2
The pairing force and seniority

2.1 Evidence for pairing correlations

The nucleus ^{208}Pb is well described by the independent particle model in terms of closed shells in both neutrons and protons. The absence of low-lying states supports this hypothesis. The shell model would then describe the low-lying levels of ^{207}Pb in terms of the states of a single neutron hole. The observed energies, angular momenta and parities are in good accord with this picture, although small admixtures of more complicated configurations must be invoked to account for some fast electromagnetic decays (see Sections 9.1 and 9.2). The next step is to interpret the levels of ^{206}Pb in terms of two holes, which are combinations of the single-hole states of ^{207}Pb, interacting through a residual interaction. The dramatic effect of the internucleon force is shown by the fact that there is only one excited state below 1.2 MeV, compared with the five we would get in the independent hole approximation (see Fig. 2.1). This becomes even more striking in ^{204}Pb, where the independent hole picture predicts about thirty levels within 1 MeV of the ground state, whereas again only one is observed (Mottelson (1996)).

Another indication of the importance of the residual interaction among nucleon pairs is the well-known difference in physical properties between even and odd nuclei. For example, studies of cosmic abundances show that nuclei with even proton and neutron numbers are much more abundant, and thus more stable, indicating stronger binding energies.

Fig. 2.2 shows a typical trend in binding energies as a function of the number of nucleons (see also Fig. 1.10). The binding energies of the even nuclei (N_0, $N_0 \pm 2$, ...) give rise to the lower line and odd nuclei to the upper line. The ordinate is $E - \lambda (N - N_0)$ where E is the energy and λ is the chemical potential. The term $\lambda (N - N_0)$ subtracts the average dependence of the energy on the particle number so that the odd–even fluctuations show more clearly. The energy
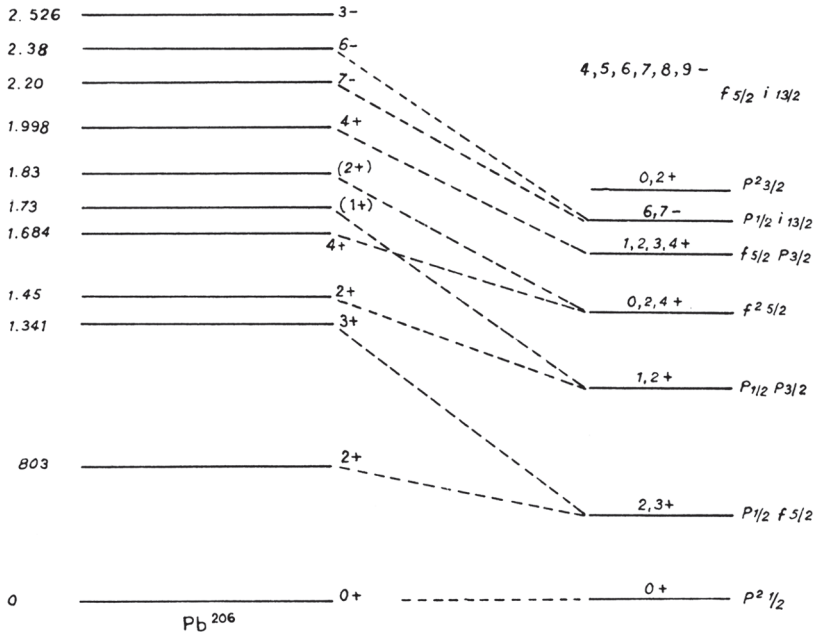
Figure 2.1. Level spectrum of ^{206}Pb. The experimental spectrum together with the observed spin and parities are shown on the left. On the right the different possible configurations $(j_1 j_2)$ are drawn with excitation energies equal to $E(j_1) + E(j_2)$ as obtained from ^{207}Pb. The spin and parities of the different levels that can be obtained from coupling j_1 and j_2 connect these asignments with the appropriate levels in the observed spectrum (after Mottelson (1996)).
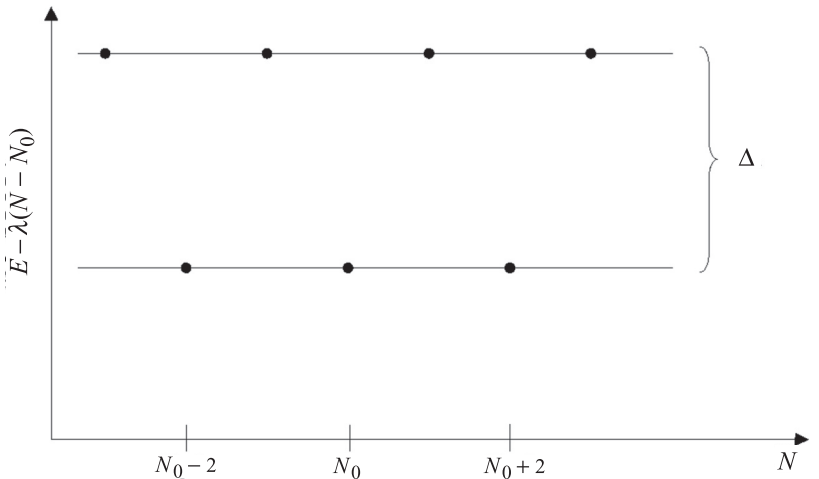

