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LETTER

Exact nonlinear solutions for three-dimensional Alfvén-wave packets in relativistic magnetohydrodynamics

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We show that large-amplitude, non-planar, Alfvén-wave (AW) packets are exact nonlinear solutions of the relativistic magnetohydrodynamic equations when the total magnetic-field strength in the local fluid rest frame (b) is a constant. We derive analytic expressions relating the components of the fluctuating velocity and magnetic field. We also show that these constant-b AWs propagate without distortion at the relativistic Alfvén velocity and never steepen into shocks. These findings and the observed abundance of large-amplitude, constant-b AWs in the solar wind suggest that such waves may be present in relativistic outflows around compact astrophysical objects.

Key words: strophysical plasmas, plasma nonlinear phenomena, plasma waves

1. Introduction

Black holes and neutron stars are among the most remarkable objects in the universe. In addition to warping space–time, they generate powerful plasma outflows, which, in the case of supermassive black holes, can manifest as radio sources that extend up to a million light years through intergalactic space. Because the plasma around these compact objects and their outflows are often relativistic, with flow velocities comparable to the speed of light and magnetic energy density comparable to (or much greater than) the rest-mass energy density of the plasma, the plasma physics of these environments has been the subject of a great deal of recent study (Thompson & Blaes 1998; Blandford 2002; Chandran, Foucart & Tchekhovskoy 2018; Li, Zrake & Beloborodov 2019; Nathanail *et al.* 2020; Ripperda, Bacchini & Philippov 2020; Yuan *et al.* 2020; Chashkina, Bromberg & Levinson 2021; Li, Beloborodov & Sironi 2021; Ripperda *et al.* 2021; TenBarge *et al.* 2021; Yuan *et al.* 2021). It is therefore of interest to study how the basic building blocks of plasma physics, for example the plasma waves, behave in a relativistic system.

One of the most important waves in non-relativistic plasma physics is the Alfvén wave (AW) (Alfvén 1942; Barnes & Suffolk 1971; Barnes & Hollweg 1974; Goldstein, Klimas & Barish 1974). This wave has prompted a great deal of study, in part because of its

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ubiquitous presence in spacecraft observations of the solar wind (Belcher & Davis 1971). The prevalence of this wave in the solar wind may be due to the fact that long-wavelength propagating fluctuations that are not AWs quickly dissipate, either through steepening into shocks, turbulent mixing or damping due to wave–particle interactions (Barnes & Hollweg 1974; Cohen & Kulsrud 1974; Vasquez & Hollweg 1996; Schekochihin *et al.* 2009). In contrast, AWs are undamped in a collisionless plasma in the long-wavelength limit (Barnes & Suffolk 1971), and they undergo only weak turbulent mixing when most of the AWs propagate in a single direction along the background magnetic field lines (as happens in the solar wind, in which most of the AWs propagate away from the Sun in the plasma rest frame). AWs in non-relativistic plasmas also possess a polarization state in which the waves do not steepen into shocks, irrespective of their amplitude. This is the 'spherical polarization state', in which the total magnetic-field strength B is a constant. Indeed, in a homogeneous, non-relativistic plasma, a nonlinear, three-dimensional AW packet in which the total magnetic-field strength, density and pressure are constant is an exact solution to the compressible magnetohydrodynamic (MHD) equations (Goldstein *et al.* 1974). In the solar wind, the observed AWs are often nearly perfectly spherically polarized.

