,.(E))\big)$ (Corollary 2.15). We shall still write $\alpha^\#$, $\bar{\alpha}^\#$, $h^\#$, etc., for the corresponding cocycles in $\operatorname{Hom}_S\big(B(A), B(E)\big)$. From our work in § 2, now it is quite easy to describe $h^\#$ in terms of the traditional higher strong homotopies. Let $\alpha^\# = (\alpha^{(i)})_{i=0,-1,\ldots}$, $\bar{\alpha}^\# = (\bar{\alpha}^{(i)})_{i=0,-1,\ldots}$, and $h^\# = (h^{(i)})_{i=0,-1,\ldots}$ be the decomposition of Proposition 2.1, where $h^{(i)} \in \operatorname{Hom}\big(C_*(A), C_*(E)\big)_{2i}$, etc.

Given any unital k-algebra A, recall that $\tilde{A}$ is the algebra with a new identity $\tilde{1}$ adjoined to A. For any algebra homomorphism $\alpha\colon A \longrightarrow A'$, denote by $\tilde{\alpha}\colon \tilde{A} \longrightarrow \tilde{A}'$ be the "unitalization" of α, and by $\pi\colon \tilde{A}' \longrightarrow A'$ the unital map sending $\tilde{1}_{A'}$ of $\tilde{A}'$ to the "old" unit $1_{A'}$. Then α is the composite

$$A \xrightarrow{I} \tilde{A} \xrightarrow{\tilde{\alpha}} \tilde{A}' \xrightarrow{\pi} A' \tag{3.7}$$

where I is the inclusion.

LEMMA 3.5. *If $\alpha\colon A \longrightarrow A'$ is a unital homomorphism, then we may choose $\alpha^{(i)} = 0$ for $i = -1, -2, \ldots$, while $\alpha^{(0)}$ is given by the obvious map: $\alpha^{(0)}(a^0, \ldots, a^m) = \big(\alpha(a^0), \ldots, \alpha(a^m)\big)$, for any $m \geq 0$.*

Thus in the decomposition $\tilde{\alpha}^\# = (\tilde{\alpha}^{(i)})_{i=0,-1,\ldots}$ and $\tilde{\pi}^\# = (\tilde{\pi}^{(i)})_{i=0,-1,\ldots}$, of the induced maps $\tilde{\alpha}^\# \in \mathrm{Hom}_S\big(B(\tilde{A}), B(\tilde{A}')\big)^0$ and $\pi^\# \in \mathrm{Hom}_S\big(B(\tilde{A}'), B(A)\big)^0$ the only nonzero maps are the obvious maps $\tilde{\alpha}^{(0)}$ and $\pi^{(0)}$.

PROPOSITION 3.6. *In the decomposition $I^\# = (I^{(-i)})_{i=0,-1,\ldots}$, one may choose $I^{(-i)} = 0$ for $i \geq 2$, while $I^{(0)} = \mathrm{Id}$ is the inclusion, and $I^{(-1)} = (1-t)\tilde{s}sN$. Here the standard notation is as in § 2. The degeneracy operators $\tilde{s}$, s are given by $s(a^0, \ldots, a^n) = (1, a^0, \ldots, a^n)$ and $\tilde{s}(a^0, \ldots, a^n) = (\tilde{1}, a^0, \ldots, a^n)$.*

PROOF. Consider the following commuting diagram of d.g. complex maps:

$$\begin{array}{ccc} \mathrm{Tot}\, C_{\cdot\,\cdot}(A) & \xrightarrow{I_*} & \mathrm{Tot}\, C_{\cdot\,\cdot}(\tilde{A}) \\ \phi\downarrow & & \downarrow\psi \\ B(A) & \xrightarrow{I^\#} & B(\tilde{A}) \end{array}$$

where the inclusion I_* is a chain map, because not all the differentials in $C_{\cdot\,\cdot}(A)$ and $C_{\cdot\,\cdot}(\tilde{A})$ involve the degeneracy map. Recall the Loday-Quillen map ϕ and the quasi-inverse ψ we constructed in Theorem 2.14. We get

$$\begin{aligned} \psi \circ I_* \circ \phi(u^n \otimes m_\ell) &= \psi \circ I_*(m_\ell \oplus sNm_\ell)_{2n+\ell} \\ &= \psi(m_\ell \oplus sNm_\ell)_{2n+\ell} \\ &= u^n \otimes m_\ell + u^n \otimes (1-t)\tilde{s}sNm_\ell, \end{aligned}$$

where $m_\ell \in A^{\otimes(\ell+1)}$ and $(x)_{2n+\ell}$ indicates that the total degree of $x \in \mathrm{Tot}\, C_{\cdot\,\cdot}$ is $2n+\ell$.

One can check $[b, I^{(0)}] = 0$, $[B, I^{(0)}] + [b, I^{(-1)}] = 0$ and $[B, I^{(-1)}] = 0$, so $(I^{(i)})_{i=0,-1,\ldots}$ defines a cocycle. Note that the inclusion $I^{(0)}$ does not commute with B; however

$$\begin{aligned} [b, I^{(-1)}] &= b(1-t)\tilde{s}sN - (1-t)\tilde{s}sNb \\ &= (1-t)(b'\tilde{s}s - \tilde{s}sb')N \\ &= (1-t)(s - \tilde{s}b's - \tilde{s}sb')N \\ &= (1-t)(s - \tilde{s})N \\ &= B - \tilde{B} \\ &= -[B, I^{(0)}]. \end{aligned}$$

■

There is a parallel statement to Theorem 3.4 concerning $\mathrm{Hom}_S\big(B(A), B(E)\big)$. It requires a separate proof. First we notice that if we replace A' by E in (3.7), we obtain easily that $\alpha^{(-i)} = \pi^{(0)} \circ \tilde{\alpha}^{(0)} \circ I^{(-i)}$, for $i = 0, 1, \ldots$. Thus

$$(\alpha^{(-i)} - \bar{\alpha}^{(-i)})(a^0, \ldots, a^n) = \sum \lambda_k^i \pi^{(0)} \circ (\tilde{\alpha}^{(0)} - \tilde{\bar{\alpha}}^{(0)})(b_{0,i}^K, \ldots, b_{n+2i,i}^K);$$

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in the sum in every tensor each a_ℓ, $\ell = 0, \dots n$, appears exactly once as $b^K_{j,i}$, and the rest either are 1 or $\tilde{1}$. Of course, Proposition 3.6 says that the coefficients λ^i_K are 0 if $i \geq 2$. Every term can be expressed as a sum of $(n+2i)$ differences, each of them an $(n+2i)$ tensor with at least one factor being either $h(qa_\ell)$ or $h(q1)$ (see Lemma 3.2). We have obtained:

LEMMA 3.7. *The images of $h^{(i)}$, $i = 0, -1, \dots$, are contained in the generalized cyclic complex $K_*(E)$. Thus, an I-summable quasihomomorphism from A to $A' \otimes K$ defines a unique element* $[h] \in \mathrm{HC}^0\big(A, K_*(E)\big)$.

In conclusion, we have the main theorem of [W1].

THEOREM 3.8. *Let A, A′ be unital k-algebras. Let h be an I-summable quasihomomorphisms from A to $A'\tilde{\otimes}K$. Then h defines an element* $\mathrm{ch}(h) = [h] \in \mathrm{HC}^0\big(A, K_*(E)\big)$, *which is represented by either of*

(i) a cycle $h^\#$ in $\mathrm{Hom}_S\big(C..(A), K..(E)\big)^{0,0}$, given by Corollary 3.4, Theorem 3.5;

(ii) a cycle $h^\#$ in $\mathrm{Hom}_S\big(B(A), B(K_*(E))\big)^0$, ($K_*(E)$ is H-unital), given by the family of strong higher homotopy maps $(h^{-i})_{i=0,1,2,\dots}$, $h^{-i}\colon A^\ell \longrightarrow K_{\ell+2i}(E), \cdot \ell \geq 0$, such that $h^{(-i)} = 0$ for $i \geq 2$, while

(a) $h^0(a^0) = h(qa^0)$, $h^0(a^0, \dots, a^n) = \big(h(qa^0), \alpha(a^1), \dots, \alpha(a^n)\big) + \big(\alpha(a^0), h^0(a^1, \dots, a^n)\big) - \big(h(qa^0), h^{(0)}(a^1, \dots, a^n)\big)$.

(b) $h^1 = \pi^{(0)} \circ \tilde{h}^0 \circ (1-t)\tilde{s}sN$ *($\tilde{h}^0$ is the "unitalization" of h^0).*

COROLLARY 3.9 (THEOREM 1, [W1]). *Suppose that h is a 1-summable quasihomomorphism from A to $A'\tilde{\otimes}K$. Then there is a canonical element* $[\mathrm{Tr}^\#] \in \mathrm{HC}^0\big(K_*(E), A'\big)$ *which is an "HC-equivalence". $\mathrm{Tr}^\#\colon K_\ell(E) \longrightarrow A'^{\otimes(\ell+1)}$ is just given by the continuous extension on tensor products of $M_n(A)$ of the ordinary trace.*

Let $\mathrm{ch}^0(h) = [\mathrm{Tr}^\# \circ h^\#]$, with $h^\#$ given in Theorem 3.9 (i) or (ii). Then $\mathrm{ch}^0(h)$ is a well defined 0-cycle in $\mathrm{HC}^0(A, A')$.

Now we return to general I-summable quasihomomorphisms.