Figure 2.2. Binding energies $(-E)$ as a function of the number of neutrons. The quantity $-\lambda$ is the average binding energy per added neutron, that is λ = chemical potential.

difference between the lines is Δ. Whether this quantity reflects the binding energy associated with a pair interaction between nucleons or not depends on its magnitude. This is because the quantization of the independent particle model also implies an extra binding energy for the even system compared with the odd system. The magnitude of this effect can be estimated in the Fermi gas model (see equations (2.4)–(2.7)).

The spacing d between states at the Fermi energy with the same spin and isospin quantum numbers can be written in terms of the total level density sum of the proton and neutron level densities $\rho = 3A/2\varepsilon_F$ as

$$d = \frac{4}{\rho(\varepsilon_F)} = \frac{8\varepsilon_F}{3A}. \tag{2.1}$$

The factor of 4 is due to the fact that each level can be occupied by two protons and two neutrons (with spin up and spin down). If the levels are equally spaced the total energy is

$$E(N) = \tfrac{1}{4}d(N-1)^2 - \tfrac{1}{4}d\,\delta(N, \text{even}), \tag{2.2}$$

for each type of particle, as can be seen from Fig. 2.3. Making use of the relations given in equations (1.28) and (2.2) and of the fact that $B(N) = -E(N)$, one finds
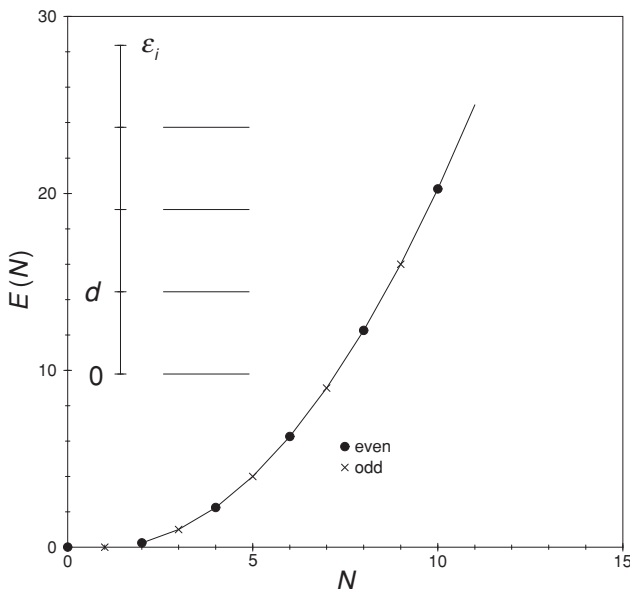

Figure 2.3. Energy $E(N) = \sum_{i=1}^{N} \varepsilon_i$ associated with the motion of N identical independent particles moving in the set of equidistant two-folded single-particle levels shown in upper left corner. Under close inspection, there is an odd–even staggering as described by equation (2.2). The dots and the crosses represent the even and odd systems respectively. The continuous curve corresponds to the expression given by equation (2.2).

that the effective gap parameter in this model is (Bohr and Mottelson (1969))

$$(\Delta)_{\text{kin}} = 2 \times \frac{d}{8} \simeq \frac{2\varepsilon_{\text{F}}}{3A} \sim \frac{25}{A} \text{MeV}. \tag{2.3}$$

The observed pairing energies are shown in Fig. 1.11 and can be parametrized according to equation (1.30). For a ^{160}Dy nucleus ($A = 160$) we have $(\Delta)_{\text{kin}} \simeq 0.16$ MeV and $\Delta \approx 0.95$ MeV. Thus Δ is almost an order of magnitude larger than $(\Delta)_{\text{kin}}$ (see also Satula *et al.* (1998) and Rutz *et al.* (1999)).