AWs play an important role in space and astrophysical plasmas. For example, they contribute substantially to the heating of the solar corona and the energization of the solar wind. Convective motions at the solar photosphere shake the magnetic field lines that connect the solar surface to the distant interplanetary medium, thereby launching AWs that transport energy outward from the solar surface. In many models, the dissipation of this AW energy flux is the dominant heating mechanism in the solar corona and solar wind (Cranmer, Van Ballegooijen & Edgar 2007; Verdini *et al.* 2009; van der Holst *et al.* 2014; Chandran 2021). A similar energization mechanism could arise in relativistic astrophysical plasmas, in which a dense central object (e.g. a black-hole accretion disk, or the surface of a proto-neutron star) has a turbulent surface and is threaded by a magnetic field (e.g. Metzger, Thompson & Quataert 2007). Relativistic AW turbulence has been the subject of several recent studies (e.g. Thompson & Blaes 1998; Cho 2005; Chandran *et al.* 2018; Ripperda *et al.* 2021; TenBarge *et al.* 2021) and relativistic AW in the magnetically dominated regime have been implicated in the energization of pulsar and magnetar magnetospheres (Bransgrove, Beloborodov & Levin 2020; Yuan *et al.* 2020, 2021; Beloborodov 2021).

AWs also play a crucial role in the transport and confinement of cosmic rays. When the average cosmic-ray drift velocity through a plasma exceeds the Alfvén speed, the AW becomes unstable and grows, leading to wave pitch-angle scattering of the cosmic rays (Lerche 1966; Wentzel 1968; Kulsrud & Pearce 1969). This same process plays a critical role in diffusive shock acceleration. The streaming of cosmic rays away from a shock in the upstream direction amplifies AWs, which scatter the cosmic rays, causing them to return to the shock, thereby enabling the repeated shock crossings required to accelerate particles to high energies (Bell 1978).

The tendency for AWs to develop spherical polarization in non-relativistic plasmas has important implications for the way that AWs affect the transport of energetic particles and energize plasma outflows. For example, in contrast to large-amplitude, linearly polarized AWs, large-amplitude spherically polarized AWs do not cause magnetic mirroring of cosmic rays. In addition, numerical simulations suggest that when the amplitudes of the fluctuating and background magnetic fields are comparable, spherically polarized AWs necessarily develop discontinuous magnetic-field rotations (Valentini *et al.* 2019; Squire, Chandran & Meyrand 2020; Shoda, Chandran & Cranmer 2021). Copious abrupt magnetic-field rotations are indeed observed in the solar wind close to the Sun (Bale *et al.* 2019; Kasper *et al.* 2019; Horbury *et al.* 2020), but fewer are observed farther away, implying that these discontinuities erode over time, possibly via plasma instabilities (Tenerani *et al.* 2020). The development and decay of these discontinuities provide a dissipation channel for AWs that can alter the rate at which wave energy is thermalized and, in principle, the way that the dissipated wave energy is apportioned among different particle species (cf. Howes 2010; Kawazura *et al.* 2020). If a relativistic analogue of the spherically polarized state exists for the relativistic AW, this could have important implications for energetic particle propagation and turbulent heating in relativistic plasmas. This possibility is the focus of this Letter.

Previous work on large-amplitude relativistic AWs has been limited to two simplified cases. First, in the magnetically dominated limit, where the magnetic energy density is much larger than the energy density of the plasma, the inertia of the plasma may be neglected and arbitrary AW are exact nonlinear solutions, travelling at the speed of light c (Thompson & Blaes 1998). Second, when the magnetic field is not so dominant and the plasma inertia may not be neglected, exact nonlinear AW solutions have been found for the so-called 'simple wave', in which the magnetic-field strength in the local fluid frame is a constant and the fluctuations depend only on a single scalar variable $\phi(x^\mu)$ (Barnes & Suffolk 1971; Greco 1972; Anile 1989). It was shown that the simple AW propagates without steepening. This was apparently rediscovered by Heyvaerts, Lehner & Mottez (2012), who also showed that the simple AWs are necessarily planar (1 + 1-dimensional).