The inclusion $I \overset{i}{\hookrightarrow} E$ induces an inclusion of generalized cyclic complexes $i_\#$: $C_*(I) \hookrightarrow K_*(E)$ and, further,

$$\tilde{\imath}_\#\colon \mathrm{Hom}_S\big(\mathrm{Tot}\, C_*(A), \mathrm{Tot}\, C_*(I)\big) \longrightarrow \mathrm{Hom}_S\big(\mathrm{Tot}\, C_*(A), \mathrm{Tot}\, K_*(E)\big)$$

induces a group homomorphism $i_*\colon \mathrm{HC}^*(A, I) \longrightarrow \mathrm{HC}^*\big(A, K_*(E)\big)$. A natural question is: when is $\tilde{\imath}_\#$ a quasi-isomorphism?

The necessary and sufficient condition for $i_\#$ to be a quasi-isomorphism was found by Wodzicki ([Wod1], [Wod2]). It is that I be *H-unital* (Definition 3.10). The filtration $\{F_k(E)\}_{k=-1,0,1\dots}$ of $K_*(E)$ defines a filtration $\{F_k \mathrm{Hom}_S(A, E) := \mathrm{Hom}_S\big(\mathrm{Tot}\, C_*(A), \mathrm{Tot}\, F_k(E)\big)\}_{k=-1,0,1,\dots}$ of $\mathrm{Hom}_S\big(\mathrm{Tot}\, C_*(A), \mathrm{Tot}\, K_*(E)\big)$, because the inclusion $F_{k-1}(E) \hookrightarrow F_k(E)$ induces an inclusion

$$\mathrm{Hom}_S\big(\mathrm{Tot}\, C_*(A), \mathrm{Tot}\, F_{k-1}(E)\big) \hookrightarrow \mathrm{Hom}_S\big(\mathrm{Tot}\, C_*(A), \mathrm{Tot}\, F_k(E)\big),$$

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and the functor Hom_S commutes with direct limit:

$$\bigcup_k \mathrm{Hom}_S\big(\mathrm{Tot}\, C_*(A), \mathrm{Tot}\, F_k(E)\big) = \mathrm{Hom}_S\big(\mathrm{Tot}\, C_*(A), \mathrm{Tot}\, K_*(E)\big).$$

Wodzicki's notion of H-unitality can be generalized to categories with general admissable topological tensor products. Let k be a unital topological ring and A a k-algebra, not necessarily unital. Write $\tilde{A} = A \oplus k$. Given an admissible topological tensor product $\tilde{\otimes}$, the reduced torsion group $\overline{\mathrm{Tor}}_*^{(\tilde{A},\tilde{\otimes})}(k,k)$ is the homology group of the reduced bar resolution $\big(\bar{C}_*(A), b'\big)$ with respect to the topological tensor product $\tilde{\otimes}$, where $\bar{C}_0(A) = 0$, and $\bar{C}_n(A) = A^{\tilde{\otimes} n}$, $n \geq 0$.

DEFINITION 3.10 (SEE [WOD1]). A k-algebra A is called *H-unital with respect to* $\tilde{\otimes}$, if $\overline{\mathrm{Tor}}_*^{(\tilde{A},\tilde{\otimes})}(k,k) = 0$. We may simply say that $(A, \tilde{\otimes})$ is H-unital.

A is said to be *algebraically H-unital*, if $(A, \otimes)$, is H-unital where $\otimes$ is the algebraic tensor product. We simply say A is *H-unital*.

REMARK 3.11. If A is an H-unital k-algebra, then $\big(C_*(A), b'\big)$ is acyclic. Since A is k-projective, the complex $\big(C_*(A), b'\big)$ is contractible (Theorem 5, p. 164 [Span]). Thus there is homotopy s of degree 1 such that $sb' + b's = 1$. So one can still define $B = (1-t)sN$ on $C_*(A)$ and make $B(M)$ a $B(\Lambda)$-comodule.

PROPOSITION 3.12 ([WOD1], PROPOSITION 1). *(i) If there is a right or left A-linear splitting of the multiplication map $A\tilde{\otimes}A \longrightarrow A$, then A is H-unital with respect to $\tilde{\otimes}$.*

(ii) If, for any finite number $a_1, \ldots, a_m$ of elements A there is $e \in A$ such that $a_1 = ea_1, \ldots, a_m = ea_m$, or $a_1 = a_1e, \ldots, a_m = a_me$, then A is algebraically H-unital.

COROLLARY 3.13. *If a locally convex ($\mathbb{C}$ or $\mathbb{R}$) algebra A is algebraically H-unital then A is also H-unital for any complete topological tensor product $\tilde{\otimes}$.*

REMARK 3.14. Wodizicki has given a variety of examples of H-unital topological algebras in §§4-8, [Wod2]. For any C^∞-manifold X and closed subset Y, the algebra $C^\infty(X,Y)$ of smooth functions "flat" (i.e. vanishing with all derivatives) along Y is H-unital. In particular, the algebra of Schwartz functions $\mathcal{S}(\mathbb{R}^n)$ is H-unital. The algebra $\mathcal{D}(X,Y)$ of smooth differential operators "flat" along Y is H-unital. On a regular affine variety V, every (one-sided) ideal in the algebra $\mathcal{D}^{\mathrm{alg}}(V)$ of algebraic differential operators is H-unital. Every Banach algebra A with left or right bounded approximate unit is algebraically H-unital.

EXAMPLE 3.15. The algebra of nuclear operators in a Banach space with the approximation property is H-unital [Wod2]. In particular, the algebra of trace class operators $L^1(H)$ is H-unital with respect to the projective tensor product $\hat{\otimes}$. The algebra of compact operators $L^\infty(H) = \mathcal{K}(H)$ on a Hilbert space is H-unital. For any complete locally convex topological algebra B, the projective tensor products $B\hat{\otimes}L^1(H)$ and $B\hat{\otimes}\mathcal{K}(H)$ are H-unital with respect to $\hat{\otimes}$.

The Schatten ideal $L^p(H)$ are not H-unital for $1 < p < \infty$ (see §5).

H-unital algebras are characterized by the excision property:

THEOREM 3.16 (THEOREM 3, [WOD1]). *For any algebra A over a field k of characteristic zero, the following two conditions are equivalent:*

(a) every extension of k-algebras $0 \longrightarrow A \longrightarrow E \longrightarrow R \longrightarrow 0$ *induces the long exact sequence in cyclic homology*

$$\cdots \longrightarrow \mathrm{HC}_q(E) \longrightarrow \mathrm{HC}_q(R) \xrightarrow{\partial} \mathrm{HC}_{q-1}(A) \longrightarrow \mathrm{HC}_{q-1}(E) \longrightarrow \cdots;$$

(b) A is H-unital.

Let $\mathrm{Tot}\, K_*$ be the chain complex defined by the short exact sequence

$$0 \longrightarrow \mathrm{Tot}\, K_* \longrightarrow \mathrm{Tot}\, C_{\cdot\,\cdot}(E) \longrightarrow \mathrm{Tot}\, C_{\cdot\,\cdot}(R) \longrightarrow 0.$$

REMARK 3.17. The chain complex $\mathrm{Tot}\, K_*$ is identical to $\mathrm{Tot}(K_{\cdot\,\cdot})$, where $K_{\cdot\,\cdot}$ is the bicomplex associated to the generalized cyclic complex K_*; (see Definition 2.3).

THEOREM 3.18. *Let k be a field of characteristic zero. An algebra* $(I, \tilde{\otimes})$ *is H-unital if and only if for every extension* $0 \longrightarrow I \longrightarrow E \longrightarrow R \longrightarrow 0$ *of k-algebras the natural inclusion* $\mathrm{Tot}\, C_{\cdot\,\cdot}(I) \hookrightarrow \mathrm{Tot}\, K_*$ *is a quasi-isomorphism.*

PROOF. We define a filtration $F_p \,\mathrm{Tot}\, K_*$ as follows:

$$F_{-1}\,\mathrm{Tot}\, K_* = 0,\ F_0\,\mathrm{Tot}\, K_* = \mathrm{Tot}\, C_{\cdot\,\cdot}(I), \ldots$$
$$F_k\,\mathrm{Tot}\, K_* = \mathrm{Tot}(F_k K)_{\cdot\,\cdot}, \text{ where } (F_p K)_{m,n} = (F_p K)_n, \quad m, n \geq 0,\ p \geq 1.$$

We notice

$$\begin{aligned} E^0_{p,q}(F\,\mathrm{Tot}\, K_*) &\simeq \big(\mathrm{Tot}(F_p K)_{\cdot\,\cdot}\big)_{p+q} \big/ \big(\mathrm{Tot}(F_{p-1} K)_{\cdot\,\cdot}\big)_{p+q} \\ &\simeq [(F_p K_*)_0 + \cdots + (F_p K_*)_{p+q}] \big/ [F_{p-1} K_*)_0 + \cdots + (F_{p-1} K_*)_{p+q}] \\ &\simeq (F_p K_* / F_{p-1} K_*)_0 + \cdots + (F_p K_* / F_{p-1} K_*)_{p+q} \\ &\simeq \bigoplus_{0 \leq k \leq q} E^0_{p,k}(FK_*). \end{aligned}$$

By Corollary 2 of [Wod1], if A is H-unital, the $E^0_{p,*}(FK_*)$ is acyclic. Wodzicki's proof shows that $E^0_{p,*}(F\,\mathrm{Tot}\, K_*)$ is acyclic if $p \geq 1$. Thus the spectral sequence $E^r_{p,*}$ collapses at the E^1-term and we have

$$E^1_{p,*}(F\,\mathrm{Tot}\, K_*) \simeq E^\infty_{p,*}(F\,\mathrm{Tot}\, K_*), \text{ for } p \geq 1. \qquad \blacksquare$$

LEMMA 3.19. *Let* $P, Q \overset{i}{\hookrightarrow} Q'$ *be d.g. S-modules, where the inclusion is a quasi-isomorphism. Then the induced map*

$$\mathrm{Hom}_S(P, Q) \overset{i_*}{\hookrightarrow} \mathrm{Hom}_S(P, Q')$$

is also a quasi-isomorphism.