In this Letter, we extend this work to more general 3 + 1-dimensional structures, without assuming plane polarization or that the system is magnetically dominated. We show that any fluctuations in the magnetic-field 4-vector and velocity 4-vector that are proportional to each other in the same way as linear AWs are exact nonlinear solutions to the relativistic MHD equations in flat space–time, provided that the mass density, internal energy, pressure and background magnetic field are constants. In these solutions, the magnetic-field strength in the local fluid rest frame is a constant. The resulting wave packets propagate through the plasma at the relativistic Alfvén velocity without steepening into shocks.

2. Elsasser formulation of general relativistic magnetohydrodynamics

The equations of general relativistic magnetohydrodynamics (GRMHD) (Anile 1989) describe the motion of a perfectly conducting fluid under the influence of the electromagnetic fields and gravity,[1] and may be derived assuming that the electric field vanishes in the local fluid rest frame. These equations are, first, the conservation of mass

$$\nabla_\nu(\rho u^\nu) = 0, \tag{2.1}$$

the stress–energy equation

$$\nabla_\nu T^{\mu\nu} = 0, \tag{2.2}$$

and the induction equation

$$\nabla_\nu(b^\mu u^\nu - b^\nu u^\mu) = 0. \tag{2.3}$$

In these equations, ∇_ν denotes the covariant derivative, ρ is the mass density, u^μ is the fluid 4-velocity, the GRMHD stress–energy tensor is

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu - b^\mu b^\nu + \left(p + \frac{b^2}{2}\right) g^{\mu\nu}, \tag{2.4}$$

[1]In this paper, we will in fact neglect gravity, but it may be useful to write the general relativistic equations for future work.

where $g^{\mu\nu}$ is the metric tensor, the magnetic-field 4-vector is

$$b^\mu = \tfrac{1}{2}\epsilon^{\mu\nu\kappa\lambda}u_\nu F_{\lambda\kappa}, \tag{2.5}$$

with $b^2 = b^\mu b_\mu > 0$, $F_{\lambda\kappa}$ the Faraday tensor divided by $\sqrt{4\pi}$, $\epsilon^{\mu\nu\kappa\lambda}$ the Levi-Civita tensor, and

$$\mathcal{E} = \rho + U + p + b^2, \tag{2.6}$$

where U is the internal energy and p is the thermal pressure. We use units such that the speed of light $c = 1$. We use the notation

$$A^2 = A^\mu A_\mu, \tag{2.7}$$

to denote the magnitude squared of any 4-vector A^μ; for spacelike 4-vectors, we also write $A = \sqrt{A^2}$.

The 4-velocity satisfies

$$u^2 = -1, \tag{2.8}$$

and (2.5) implies that

$$u_\mu b^\mu = 0. \tag{2.9}$$

Chandran *et al.* (2018) noticed that, just as in non-relativistic MHD, (2.2) and (2.3) may be cast in a useful pseudo-symmetric Elsasser (1950) form by multiplying (2.3) by $\pm\mathcal{E}^{1/2}$, adding to (2.2) and dividing the two resulting equations by $\mathcal{E}$. This results in

$$\nabla_\nu(z_\pm^\mu z_\mp^\nu + \Pi g^{\mu\nu}) + \left(\frac{3}{4}z_\pm^\mu z_\mp^\nu + \frac{1}{4}z_\mp^\mu z_\pm^\nu + \Pi g^{\mu\nu}\right)\frac{\partial_\nu\mathcal{E}}{\mathcal{E}}, \tag{2.10}$$

where

$$z_\pm^\mu = u^\mu \mp \frac{b^\mu}{\mathcal{E}^{1/2}}, \quad \Pi = \frac{2p + b^2}{2\mathcal{E}}, \tag{2.11a,b}$$

and ∂_ν refers to differentiation with respect to the coordinate ν. Equation (2.10), along with (2.1) and an equation of state, comprise the Elsasser formulation of GRMHD. These equations have been used recently by TenBarge *et al.* (2021) to study weak AW turbulence in the small-amplitude, magnetically dominated, anisotropic limit.