PROOF. There is an isomorphism of d.g. S-modules:

$$\mathrm{Hom}_S(P, Q') / i_*\big(\mathrm{Hom}_S(P, Q)\big) \simeq \mathrm{Hom}_S(P, Q'/Q).$$

Since $Q \overset{i}{\hookrightarrow} Q'$ is a quasi-isomorphism, the above d.g. module is acyclic. Thus in the long exact sequence

$$\cdots \longrightarrow H_n\big(\mathrm{Hom}_S(P,Q)\big) \xrightarrow{i_*} H_n\big(\mathrm{Hom}_S(P,Q')\big) \longrightarrow \cdots$$

the induced maps i_* is an isomorphism. ∎

As a consequence we have

THEOREM 3.20. *If I is H-unital, then the natural inclusion induces isomorphisms*

$$\mathrm{HC}^*(A,I) \simeq \mathrm{HC}^*(A,F_1K_*) \simeq \mathrm{HC}^*(A,F_2K_*) \simeq \cdots \simeq \mathrm{HC}^*(A,K_*).$$

DEFINITION 3.21. Suppose that $h = (\alpha,\bar{\alpha})\colon A \overrightarrow{\rightarrow} M(A'\tilde{\otimes}K)$ is an I-summable quasi-homomorphism from A to $A'\tilde{\otimes}K$, such that I is H-unital with respect to $\tilde{\otimes}$. Let $[h^\#] \in \mathrm{HC}^0(A,K_*)$ be the class of h defined in Theorem 3.5. The the Chern character of h in $\mathrm{HC}^{2m}(A,I)$, $m = 0,1,2,\ldots$, is defined to be the image of $S^m[h^\#]$ under the natural isomorphism from $\mathrm{HC}^*(A,K_*)$ to $\mathrm{HC}^*(A,I)$, denoted by $\mathrm{ch}^{2m}(h)$.

REMARK 3.22. Consider the universal quasihomomorphism

$$A \underset{\bar{i}}{\overset{i}{\rightrightarrows}} A * A \rhd qA,$$

which we denote by $q\colon a \longmapsto qa = i(a) - \bar{i}(a)$. Suppose that Cuntz's algebra qA (completed or algebraic) is H-unital. The the exact sequence ($QA = A*A$ is completed or algebraic according to qA)

$$0 \longrightarrow qA \longrightarrow QA \longrightarrow A \longrightarrow 0$$

has the excision property. Let $K_*(QA)$ be the kernel of $C_*(QA) \longrightarrow C_*(A)$. Then $\mathrm{HC}^*\big(A, K_*(QA)\big)$ and $\mathrm{HC}^*(A,qA)$ are isomorphic. The universal quasihomorphism q defines an element $[q^\#] \in \mathrm{HC}^0(A,K_*)$, and thus an element $\mathrm{ch}^m(q) \in \mathrm{HC}^{2m}(A,qA)$ under this isomorphism. For any I-summable quasihomomorphism $h\colon qA \longrightarrow I$, we may define the Chern character $\mathrm{ch}^{2m}(h) = [h^\#] \circ \mathrm{ch}^{2m}(q) \in \mathrm{HC}^{2m}(A,I)$. For instance, when A is a subalgebra of a C^*-algebra, qA the C^*-algebraic completion (of qA), then qA is H-unital. Thus for any I-summable quasihomomorphism h from A to $A' \otimes K$, the Chern character $\mathrm{ch}_0(h) \in \mathrm{HC}^0(A,\bar{I})$ is defined.

If, in addition, I is also H-unital, this definition clearly coincides with Definition 3.21.

This definition does not offer less restriction than Definition 3.21. The algebraic version of qA is actually never H-unital (for instance, $q1$ cannot be written as finite linear combinations of products of elements in qA). When we complete it with a certain locally convex topology so that $\overline{qA}$ becomes H-unital, we have to complete I accordingly so that the Cuntz homomorphism $\overline{qA} \longrightarrow \bar{I}$ is defined. Such a completion usually makes $\bar{I}$ H-unital already.

4. *I*-summable extensions and the bivariant Chern character—the odd case. Now we turn our attention to the construction of a bivariant Chern character for the odd dimensional Kasparov group KK^1, or rather, generators of "KK^1-type" invariants. Given the instruction in § 3 of a Chern character for generators of $KK^0(A, A')$, we need only replace A or A' by its suspension algebras ΣA or $\Sigma A'$. Then use the KK-periodicity, $KK^1(A, A') \simeq KK^0(\Sigma A, A') \simeq KK^0(A, \Sigma A')$, and we get the corresponding construction. This is precisely the idea behind the first definition of the Chern character (Definition 4.4, also in [W2]).

A natural question arises here: what is a suspension of an algebra? Just like the question of which topological tensor product to choose (see § 0), the answer depends on what category of algebras we are dealing with. For a given category of algebras, the suspension may not be unique. In the category of C^*-algebras, according to Kasparov (Theorem 7, p. 547, [Kasp2]), $C_0(\mathbb{R})\hat{\otimes}A$ plays the role of ΣA. For other categories of locally convex topological algebras, primarily dense smooth subslgebras of C^*-algebras, we replace $C_0(\mathbb{R})$ by its various dense subalgebras, e.g., $C_0^\infty(\mathbb{R})$, $C_c^\infty(\mathbb{R})$, $C_0^{\omega}(\mathbb{R})$, Schwartz functions $\mathcal{S}(\mathbb{R})$, just to name a few. In the category of pure algebras, one may choose $\Sigma A = C \otimes A$, where C is the Karoubi-Villamayor algebra (p. 319, [Con1]), or choose $\Sigma A = k[t] \otimes A$, etc.

Due to the work of [B-D-F], [Kasp2], we know that for C^*-algebras, A, A', the Kasparov groups $KK^1(A, A')$ are generated by short exact sequences:

$$0 \longrightarrow A\hat{\otimes}K \longrightarrow E \longrightarrow A \longrightarrow 0. \tag{4.1}$$

(4.1) corresponds to a $*$-homomorphism (the *Busby invariant* [Bus]):

$$\tilde{\tau}_E: A \longrightarrow O(A'\tilde{\otimes}K) \equiv M(A'\tilde{\otimes}K)/A'\tilde{\otimes}K. \tag{4.2}$$

Thus the (nonseparable) Calkin algebra of A, $O(A'\tilde{\otimes}K)$, plays the role of a "K-homological universal suspension algebra" of A. For dense subalgebras of A, we may replace $\mathcal{K}$ by its dense ideals and change the minimal C^*-tensor product to other appropriate topological tensor products. For discrete, or "pure" algebras over a field k, replace $\mathcal{K}$ by $K = \varinjlim_n M_n(K)$ in (4.2). Recall that the same notation K stands for "the abstract algebra of compact operators" in the general sense.

PROPOSITION 4.1. *Let I be a dense ideal in $A'\tilde{\otimes}K$ and a closed ideal in E. Any I-summable extension E from A to $A'\tilde{\otimes}K$,*

$$0 \longrightarrow I \longrightarrow E \longrightarrow A \longrightarrow 0, \tag{4.3}$$

induces a Busby invariant

$$\tilde{\tau}_E: A \longrightarrow M(A'\tilde{\otimes}K)/A'\tilde{\otimes}K.$$

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PROOF. Given an I-summable extension E from A to $A'\tilde{\otimes}K$, there is a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \sigma_E} & & \downarrow{\scriptstyle \tau_E} & & \\ 0 & \longrightarrow & A\tilde{\otimes}K & \longrightarrow & M(A'\tilde{\otimes}K) & \longrightarrow & O(A'\tilde{\otimes}K) & \longrightarrow & 0 \end{array}$$

Since I is a dense ideal of $A\tilde{\otimes}K$, there is a canonical quotient map $M(I) \longrightarrow M(A\tilde{\otimes}K)$. Thus, the homomorphism $\pi_I: M(I)/I \longrightarrow O(A'\tilde{\otimes}K)$ is well defined and onto. Any I-summable extension from A to $A'\tilde{\otimes}K$ induces $\tau_E: A \longrightarrow M(I)/I$. We define $\tau_E = \tilde{\tau}_E \cdot \pi_I$. ∎

REMARK 4.2. In the category of C^*-algebras, if $K = \mathcal{K}(H)$ and $\tilde{\otimes}$ is the projective tensor product, the $\mathrm{HC}^*(A\hat{\otimes}\mathcal{K}) = 0$, $H^*(A\hat{\otimes}\mathcal{K}) = 0$ for any unital C^*-algebra A [Wod3]. The long exact sequences associated (since $A\hat{\otimes}\mathcal{K}$ is H-unital) to the universal short exact sequence

$$0 \longrightarrow A\hat{\otimes}\mathcal{K} \longrightarrow M(A\hat{\otimes}\mathcal{K}) \longrightarrow O(A\hat{\otimes}\mathcal{K}) \longrightarrow 0$$

then imply that

$$\mathrm{HC}^*\big(M(A\hat{\otimes}\mathcal{K})\big) \simeq \mathrm{HC}^*\big(O(A\hat{\otimes}\mathcal{K})\big)$$

and

$$H^*\big(M(A\hat{\otimes}\mathcal{K})\big) = H^*\big(O(A\hat{\otimes}\mathcal{K})\big).$$

This is not exactly what we want. In general, $\tilde{\otimes}$ is not $\hat{\otimes}$. We prefer the dense ideal I to $A'\tilde{\otimes}\mathcal{K}$, and the lift $\tilde{\tau}_E$ to τ_E. As a consequence of this preference we need to assume $(I, \tilde{\otimes})$ to be H-unital, and so this will be a standing assumption throughout this section.

Therefore the extension (4.1) defines an element $\mathrm{ch}_0(\tau_E) \in \mathrm{HC}^0\big(A, O(I)\big)$. We are looking for a natural transformation $\partial_I: \mathrm{HC}^0\big(A, O(I)\big) \longrightarrow \mathrm{HC}^1(A, I)$; then the *Chern character* $\mathrm{ch}^1(\tau_E)$ of (4.1) is $\partial_I\big(\mathrm{ch}^0(\tau_E)\big)$. The following fact is an easy extension of Theorem 3.15 to the bivariant situation.

THEOREM 4.3 (THEOREM 3.1 [KASS3]). *Suppose that* $0 \longrightarrow I \longrightarrow E \longrightarrow A \longrightarrow 0$ *is an extension such that I is H-unital. Then for any algebra B there are natural long exact sequences*

$$(4.5) \qquad \cdots \longrightarrow \mathrm{HC}^n(B, I) \longrightarrow \mathrm{HC}^n(B, E) \longrightarrow \mathrm{HC}^n(B, A) \xrightarrow{\partial_E} \mathrm{HC}^{n+1}(B, I) \longrightarrow \cdots$$

and

$$(4.6) \qquad \cdots \longrightarrow \mathrm{HC}^n(A, B) \longrightarrow \mathrm{HC}^n(E, B) \longrightarrow \mathrm{HC}^n(I, B) \xrightarrow{\partial'_E} \mathrm{HC}^{n+1}(A, B) \longrightarrow \cdots.$$

PROOF. Recall Example 2.4. There is a short-exact sequence of generalized cyclic complexes.

$$0 \longrightarrow K_* \longrightarrow C_*(E) \longrightarrow C_*(A) \longrightarrow 0,$$

which induces an exact sequence of d.g. S-modules

(4.7) $0 \longrightarrow \mathrm{Hom}_S\big(C_*(B), K_*\big) \longrightarrow \mathrm{Hom}_S\big(C_*(B), C_*(E)\big) \longrightarrow \mathrm{Hom}_S\big(C_*(B), C_*(A)\big) \longrightarrow 0.$

Now apply Theorem 3.20. The long exact sequence of (4.7) is (4.5). The proof of (4.6) is similar. ∎

DEFINITION 4.4. The *Chern character* of the I-summable extension (4.1) from A to $A'\tilde{\otimes}K$ is $\mathrm{ch}^{2m+1}(\tau_E) = S^m \circ \mathrm{ch}^1(\tau_E)$, (Proposition 2.2), $m = 0, 1, 2, \ldots$, with $\mathrm{ch}^1(\tau_E) = \partial_{M(I)}\big(\mathrm{ch}^0(\tau_E)\big)$.

We shall write $\partial_I = \partial_{M(I)}$, $\tau_I = \tau_{M(I)}$, etc. There is a classical concept which can be traced back to Cartan and Eilenberg (p. 226, [C-E]). Let Λ be any ring. Consider the category of d.g. Λ-modules. Let $\mathrm{id}_I \in \mathrm{Hom}_\Lambda(I, I) = \mathrm{Ext}^0_\Lambda(I, I)$ be the identity map. Given an exact sequence

$$0 \longrightarrow I \longrightarrow E_{q-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow A \longrightarrow 0 \tag{4.8}$$

of left Λ-modules, there is an iterated connecting homomorphism

$$\partial_q \colon \mathrm{Hom}_\Lambda(I, I) \longrightarrow \mathrm{Ext}^q_\Lambda(A, I).$$

The image $\partial_q[\mathrm{id}_I]$ is the *characteristic element* of the sequence (4.8). The bivariant cyclic theory $\mathrm{HC}^*(\cdot,\cdot)$ can be defined as the right derived functor Ext of the left exact functor Hom_Λ on the category of cyclic complexes [J-K] where $\Lambda = k[\varepsilon]$ is the algebra of dual numbers (see the beginning of § 2). So the notion of characteristic element makes sense.

DEFINITION 4.5 [KASS3]. Under the assumption of Theorem 4.3, the character $\mathrm{Ch}^1(E \longrightarrow A)$ of an extension (4.1) is $\partial_E(\mathrm{id}_A)$, (see (4.5)).

THEOREM 4.6 (THEOREM 3.1, [KASS3]). *In Theorem 4.3*

$$\partial_E(\mathrm{id}_A) = -\partial'_E(\mathrm{id}_I).$$

The reader may consult p. 308, XIV.1, [C-E] for the idea of a proof. Not surprisingly, the two characters defined in Definition 4.4 and Definition 4.5 coincide.

PROPOSITION 4.7. *Assume that* $(I, \tilde{\otimes})$ *is H-unital. Then* $\mathrm{ch}^1(\tau_E) = \mathrm{Ch}^1(E \longrightarrow A)$.

PROOF. Consider the commuting diagram (4.4).

Because of the naturality, there is a morphism between the two associated long exact sequences:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{HC}^0\big(A, O(I)\big) & \xrightarrow{\partial_I} & \mathrm{HC}^1(A, I) & \longrightarrow \cdots \\
& \uparrow \tau_{E*} & & \| & \\
\cdots \longrightarrow & \mathrm{HC}^0\big(A, A\big) & \xrightarrow[\partial_E]{} & \mathrm{HC}^1(A, I) & \longrightarrow \cdots .
\end{array}$$

Thus $\partial_I[\mathrm{ch}^0(\tau_E)] = \partial_I \circ \tau_{E*} \circ [\mathrm{id}_A] = \partial_E[\mathrm{id}_A]$. ∎

5. **Topological tensor products of L^p-operator spaces.** Given the results in § 3 and § 4, an inevitable question is to decide the H-unitality of $L^p(H)$ with respect to various tensor products. Recall (Proposition 5 and Example of [Wod1]) that $L^\infty(H) = \mathcal{K}(H)$ is H-unital with respect to all tensor products and $L^1(H)$ is H-unital with respect to the projective tensor product. Here it is shown that

(a) with respect to the algebraic tensor product, $L^p(H)$ is not H-unital for all $0 < p < \infty$.

(b) with respect to the projective tensor product, $L^p(H)$ is not H-unital for $1 < p < \infty$.

Then we generalize the study of operator spaces and their Haagerup tensor products (see [E] and its references) to L^p-operator spaces and Haagerup L^p-tensor products $\otimes_{hp}$ for all $1 < p \leq \infty$. We show that $L^p(H)$ is H-unital with respect to $\otimes_{hp}$ for all $1 \leq p \leq \infty$, although $\otimes_{hp}$ is not admissible. This result is sharp because it is then shown that $L^p(H)$ is not H-unital with respect to any admissible tensor product.

In this section all algebras and vector spaces are over $\mathbb{C}$ or $\mathbb{R}$. "Unitary" is understood as "orthogonal" when $k = \mathbb{R}$.

Let V_1 and V_2 be Banach spaces. Recall [Gro] that for $x \in V_1 \otimes V_2$, the projective tensor norm is

$$\|x\| = \inf \sum \|a_i\|_1 \|b_i\|_2$$

where the infimum is taken over all finite sums $x = \sum a_i \otimes b_i$. It defines the finest topology such that the canonical bilinear map $V_1 \times V_2 \longrightarrow V_1 \otimes V_2$ is jointly continuous.

PROPOSITION 5.1. *For $0 < p < \infty$, the Schatten class $L^p(H)$ is not H-unital, with respect to the algebraic tensor product. For $1 < p < \infty$, the Schatten class $L^p(H)$ is not H-unital even with respect to the projective tensor product.*

PROOF. For any $T, S \in L^p(H)$, by the Hölder-Horn inequality (p. 10, Chapter 2, [Gro]), one has $\|TS\|_{p/2} \leq \|T\|_p \|S\|_p$. Thus the multiplication $b'\colon L^p(H) \otimes L^p(H) \longrightarrow L^p(H)$ has its image contained in $L^{p/2}(H)$. Therefore, with respect to the algebraic tensor product, $L^p(H)$ is not H-unital for all $0 < p < \infty$.

Now consider the projective tensor product. Any element $x \in L^p(H) \hat{\otimes} L^p(H)$ has an expression $x = \sum_i T_i \hat{\otimes} S_i$ such that $\sum_i \|T_i\|_p \|S_i\|_p < \infty$. Thus, $\sum_i \|T_i S_i\|_{p/2} \leq \sum_i \|T_i\|_p \|S_i\|_p < \infty$. There are two cases:

(a) If $2 \leq p < \infty$, then $\| \sum_i T_i S_i\|_{p/2} \leq \sum_i \|T_i S_i\|_{p/2} < \infty$. Again, image$(b') \subset L^{p/2}(H)$.