In the following, we restrict ourselves to the case of special relativity, for which the Minkowski metric may be written in Cartesian coordinates $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, and the covariant derivative reduces to the simpler 4-gradient operator $\nabla_\nu = \partial_\nu$. Thus, our results only apply when the length and time scales of the fluctuations are small compared with the scales over which the metric changes significantly.

3. Fluctuations on a uniform background

We take each quantity to be the sum of a background value plus a fluctuation

$$\left.\begin{array}{c} \rho = \bar{\rho} + \tilde{\rho} \quad p = \bar{p} + \tilde{p} \quad U = \bar{U} + \tilde{U}, \\ u^\mu = \bar{u}^\mu + \tilde{u}^\mu \quad b^\mu = \bar{b}^\mu + \tilde{b}^\mu \quad z_\pm^\mu = \bar{z}_\pm^\mu + \tilde{z}_\pm^\mu. \end{array}\right\} \tag{3.1}$$

We take the background quantities to be uniform in space and time,

$$\{\bar{\rho}, \bar{\mathcal{E}}, \bar{\Pi}, \bar{u}^\mu, \bar{b}^\mu\} = \text{const.}, \tag{3.2}$$

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and consider fluctuations satisfying

$$\tilde{\rho} = \tilde{\mathcal{E}} = \tilde{\Pi} = 0, \quad \tilde{u}^{\mu} = -\frac{\tilde{b}^{\mu}}{\mathcal{E}^{1/2}}. \tag{3.3a,b}$$

The final equation of (3.3*a*,*b*) implies that $\tilde{z}_{-}^{\mu} = 0$. We assume that the fluctuations are localized in space–time around a sequence of events X^{μ}; at another sequence of events X'^{μ}, $b^{\nu}(X'^{\mu}) \rightarrow \overline{b}^{\mu}$ and $u^{\mu}(X'^{\mu}) \rightarrow \overline{u}^{\mu}$ as $|(X - X')^2| \rightarrow \infty$.

It follows from (2.8) that

$$u^{\mu}u_{\mu} = \overline{u}^2 + \tilde{u}^2 + 2\overline{u}^{\mu}\tilde{u}_{\mu} = -1. \tag{3.4}$$

The space–time localization of the fluctuations combined with the constancy of $\bar{u}^{\mu}$ implies that $\overline{u}^2 = -1$, and hence

$$\tilde{u}^2 = -2\overline{u}^{\mu}\tilde{u}_{\mu}. \tag{3.5}$$

Equation (2.9) further restricts the solution by requiring that

$$\frac{1}{\mathcal{E}^{1/2}}u^{\mu}b_{\mu} = \frac{1}{\mathcal{E}^{1/2}}\overline{u}^{\mu}\overline{b}_{\mu} - \tilde{u}^2 - \tilde{z}_{+}^{\mu}\tilde{u}_{\mu} = 0. \tag{3.6}$$

The localization of the fluctuations and the constancy of $\bar{u}^{\mu}$ and $\bar{b}^{\mu}$ imply that $\overline{u}^{\mu}\overline{b}_{\mu} = 0$, so (3.6) becomes

$$\tilde{u}^2 = -\tilde{z}_{+}^{\mu}\tilde{u}_{\mu}. \tag{3.7}$$

Subtracting (3.5) from twice (3.7), we find that

$$\tilde{u}^2 = \frac{2}{\mathcal{E}^{1/2}}\overline{b}^{\mu}\tilde{u}_{\mu}. \tag{3.8}$$

Finally, we calculate the scalar b^2. This is

$$b^2 = b^{\mu}b_{\mu} = \overline{b}^2 + \mathcal{E}\tilde{u}^2 - 2\mathcal{E}^{1/2}\overline{b}^{\mu}\tilde{u}_{\mu} = \overline{b}^2, \tag{3.9}$$

a constant, where we have used (3.8) in the last equality. Thus, the wave packet has constant 4-magnetic-field magnitude, analogous to the constant-B^2 constraint for a large-amplitude AW in non-relativistic MHD (Barnes & Hollweg 1974; Goldstein *et al.* 1974).

At each point in space–time, we may boost into an accelerating frame moving with the instantaneous local fluid velocity u^{μ}, the local fluid rest frame. In this frame, $b^t = 0$ and, therefore, the magnetic-field 3-vector has magnitude squared $B^2 = b^2$, which is a constant and therefore the same at each point; B^2 is not spatially constant in an arbitrary fixed inertial frame.

Equation (2.1) with ρ constant and flat space–time gives

$$\partial_{\nu}\tilde{u}^{\nu} = \partial_{\nu}\tilde{z}_{+}^{\nu} = 0, \tag{3.10}$$

and the + Elsasser equation (2.10) then gives

$$\tilde{z}_{-}^{\nu}\partial_{\nu}\tilde{z}_{+}^{\mu} = 0, \tag{3.11}$$

with the − Elsasser equation vanishing by virtue of (3.10) and (3.3*a*,*b*). (3.11) is a linear wave equation for the evolution of $\tilde{z}_{+}^{\mu}$; thus, a three-dimensional Alfvénic wavepacket of (apparently; see (5.1)) arbitrary amplitude and arbitrary shape propagates without distortion on a homogeneous background.

4. Components in the background rest frame

We define a background rest frame (BRF)[2] in which the homogeneous background (3.2) is at rest. In this frame $\bar{u}^\mu = (1, 0, 0, 0)$, and $\bar{b}^\mu = (0, 0, 0, b)$, where we have chosen to align the z direction with the background magnetic field. Since u^μ is a future-directed 4-velocity, it is straightforward to show, working in the BRF and using (3.4), that $\tilde{u}^2 \geq 0$, a relation that holds in all frames since $\tilde{u}^2$ is a scalar. (The equality $\tilde{u}^2 = 0$ is obtained only when $\tilde{u}^\mu = 0$.) Calculating (3.5) and (3.8) in the BRF, the t and z components of the fluctuation are given by

$$\tilde{u}^t = \frac{1}{2}\tilde{u}^2, \quad \tilde{u}^z = -\frac{\mathcal{E}^{1/2}}{2b}\tilde{u}^2 \text{ (BRF)}, \tag{4.1a,b}$$

and the magnitude of the remaining (perpendicular) fluctuation components $\tilde{u}_\perp = \sqrt{(\tilde{u}^x)^2 + (\tilde{u}^y)^2}$ is

$$\tilde{u}_\perp = \sqrt{\tilde{u}^2 + (\tilde{u}^t)^2 - (\tilde{u}^z)^2} = \tilde{u}\sqrt{1 - \frac{\tilde{u}^2}{4\sigma}} \text{ (BRF)}, \tag{4.2}$$

where $\sigma = b^2/(\rho + U + p)$. Providing $\tilde{u}^2$ thus gives us nearly all the information in the fluctuation 4-vector, apart from the direction in the y-z plane that the perpendicular fluctuation points. To determine this, we must use (3.10) and a particular functional form for $\tilde{u}^2$. This amounts to solving a two-dimensional first-order quasilinear partial differential equation with analytic coefficients for one of the components (say, $\tilde{u}^x$).

Evaluating the fluid 3-velocity $v^i = u^i/u^t$ and magnetic-field 3-vector $B^i = b^i u^t - b^t u^i$, it is clear that, in the BRF, B^2 is not constant, the 3-vector magnetic and velocity fluctuations are not related to each other by a constant of proportionality, and do not in general even point in the same direction.