(b) If $1 < p < 2$, then

$$\| \sum_i T_i S_i\|_1 \leq \sum_i \|T_i S_i\|_1 \leq \sum_i \|T_i S_i\|_{p/2} < \infty.$$

Thus, image$(b') \subset L^1(H) \subseteq L^p(H)$.

In both cases (a) and (b), it follows that $L^p(H)$ is not H-unital with respect to the projective tensor product $\hat{\otimes}$. ■

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Recall [E] that a vector space V is called a *matricially normed space* if norms are provided on the matrix spaces $M_n(V)$, $n \in N$, and for any $\alpha \in M_N(\mathbb{C})$ and $v \in M_n(V)$, one has

$$(5.1) \qquad \|\alpha v\| \leq \|\alpha\|\|v\|, \quad \|v\alpha\| \leq \|\alpha\|\|v\|,$$

where $M_n(\mathbb{C})$ is equipped with usual operator norm. For $1 < p \leq \infty$, if in addition

$$(5.2) \qquad \|v \oplus w\| \leq (\|v\|^p + \|w\|^p)^{1/p}$$

for any $v \in M_n(V)$ and $w \in M_n(W)$, then V is called an *L^p-matrically normed space*. When $p = \infty$, (5.2) is replaced by its limit,

$$(5.3) \qquad \|v \oplus w\| \leq \max(\|v\|, \|w\|).$$

A linear subspace of $B(H)$ is called a (*matricial*) *operator space*. Analogously, we shall call a linear subspace of $L^p(H)$ a (*matricial*) *L^p-operator space*, for $1 \leq p \leq \infty$. We always identify $L^\infty(H)$ with $B(H)$. Let $M_n\big(B(H)\big)$ act on $H \oplus \cdots \oplus H$ (n-times). Then in an obvious way L^p-operator spaces are L^p-matricially normed spaces. Ruan has shown that any L^∞-matricially normed space is completely isometric to an operator space [R]. It is very probable that the same statement holds for all $1 \leq p \leq \infty$, i.e., an L^p-matricially normed space is completely isometric to an L^p-operator space.

The Haagerup tensor product for operator spaces is introduced by Effros and Kishimoto ([E-K], [E]). Now we extend naturally their definition to all L^p-operator spaces. Let $V_1, \ldots, V_m$ be L^p-operator spaces. Let $v_i \in M_{n_in_{i+1}}(V_i)$, $i = 1, \ldots, n-1$, with $n_1 = n_m = n$. Following Effros, we denote $v_1 \odot \cdots \odot v_m$ in $M_n(V_1 \otimes \cdots \otimes V_m)$ by

$$(5.4) \qquad (v_1 \odot \cdots \odot v_m)(i,j) = \sum_{k,\ell,\ldots,r} v_1(i,k) \otimes v_2(k,\ell) \otimes \cdots \otimes v_m(r,j).$$

DEFINITION 5.2. Let $V_1, \ldots, V_m$ be L^p-operator spaces, $1 \leq p \leq \infty$. If $y \in M_n(V_1 \otimes \cdots \otimes V_m)$, then the (*weak*) Haagerup L^p-tensor norm $\|\ \|_{hp}$ is defined by

$$(5.5) \qquad \|y\|_{hp} = \inf\{\sum_i \|v_{1j}\|_{2p} \cdots \|v_{mj}\|_{2p} : y = \sum v_{1j} \odot \cdots \odot v_{mj}\}.$$

where $v_{ij} \in M_{n_in_{i+1}}(V_i)$, n_i varies, but $n_1 = n_m = n$.

The *strong Haagerup L^p-tensor norm* $\|\ \|_{shp}$ is also obtained from (5.5) after replacing all $\|\ \|_{2p}$ by $\|\ \|_p$. We introduce $\|\ \|_{shp}$ only for comparison.

When taking $\|\ \|_p$ of a rectangular matrix, we add appropriate rows or columns of zeros to make it a square matrix and use the matricial norm.

Clearly both $\|\ \|_{hp}$ and $\|\ \|_{shp}$ are seminorms. When $p = \infty$, the two coincide and reduce to the usual Haagerup norm $\|\ \|_h$ for operator spaces of [E-K]. Since $\|\ \|_{shp} \geq \|\ \|_{hp} \geq \|\ \|_h$, both seminorms have zero kernel and are actually Banach norms. The completed tensor products with the corresponding norms are called the (*weak*) *Haagerup L^p-tensor product* and the *strong Haagerup L^p-tensor product*, denoted by $\otimes_{hp}$ and $\otimes_{shp}$

respectively. It is easly to see that the projective tensor norm $\| \ \|_{\wedge}$ is at least $\| \ \|_{shp}$. Thus the canonical multilinear map from $V_1 \times \cdots \times V_m$ into $L^p \otimes_{hp} \cdots \otimes_{hp} L^p$ (m-times) and $L^p \otimes_{shp} \cdots \otimes_{shp} L^p$ (m-times) are jointly continuous.

In general, for $S, T \in L^p(H)$, if $1 \leq p < \infty$, then

$$\|S \otimes T\|_{hp} \leq \|SS^*\|_p^{1/2} \|T^*T\|_p^{1/2} < \|S\|_p \|T\|_p = \|S \otimes T\|_{shp},$$

so $\| \ \|_{shp}$ is a cross-norm and $\| \ \|_{hp}$ is not.

When $m = 2$, the definition (5.5) can be simplified.

PROPOSITION 5.3. *Let V_1, V_2 be L^p-operator spaces. Let $y \in M_n(V_1 \otimes V_2)$, and*

$$\|y\|'_{hp} = \inf\{ \|v_1\|_{2p} \|v_2\|_{2p} : y = v_1 \odot v_2 \}, \tag{5.6}$$

where $v_1 \in M_{n,k}(V_1)$ and $v_2 \in M_{k,n}(V_2)$. Then

(a) $\| \ \|'_{hp}$ is a submultiplicative norm;

(b) $\|y\|'_{hp} = \|y\|_{hp}$.

PROOF. (a) We shall verify the two inequalities

$$\|y_1 + y_2\|'_{hp} \leq \|y_1\|'_{hp} + \|y_2\|'_{hp} \tag{5.7}$$

and if both V_1 and V_2 are algebras,

$$\|y_1 y_2\|'_{hp} \leq \|y_1\|'_{hp} \|y_2\|'_{hp}. \tag{5.8}$$

To check (5.7), let $y_i = u_i \odot v_i$, $u_i \in M_{n,n_i}(V_1)$, $v_i \in M_{n_i,n}(V_2)$, $i = 1, 2$, be such that

$$\|y_i\|'_{hp} \geq \|u_i\|_{2p} \|v_i\|_{2p} - \varepsilon, \quad i = 1, 2.$$

We may rescale u_i and v_i so that $\|u_i\|_{2p} = \|v_i\|_{2p}$, $i = 1, 2$. Since $y_1 + y_2 = (u_1, u_2) \odot \binom{v_1}{v_2}$,

$$\begin{aligned}\|y_1 + y_2\|'_{hp} &= \|(u_1, u_2)\|_{2p} \left\| \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\|_{2p} \leq \|u_1 u_1^* + u_2 u_2^*\|_p^{1/2} \|v_1^* v_1 + v_2^* v_2\|_p^{1/2} \\ &\leq \frac{1}{2}(\|u_1 u_1^* + u_2 u_1^*\|_p + \|v_1^* v_1 + v_2^* v_2\|_p) \\ &\leq \frac{1}{2}(\|u_1 u_1^*\|_p + \|v_1^* v_1\|_p) + \frac{1}{2}(\|u_2^* u_2\|_p + \|v_2^* v_2\|_p) \\ &\leq \|y_1\|'_{hp} + \|y_2\|'_{hp} + 2\varepsilon.\end{aligned}$$

To verify (5.8), it is enough to check that, in $M_n\big(L^p(H)\big)$,

$$\|u_1 u_2 u_2^* u_1^*\|_p \leq \|u_1 u_1^*\|_p \|u_2 u_2^*\|_p. \tag{5.9}$$

Applying Theorem 2.7 of [Sim], we have

$$\|u_1 u_2 u_2^* u_1^*\|_p \leq \|u_1\|_\infty^2 \|u_2 u_2^*\|_p. \tag{5.10}$$

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Since $\|u_1\|_\infty \leq \|u_1\|_p$, the inequality (5.9) follows.

(b) It is enough to check that

$$\|y\|_{hp} \geq \|y\|'_{hp}. \tag{5.11}$$

Write $y = \sum_1^n u_j \odot v_j$ with

$$\|y\|_{hp} \geq \sum_1^n \|u_j\|_{2p}\|v_j\|_{2p} - \varepsilon.$$

Since

$$\begin{aligned}\sum_1^n \|u_j\|_{2p}\|v_j\|_{2p} &\geq \sum_1^n \|u_j \odot v_j\|'_{hp} \\ &\geq \|\sum_1^n u_j \odot v_j\|'_{hp} = \|y\|'_{hp}, \qquad \text{(by (a))}\end{aligned}$$

the inequality (5.11) is immediate. ∎

PROPOSITION 5.4. *The acyclic Hochschild boundary $b'\colon L^p(H) \otimes_{shp} L^p(H) \longrightarrow L^p(H)$ is a contraction, with the image contained in $L^{p/2}(H)$ if $p \geq 2$ and in $L^1(H)$ if $1 < p \leq 2$.*

PROOF. We may assume that

$$x = \sum s_{ij} \otimes t_{ij} = \sum S_i \odot T_i,$$

where

$$S_i = (s_{i1}, \ldots, s_{in_i}), \quad T_i = (t_{i1}, \ldots, t_{in_i})^T.$$

Then, if $p \geq 2$,

$$\|b'(x)\|_{p/2} = \|\sum_i \Big(\sum_j s_{ij}t_{ij}\Big)\|_{p/2} \leq \sum_i \|S_iT_i\|_{p/2} \leq \sum_i \|S_i\|_p\|T_i\|_p < \infty.$$

Similarly, if $1 \leq p < 2$, then

$$\|b'(x)\|_1 \leq \sum_i |\operatorname{Tr}(S_iT_i)| \leq \sum_1 \|S_i\|_2\|T_i\|_2 < \infty.$$ ∎

Write $L^p(H)^{\otimes_{hp}n} = L^p(H) \otimes_{hp} \cdots \otimes_{hp} L^p(H)$ (n-times).