5. Maximum amplitude

Equation (4.2) implies an upper limit on the magnitude of the fluctuations,

$$\tilde{u} \leq \tilde{u}_{\max} = \frac{2b}{\sqrt{\rho + U + p}}. \tag{5.1}$$

In the non-relativistic case, this has been recently noticed in solar wind AWs by (Matteini *et al.* 2018). The observed magnitude of the fluid 3-velocity in the BRF is

$$v = \frac{\sqrt{u^i u_i}}{u^t} = \frac{\tilde{u}\sqrt{1 + \tilde{u}^2/4}}{1 + \tilde{u}^2/2} \text{ (BRF)}, \tag{5.2}$$

an increasing function of $\tilde{u}$; $v < 1$ and $v \to 1$ as $\tilde{u} \to \infty$.

6. Alfvén velocity and wave frame

The propagation of the wave is controlled by the constant time-like 4-vector $\bar{z}_-^\mu$, with

$$\bar{z}_-^2 = \frac{b^2}{\mathcal{E}} - 1 < 0. \tag{6.1}$$

(In the limit $b^2 \to \infty$ while keeping ρ, U and p constant, $\bar{z}_-^2 \to 0$.) An observer moving with 4-velocity $\bar{z}_-^\mu/\sqrt{-\bar{z}_-^2}$ sees a time-independent structure; such an observer is in the

[2]In Chandran *et al.* (2018), this was called the average fluid rest frame.

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wave frame (WF). The WF 3-velocity relative to another frame is $v_w^i = \bar{z}_-^i/\bar{z}_-^t$, and specifically, relative to the BRF is

$$v_w^i = v_A^i = \frac{b}{\mathcal{E}^{1/2}}(0,0,1)\ (\mathrm{BRF}). \tag{6.2}$$

Thus, the three-dimensional relativistic AW propagates along the background field lines at this relativistic Alfvén velocity, just like the planar AW (Barnes & Suffolk 1971; Heyvaerts *et al.* 2012). Note that this does not depend on $\tilde{u}^\mu$, so the wave does not steepen into a shock. If all of $b^2/\rho, U/\rho, p/\rho \ll 1$, we recover the usual non-relativistic Alfvén velocity. As $b^2 \to \infty$, $v_A^2 \to 1$, the ultra-relativistic limit of the AW previously studied by (for example) Thompson & Blaes (1998) and Heyl & Hernquist (1999).

7. Structure in the WF

In the WF, the spatial components of $z_-^\mu = \bar{z}_-^\mu$ are zero and so

$$u^i = -b^i/\mathcal{E}^{1/2}\ (\mathrm{WF}). \tag{7.1}$$

Using (2.9), we may obtain

$$b^t = -\mathcal{E}^{1/2}\gamma v^2\ (\mathrm{WF}), \tag{7.2}$$

where v^2 is the square of the 3-velocity $v^i = u^i/\gamma$ and $\gamma = u^t$. Then, we may calculate the magnetic-field 3-vector,

$$B^i = b^i u^t - b^t u^i = -\mathcal{E}^{1/2} v^i\ (\mathrm{WF}), \tag{7.3}$$

so in the WF the 3-velocity is parallel and proportional to the magnetic-field 3-vector, just like in the non-relativistic case. This also implies $\partial_i v^i = 0$ in the WF. We may also calculate

$$b^2 = b^i b_i - (b^t)^2 = \mathcal{E} v^2 = B^2\ (\mathrm{WF}), \tag{7.4}$$

and so, since b^2 is constant, in the WF v^2 and B^2 are both constant, just like in the non-relativistic case (Matteini *et al.* 2015).

Let us consider the components of the stress–energy tensor (2.4) in the WF. First, using (7.2) and (7.4),

$$T^{tt} = \rho + U + \frac{3b^2}{2}\ (\mathrm{WF}), \tag{7.5}$$

a space–time constant for our solution. Applying (7.2) again,

$$T^{ti} = T^{it} = \mathcal{E} v^i\ (\mathrm{WF}), \tag{7.6}$$

which from (7.3) has no spatial divergence, maintaining the constancy of T^{tt} in the equation $\partial_\nu T^{t\nu} = 0$. Finally,

$$T^{ij} = \left(p + \frac{b^2}{2}\right)\delta^{ij}\ (\mathrm{WF}), \tag{7.7}$$

a space–time constant, thus enforcing the constancy in time of T^{it} in the equations $\partial_\nu T^{i\nu} = 0$. The cancellation of the first two terms in the space–space components of (2.4) in the WF generalizes the result for the non-relativistic AW that the centrifugal force exactly balances the tension force in the magnetic field, keeping the fluid flowing exactly along the field lines in the WF.