THEOREM 5.5. *Let $n \geq 1$. The linear map*

$$b_i\colon L^p(H)^{\otimes_{hp}(n+1)} \longrightarrow L^p(H)^{\otimes_{hp}n}, \quad i = 1, \ldots, n,$$

defined by multiplying the i-th tensor factor to the $(i+1)$-st factor is well defined, onto and completely contractive. With respect to the Haagerup L^p-tensor product, $L^p(H)$ is H-unital.

PROOF. Let $y \in M_n\big(L^p(H)^{\otimes m}\big)$ and $\|y\|_{whp} < 1$. Then $y = \sum_j v_{1j} \odot v_{2j} \odot \cdots \odot v_{mj}$ with $\sum_j \|v_{1j}\|_{2p} \cdots \|v_{mj}\|_{2p} < 1$. Since $b_i(y) \in M_n\big(L^p(H)^{\otimes(n-1)}\big)$ with

$$\begin{aligned}\|b_i(y)\|_{hp} &\leq \sum_j \|v_{1j}\|_{2p} \cdots \|v_{ij}v_{i+1,j}\|_{2p} \cdots \|v_{mj}\| \\ &\leq \sum_j \|v_{1j}\|_{2p} \cdots \|v_{ij}\|_{2p}\|v_{i+1,j}\|_{2p} \cdots \|v_{mj}\| < 1,\end{aligned}$$

we see that b_i is completely contractive.

To show that $\left(L^p(H), \otimes_{hp}\right)$ is H-unital, we use Wodzicki's criterion (Proposition 3.1.2). There is a linear splitting Γ of the multiplication μ

$$M_\infty(\mathbb{C}) \otimes M_\infty(\mathbb{C}) \underset{\Gamma}{\overset{\mu}{\rightleftarrows}} M_\infty(\mathbb{C}). \tag{5.12}$$

If we write $e_{ij} = \langle e_j, \cdot \rangle e_i$, then

$$\Gamma\Big(\sum_{i,j} x_{ij} e_{ij}\Big) = \sum_{i,j} x_{ij} e_{i1} \otimes e_{1j}.$$

We claim that Γ is a $M_\infty(\mathbb{C})$-bimodule map, i.e., for any $P, Q, x \in M_\infty(\mathbb{C})$, one has $\Gamma(PxQ) = P\Gamma(x)Q$. Let $P = (p_{ij})$, $Q = (q_{ij})$, and $x = (x_{ij})$. Then

$$Pe_{ij} = \sum_k P_{ki} e_{kj}, \quad e_{ij}Q = \sum_k q_{jk} e_{ik}$$

for all $i \geq j \geq 1$. Thus

$$\Gamma(Pe_{ij}) = \sum_k p_{ki}\Gamma(e_{kj}) = \sum_k p_{ki} e_{k1} \otimes e_{1j} = (Pe_{i1}) \otimes e_{1j} = P\Gamma(e_{ij}) \tag{5.13}$$

and

$$\Gamma(e_{ij}Q) = \sum_k q_{jk}\Gamma(e_{ik}) = \sum_k q_{jk} e_{i1} \otimes e_{1k} = Pe_{i1} \otimes (e_{1j}Q) = (\Gamma e_{ij})Q. \tag{5.14}$$

So

$$\Gamma(PxQ) \sum x_{ij} \Gamma(Pe_{ij}Q) = P\Gamma(x)Q. \tag{5.15}$$

By definition, $\|x\|_p = \left(\sum \mu_n(x)^p\right)^{1/p}$ for any $x \in L^p(H)$ with canonical expansion $x = \sum \mu_n(x) \langle \phi_n, \cdot \rangle \psi_n$, where $\{\phi_n\}$ and $\{\psi_n\}$ are orthonormal families and $\mu_1(x) \geq \mu_2(x) \geq \cdots \geq 0$. For any unitary operators P and Q, one has

$$\|PxQ\|_p = \sum \mu_n(x) \langle Q^* \phi_n, \cdot \rangle P\psi_n.$$

So

$$\|PxQ\|_p = \|x\|_p. \tag{5.16}$$

Let $y \in L^p(H) \otimes_{hp} L^p(H)$ and $P, Q \in B(H)$. From (5.6) one may assume that $y = u \odot v$ with $\|y\|_{hp} \geq \|u\|_{2p} \|v\|_{2p} - \varepsilon$. By (5.10) one has

$$\begin{aligned} \|PyQ\|_{hp} &\leq \|Pu \cdot u^* p^*\|_p^{1/2} \|Q^* v^* vQ\|_p^{1/2} \\ &\leq \|P\|_\infty \|Q\|_\infty \|uu^*\|_p^{1/2} \|v^* v\|_p^{1/2} \\ &\leq \|P\|_\infty \|Q\|_\infty (\|y\|_{hp} + \varepsilon). \end{aligned}$$

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Thus

$$\|PyQ\|_{hp} \leq \|P\|_\infty \|Q\|_\infty \|y\|_{hp}. \tag{5.17}$$

Any $x \in M_\infty(\mathbb{C})$ has form $x = P\Lambda Q$, with $P, \Lambda, Q \in M_\infty(\mathbb{C})$, where $\Lambda = \sum \lambda_i e_{ii}$ is diagonal with $\lambda_i \geq 0$, $\|P\|_\infty \leq 1$, $\|Q\|_\infty \leq 1$. Notice that

$$\|\Gamma(\Lambda)\|_{hp} = \|\sum \sqrt{\lambda_i} e_{i1} \otimes \sqrt{\lambda_i} e_{1i}\|_{hp} \leq \|\sum \lambda_i^p\|^{1/p} = \|\Lambda\|_p.$$

From (5.15), (5.16) and (5.17),

$$\|\Gamma(x)\|_{hp} = \|P\Gamma(\Lambda)Q\|_{hp} \leq \|\Gamma(\Lambda)\| \leq \|\Lambda\|_p = \|x\|_p.$$

Thus Γ is a contraction. ∎

REMARK 5.6. All the Haagerup L^p-tensor products $\otimes_{hp}$, for $1 < p \leq \infty$ are non-symmetric, i.e., the flip map $A \otimes B \longrightarrow B \otimes A$ is unbounded with respect to the norm $\| \ \|_{hp}$. Thus $\otimes_{hp}$ is not admissible. Is there any admissible topological tensor product $\tilde{\otimes}$ such that $\big(L^p(H), \tilde{\otimes}\big)$ is H-unital for $1 < p < \infty$? The answer is "No". In a certain sense Theorem 5.5 provides the best possible result. Any tensor product making $L^p(H)$ H-unital has to be non-symmetric.

To see this, let $\| \ \|_\sim$ be a tensor norm on $L^p(H) \otimes L^p(H)$ and let $\tilde{\otimes}$ be the complete tensor product such that the multiplication

$$b' \colon L^p(H) \tilde{\otimes} L^p(H) \longrightarrow L^p(H)$$

is bounded and onto.

Note that H is self-dual, and recall Grothendieck's formulation of $L^p(H)$ as the complete tensor product $H \overset{(p)}{\otimes} H$ (see Chapter 2, [Gro]). Thus

$$L^p(H) \tilde{\otimes} L^p(H) = H_1 \overset{(p)}{\otimes} H_2) \tilde{\otimes} (H_3 \overset{(p)}{\otimes} H_4).$$

where all $H_i = H$.

Observe that (a) For fixed $\xi_1 \in H_1$ and $\xi_4 \in H_4$, $\| \ \|_\sim$ assigns a tensor topology on $H_2 \otimes H_3$ which is no coarser than the projective topology, because the multiplication b' corresponds to taking the trace on $H_2 \otimes H_3$;

(b) for fixed $\xi_2 \in H_2$ and $\xi_3 \in H_3$, $\| \ \|_\sim$ assigns a tensor topology on $H_1 \otimes H_4$ which must be strictly coarser then the projective topology, because $L^p(H) \tilde{\otimes} L^p(H)$ is strictly "larger" than $L^p(H) \hat{\otimes} L^p(H)$. In fact, $b'\big(L^p(H) \hat{\otimes} L^p(H)\big)$ is a proper subspace of $L^p(H)$ (Proposition 5.1).

Comparing (a) and (b), one concludes that the flip map induced by

$$(\xi_1 \otimes \xi_2) \otimes (\xi_3 \otimes \xi_4) \longmapsto (\xi_3 \otimes \xi_4) \otimes (\xi_1 \otimes \xi_2)$$

is unbounded with respect to $\| \ \|_\sim$.