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8. Discussion

Our analysis has shown that some of the unique properties of the AW survive, even with relativistic fluctuation velocities and arbitrarily strong magnetic-field strength. Specifically, just as in the non-relativistic case (Goldstein *et al.* 1974), a three-dimensional Alfvénic structure propagates in time without steepening into a shock,[3] no matter its fluctuation amplitude; equivalently, the propagation velocity in the rest frame of the background is always the relativistic Alfvén velocity (6.2), which is independent of the fluctuation amplitude. Also analogous to the non-relativistic case, the magnitude b^2 of the magnetic-field 4-vector b^μ is a space–time constant. This implies correlations between different components of the fluctuation to enforce this constraint. Unlike in the non-relativistic case, in a general inertial frame the magnetic-field 3-vector does not have constant magnitude; however, in the WF moving at the Alfvén velocity, both the velocity and magnetic field 3-vectors have constant magnitude, as in the non-relativistic case. Also in the WF, the plasma 3-velocity is parallel and proportional to the magnetic-field 3-vector.

In what situations might one see large-amplitude relativistic AWs? In a statistically homogeneous medium, one might expect equal fluxes of $\tilde{z}^\mu_\pm$ AWs, a nonlinear, turbulent situation. However, if the waves are excited by some particular event or set of events, they will mainly travel away from that event. We might postulate (inspired by non-relativistic plasma physics) that sufficiently far from the source, the other, non-Alfvénic modes largely dissipate, and then we are left with just the AWs. This situation would be relevant, for example, in outflows around a compact object like a black hole (Chandran *et al.* 2018). One caveat is that in this case the background is likely to be highly inhomogeneous, and these inhomogeneities will reflect the waves and thus drive turbulence. However, as previously mentioned, in the non-relativistic case it can be shown that even including this turbulence (Cranmer & Van Ballegooijen 2005; Verdini & Velli 2007; Perez & Chandran 2013; Van Ballegooijen & Asgari-Targhi 2016; van Ballegooijen & Asgari-Targhi 2017; Chandran & Perez 2019), in fact the normalized amplitude $\tilde{B}/\overline{B}$ of the primary outward-travelling AWs tends to grow with distance from the central object (in the solar wind case, the Sun Parker 1965; Hollweg 1974), with the other, reflected, components remaining relatively small: i.e. the fluctuations are approximately large-amplitude AWs. This is thought to be a possible origin for the 'switchbacks'; abrupt magnetic-field reversals recently observed by NASA's Parker Solar Probe in the near-Sun solar wind (Squire *et al.* 2020). If this result carries over to the relativistic case (as it appears to, cf. Chandran *et al.* 2018), large-amplitude AWs of the type discussed in this Letter may also exist in relativistic environments. Energization by large-amplitude AW have already been studied in the magnetically dominated limit (Thompson & Blaes 1998; Cho 2005; Yuan *et al.* 2020; Li *et al.* 2021; Ripperda *et al.* 2021): our results apply both in this limit and when the magnetization is less extreme, and thus may be important for the study of the heating and observed dynamics of plasma around compact objects (Akiyama *et al.* 2021*a*,*b*).

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[3]It is important to mention here that large-amplitude AWs are, however, subject to decay due to parametric instabilities (Derby 1978; Goldstein 1978; Matsukiyo & Hada 2003), which we have ignored in this paper.

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Declaration of interest

The authors report no conflict of interest.

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