If one denotes A^{op} the algebra A with the reversed multiplication, then the reversed flip map $A \otimes_{hp} B \longrightarrow B^{\mathrm{op}} \otimes_{hp} A^{\mathrm{op}}$ is an isometry. This is possible because the reversed flip map

$$(\xi_1 \otimes \xi_2) \otimes (\xi_3 \otimes \xi_4) \longmapsto (\xi_4 \otimes \xi_3) \otimes (\xi_2 \otimes \xi_1)$$

is an isometry.

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6. **Applications and examples.**

EXAMPLE 6.1. Let us derive the Hood-Jones formula for the normalized complex (p. 380 [H-J]) in the context of §3. It follows as a special case of Theorem 1[W1] (see Theorem 3.9). This observation was pointed out in Example 1 [W1]. Let $A = k$, any scalar field. Then the unitalization $\tilde{A}$ can be identified with the polynomial algebra $k[x]$ with $x^2 = x$. The inclusion $A \longrightarrow \tilde{A}$ is given by $I(1) = x$. Clearly $\tilde{A} \otimes (\tilde{A}/k)^n = \{(\alpha + \beta x) \otimes x^n \mid \alpha, \beta \in k\}$. An easy calculation shows that

$$b \mid_{\tilde{A}\otimes(\tilde{A}/k)^n} = 0 \text{ if } n \text{ is odd},$$

while

$$\begin{aligned} b(\alpha + \beta x) \otimes x^{\otimes 2n} &= 2(\alpha + \beta)x \otimes x^{\otimes 2n} + \sum_{k=1}^{2n-1} (-1)^k (\alpha + \beta x) \otimes x^{\otimes 2n-1} \\ &= \big((2\alpha + \beta)x - \alpha\big) \otimes x^{\otimes 2n-1}. \end{aligned} \tag{6.1}$$

Similarly

$$B \mid_{\tilde{A}\otimes(\tilde{A}/k)^{(2n-1)}} = 0, \quad n = 1, 2, \ldots,$$

while

$$B(\alpha + \beta x) \otimes x^{\otimes 2n} = (2n+1)\beta \otimes x^{\otimes(2n+1)}. \tag{6.2}$$

Let $I^{(-n)}(1) = (a_{2n} + b_{2n}x) \otimes x^{\otimes 2n}$, $n = 0, 1, \ldots$. Obviously, $I^{(0)}(1) = x$. The condition that $[b, I^{(-n)}] + [B, I^{(-n+1)}] = 0$ is reduced to $bI^{(-n)} + BI^{(-n+1)} = 0$. Thus, (6.1) and (6.2) yield

$$\begin{cases} 2a_{2n} + b_{2n} = 0 \\ a_{2n} = (2n-1)b_{2(n-1)}, \quad n = 1, 2, \ldots \end{cases}$$

so $a_{2n} = (-2)^{(n-1)}(2n-1)(2n-3) \cdot \cdots \cdot 3 \cdot 1 = (-2)^{(n-1)}(2n-1)!!$, and $b_{2n} = (-2)^n(2n-1)!!$. Thus,

$$I^{(-n)}(1) = (-2)^{(n-1)}(2n-1)!!\,(1 \otimes x^{\otimes 2n} - 2x^{\otimes 2n+1}). \tag{6.3}$$

Assume that $\alpha(1) = e \in A' \otimes K$ and $\bar{\alpha} = 0$. We get the Hood-Jones formula

$$\mathrm{ch}_0(e) \equiv R(\alpha^\# - \bar{\alpha}^\#)(1) = e + \sum_{n=1}^{\infty} (-2)^{(n-1)}(2n-1)!!\,(1 \otimes e^{\otimes 2n} - 2e^{\otimes 2n+1}). \tag{6.4}$$

Note that there is a sign difference with [H-J].

EXAMPLE 6.2. (Hood-Jones forumula for the unnormalized complexes). In Proposition 3.6, again let $A = k$ and $\tilde{A} = k[x]$. As in Example 6.1, we get an embedding $\tilde{I} = (\tilde{I}^{(0)}, \tilde{I}^{(-1)}, 0, \ldots)$ such that $\tilde{I}^{(0)}(1^{\otimes(n+1)}) = x^{\otimes(n+1)}$ while the degree 2 map $\tilde{I}^{(-1)} = (1-t)s s N$ gives

$$\tilde{I}^{(-1)}(1^{\otimes(n+1)}) = \begin{cases} (n+1)(1 \otimes x^{\otimes(n+2)} - x^{\otimes(n+2)} \otimes 1) & \text{if } n \text{ even}, \\ 0 & \text{if } n \text{ odd}. \end{cases} \tag{6.5}$$

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REMARK 6.3. To compare with the Hood-Jones formula for the normalized complex, we compute the decomposition $i^{\#} = (i^{(0)}, i^{(-1)}, \dots)$ of the map $i^{\#}\colon B_*(k)_{\text{norm}} \longrightarrow B_*(k)$. As in Example 6.1, we get

$$b\,|_{k^{\otimes(n+1)}} = \begin{cases} \text{id}, & n \text{ even}, n \neq 0 \\ 0, & n \text{ odd} \end{cases}$$

and

$$B\,|_{k^{\otimes(n+1)}} = \begin{cases} 2(n+1), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Thus the condition $bi^{(-n)} + Bi^{(-n+1)} = 0$ implies

$$i^{(-n)}(1) = (-2)^n(2n-1)!!\, 1^{\otimes(2n+1)}, \quad n = 1, 2, \dots . \tag{6.6}$$

Let $\pi^{\#}\colon B(\tilde{A}) \longrightarrow B(\tilde{A})_{\text{norm}}$ be the quotient map, so that $\pi^{\#} = (\pi^{(0)}, 0, \dots)$ where $\pi^{(0)}\colon \tilde{A}^{\otimes n+1} \longrightarrow \tilde{A} \otimes (\tilde{A}/k)^{\otimes n}$ is just the projection. Then the relation between the normalized and unnormalized Hood-Jones formula is

$$I^{(-n)} = \pi^{(0)} \circ \tilde{I}^{(0)} \circ i^{(-n)} + \pi^{(0)} \circ \tilde{I}^{(-1)} \circ i^{(-n+1)}.$$

From (6.5) and (6.6) we get

$$I^{(-n)}(1) = (-2)^n(2n-1)!!\, x^{\otimes 2n+1} + (-2)^{n-1}(2n-1)!!\, 1\otimes^{2n}.$$

This is just (6.3).

EXAMPLE 6.3. The definition of bivariant Chern character for the algebraic bivariant K-theory ([Kass3]), [Kass4]) can be viewed as a special case of that of the Chern character for 1-summable quasihomomorphisms construced in [W1].

Recall the definition of algebraic bivariant K-theory $K(A, A')$ [Kass3]. Let A and A' be two unital algebras over k. Let $\text{Rep}(A, B)$ be the category of A-B-bimodules which are finitely generated as a B-module. Then $K(A, A')$ is the Grothendieck group of the exact category $\text{Rep}(A, B)$.

Let $P \in \text{Rep}(A, A')$. The A action on P defines an algebra homomorphism $\phi\colon A \longrightarrow \text{End}_{A'}(P)$. Since P is a finitely generated projective A'-module, there is an embedding $e\colon \text{End}_{A'}(P) \longrightarrow M_r(B)$ as a choice of a projection from $A'^r \longrightarrow P$. The 0-cycle $\text{ch}(P, e)$ (Definition 2.1 [Kass 3]) in $\text{Hom}_S\big(C_{\cdot\cdot}(A), C_{\cdot\cdot}(A')\big)$ is defined by

$$\text{ch}(P, e) = \text{Tr}^{\#} \circ e^{\#} \circ \Phi^{\#}.$$

In Theorem 2.2 [Kass3], it is shown that the class $\text{ch}(P)$ in $\text{HC}^0(A, A')$ of $\text{ch}(P, e)$ is independent of e and only depends on the isomorphism class of P. $\text{ch}(P)$ is then defined as the Chern character of the module P.

If we associated to P a quasihomomorphism h from A to $A' \otimes K$ by $h = (\alpha, \bar{\alpha})$, $\bar{\alpha} = 0$, $\alpha = e \circ \Phi$, then the Chern character $\text{ch}^0(h)$ constructed in Corollary 3.9 is $\text{ch}(P)$ defined above.

THEOREM 6.4. *When $A = k$, the bivariant Chern character constructed in Theorem 3.8 and Corollary 3.9 is reduced to the Chern character from $K_0(A')$ to $\mathrm{HC}^-_{\mathrm{ev}}(A')$ as defined in [H-J].*

PROOF. Example 6.1 above or Example 1 in [W1]. ∎

REMARK 6.5. Let $h = (\alpha, \bar{\alpha})\colon k \rightrightarrows A' \otimes K$ be an quasihomomorphism. Then the algebra homomorphisms α and $\bar{\alpha}$ are determined by the two projections $e = \alpha(1)$ and $\bar{e} = \bar{\alpha}(1)$. Example 6.3 shows that the Chern character $\mathrm{ch}^0(h)$ is uniquely determined by the formula

$$\mathrm{ch}^0(h)(1) = (e - \bar{e}) + \sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)!!\,[1 \otimes (e^{\otimes 2n} - \bar{e}^{\otimes 2n}) - 2(e^{\otimes 2n+1} - \bar{e}^{\otimes 2n+1})]$$

for the normalized complex $\mathrm{Hom}_S\big(B(k)_{\mathrm{norm}}, B(A \otimes K)_{\mathrm{norm}}\big)$.

REMARK 6.6. For geometers who are used to defining the Chern character of vector bundles in terms of connections and curvatures, some explanation may be helpful. This would supply some intuition lost in the algebraic construction of the bivariant Chern character. Let $(\phi, F = \begin{bmatrix} 0 & P \\ Q & 0 \end{bmatrix}, H_{A'})$, be a Kasparov generator (p. 533, [Kasp2]) defining an element in $KK^0(A, A')$, where $\phi(a) = \begin{bmatrix} \phi_1(a) & 0 \\ 0 & \phi_2(a) \end{bmatrix}$ is a degree zero homomorphism of A into $\mathcal{L}(H_{A'})$, $H_{A'} = H^+_{A'} \oplus H^-_{A'}$ is a $\mathbb{Z}_2$-graded Hilbert A'-module satisfying $[\phi(a), F]$, $(F^2 - 1)\phi(a)$, $(F - F^*)\phi(a)$ all in $\mathcal{K}(H_{A'})$, the algebra of compact operator on $H_{A'}$.

We may assume that F is normalized so that $Q = P^{-1}$. To define the Chern character on the K-theory $K^0(M)$ of a compact manifold M, let $[E^+] - [E^-]$ be a virtual bundle over M. Then $A' = C(M)$, $A = \mathbb{C}$. The Hilbert $C(M)$-module $H_{A'}$ is the space of cross sections of a bundle (perhaps trivial) in which $E^+ \oplus E^-$ is embedded, and operator F plays the role of a connection. Since the image of $1 \in A$ under ϕ in $\mathcal{L}(H_{A'})$ has to be a pair of projections $e = \begin{bmatrix} e_+ & 0 \\ 0 & e_- \end{bmatrix}$, on recalling Connes' quantization of differential calculus ([Con1], p. 265), in which $de = i[F, e]$, the operator $edede$ is the quantized curvature form. Under the formulation of KK in terms of Cuntz's quasihomomorphisms, the virtual bundle $[E^+] - [E^-]$ defines a homomorphism $h = (\alpha, \bar{\alpha})\colon q\mathbb{C} \longrightarrow A' \otimes K$ by $\alpha(1) = e^+$, $\bar{\alpha}(1) = Pe^-P^{-1}$ and $h(q1) = (\alpha - \bar{\alpha})(1)$ (see Proposition 1.2 for the general case). The "curvature" $edede$ is equal to (e, e, e) as a chain in cyclic homology.

REMARK 6.7. As an extremely interesting example of applications, we note that the explicit formula of bivariant Chern character in Corollary 3.9 may be applied to Dirac operators along smooth flows to get a possible bivariant proof of the important results of Elliott-Natsume-Nest [E-N-N] and (the S^1-bundle case) of Douglas-Hurder-Kaminker [D-H-K]. Since the torsion part in the universal coefficient theorem of the bivariant cyclic theory usually does not vanish, the bivariant Chern character of the Dirac fields should yield more information, possibly about secondary invariants.

REFERENCES

[Arv] W. Arveson, *The harmonic analysis of automorphism groups*, Operator algebras and applications, Proc. Symposia Pure Math. **38**(1982), Part I, 199–269.

[B-D-F] L. G. Brown, R. Douglas and P. A. Fillmore, *Extensions of C^*-algebras and K-homology*, Ann. of Math. (2) **105**(1977), 265–324.

[Bou] N. Bourbaki, *Algèbre*, Chap. 10: Algèbre homologique, 1970.

[Bur] D. Burghelea, *Cyclic homology and algebraic K-theory of topological spaces I*, Boulder Conference on Algebraic K-theory (1983), Contemp. Math. **55** Part I, 1986.

[Bus] R. C. Busby, *Double centralizers and extensions of C^*-algebras*, Trans. Amer. Math. Soc. **132**(1968), 79–99.

[C-E] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956.

[C-E-S] E. Christensen, E. G. Effros and A. Sinclair, *Completely bounded multilinear maps and C^*-algebras*, Invent. Math., **90**(1987), 279–296.

[Con1] A. Connes, *C^*-algèbres et géometrie différentielle*, C. R. Acad. Sci. Paris **290**(1980), 599-604.

[Con2] ———, *The Chern character in K-homology*, Noncommutative Differential Geometry, I, Publ. IHES **62**(1985), 41–144.

[Con3] ———, *Cohomologie cyclique et foncteurs* Ext^n, C. R. Acad. Sci. Paris **296**(1983), 953-958.

[Co-Cu] A. Connes and J. Cuntz, *Quasihomomorphismes, cohomologie cyclique et positivité*, Commun. Math. Phys. **114**(1988), 515–526.

[Co-K1] A. Connes and M. Karoubi, *Caractère multiplicatif d'un module de Fredholm*, C. R. Acad. Sci. Paris **299**(1984), 963–968.

[Co-K2] A. Connes and M. Karoubi, *Caractère multiplicatif d'un module de Fredholm*, K-theory, (1988).

[Cun1] J. Cuntz, *Generalized homomorphisms between C^*-algebras and KK-theory*, Proc. of Math.-Phys. Conf. (ZIF Bielefeld 1981), Springer Lecture Notes in Math. **1031**, 180–195.

[Cun2] ———, *A new look at KK-theory*, K-theory (1987).

[D-H-K] R. G. Douglas, S. Hurder and J. Kaminker, *Toeplitz operators and theta-invariant: the case of S^1*, Contemporary Math. A.M.S. **19**(1988), 11–41.

[E] E. G. Effros, *Advances in quantized functional analyis*, Proc. I.C.M. 1986, 906–916.

[E-K] E. G. Effros and Kishimoto, *Module maps and Hoschschild-Johnson cohomology*, Indiana J. Math, to appear.

[E-N-N] G. A. Elliott, T. Natsume and R. Nest, *Cyclic cohomology for one-parameter smooth crossed products*, Acta Math. **160**(1988), 285–305.

[E-S] E. Getzler and A. Szenes, *On the Chern character of a theta-summable Fredholm module*, preprint.

[F-W] F. T. Farrell and J. B. Wagoner, *Infinite matrices in algebraic K-theory and topology*, Comment. Math. Helvet. **47**(1972), 474–501.

[G1] T. Goodwillie, *Cyclic homology and the free loopspace*, Topology **24**(1985), 187–215.

[G2] ———, *Relative algebraic K-theory and cyclic homology*, Ann. of Math. **24**(1986), 347–402.

[Gro] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).

[Hig] N. Higson, *A characterization of KK-theory*, PacificJ. of Math **126**(1987), 253–276.

[H-J] C. E. Hood and J. D. S. Jones, *Some algebraic properties of cyclic homology groups*, K-theory (1987) 361–384.

[J-K] J. Jones and C. Kassel, *Bivariant cyclic theory*, (Aug. 1988), to appear in K-theory.

[Kar] M. Karoubi, *Homologie cyclique et K-théorie algébrique*, I et II, C.R. Acad. Sci. Paris **297**(1983), 447–450 et 513–516.

[Kasp1] G. G. Kasparov, *Hilbert C^*-modules: Theorems of Stinespring and Voiculescu*, J. Operator Theory **4**(1980), 133–150.

[Kasp2] ———, *K-functor and extensions of C^*-algebras*, Izv. Akad. Nauk SSSR. Ser. Math **44**(1980), 571–636.

[Kass1] C. Kassel, *Cyclic homology, comodules, and mixed complexes*, J. of Algebra **107**(1987), 195–216.

[Kass2] ———, *L'homologie cyclique des algébres enveloppantes*, Invent. Math. **91**(1988), 221–251.

[Kass3] ———, *K-théorie algébrique et cohomologie cyclique bivariante*, C.R. Acad. Sci. Paris **306**(1988), 799–802.

[Kass4] ———, *Caractère de Chern bivariant*, Max Planck Institut, Bonn, Preprint December 1988.

[L-Q] J. L. Loday and D. Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment Math. Helvetici **59**(1984), 565–591.

[Mac1] S. MacLane, *Homology*, Springer-Verlag, 1975.

[Mac2] ______, *Categories for the Working Mathematician*, Springer-Verlag, 1971.

[R] Z. Ruan, *Subspaces of C^*-algebras*, J. Funct. Anal. **76**(1988), 217–230.

[Sim] B. Simon, *Trace ideals and their applications*, London Math. Soc. Lecture Notes **35**, Cambridge University Press, 1979.

[Ts] B. L. Tsygan, *Homology of matrix Lie algebras over rings and Hochschild homology*, Uspekhi Math. Nauk. 2 **38**(1983), 198–199.

[W1] X. Wang, *Bivariant Chern Character I*, preprint, May 1988; Proc. A.M.S. Summer Institute at New Hampshire, Proc. Symposia Pure Math. (2) **51**(1990), 355–360.

[W2] ______, Talk at AMS Summer Institute at Bowdoin College, Maine, Aug. 1988.

[W3] ______, *KK-theory for topological algebras*, *K*-theory, **5**(1991), 97–150.

[W4] ______, Ext-*theory for topological algebras*, in preparation.

[Wod1] M. Wodzicki, *The long exact sequence in cyclic homology associated with an extension of algebras*, C.R. Acad. Sci. Paris **306**(1988), 399–403.

[Wod2] ______, *Excision in cyclic homology and in rational algebraic K-theory*, Annals of Math., **129**(1989) 591–639.

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