The large observed odd–even effect may be described in terms of pairwise correlations of identical particles. These contribute an additional binding energy for each pair of nucleons near the top of the Fermi distribution coupled to angular momentum zero. This can be seen from the spectra shown in Fig. 2.1. Giving an extra binding energy to the $p_{1/2}^2(0^+)$ configuration is equivalent to moving up all the excited states by the same amount. In this way overall agreement between theory and experiment is obtained.

We conclude this section by collecting together some numerical values of Fermi gas parameters which will be used in this chapter and in other parts of the book. The average particle density of nuclear matter is taken to be

$$\rho(0) = 0.17 \ \text{nucleons fm}^{-3}, \tag{2.4}$$

i.e.

$$\rho(0) = 2.8 \times 10^{14} \text{gm cm}^{-3}, \tag{2.5}$$

which corresponds to a nuclear radius $R = r_0 A^{\frac{1}{3}}$, with $r_0 = 1.1$ fm. The Fermi wave number is

$$k_{\text{F}} = 1.36 \text{fm}^{-1} , \tag{2.6}$$

(an average value for neutrons and protons, $N = Z = A/2$) and the Fermi energy is

$$\varepsilon_{\text{F}} = \frac{\hbar^2 k_{\text{F}}^2}{2m} \simeq 37 \text{MeV}. \tag{2.7}$$

This value of ε_{F} has been used in equation (2.3).

2.2 The pairing interaction

The idea of a pairing interaction was already present in the early developments of the shell model (Mayer and Jensen (1955)). The purpose of this section is to identify two general properties of a pairing force interaction. The first is that it is short range and the second that it has a multipole expansion containing high angular momentum components (Belyaev (1959), Bayman (1960), Mottelson (1962)). Both of these properties are illustrated by the example of a delta function potential discussed at the end of this section and in Section 2.3. The second is important for understanding the induced pairing interaction in Chapters 8, 10 and 11.
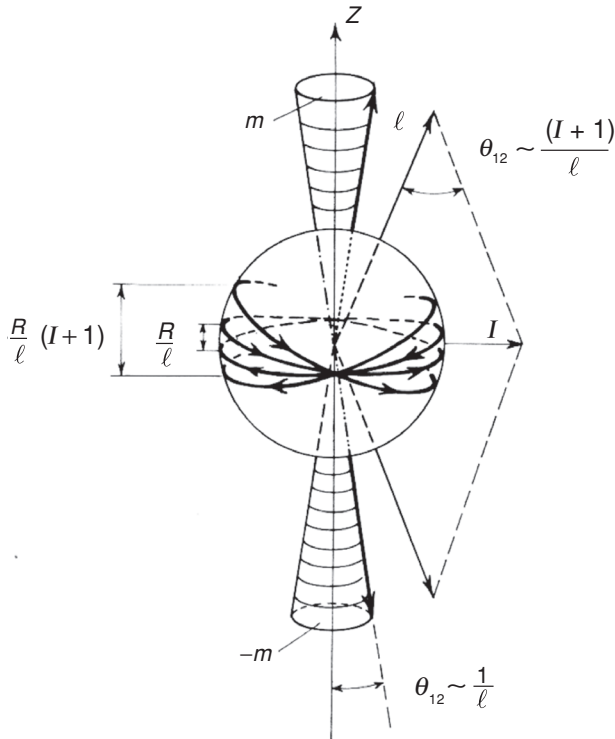
Figure 2.4. Schematic representation of two time-reversed orbits coupled to angular momentum $I = 0$ and $I \neq 0$. Two identical nucleons are assumed to move in time-reversal orbits labelled by the orbital angular momentum ℓ and the projection m. When the two particles are coupled to angular momentum zero their orbits wobble within an angle $\theta_{12} \sim 1/\ell$. This is required by the Heisenberg uncetainty principle within conjugated variables, in the present case $\Delta\ell_z\Delta\theta_{12} \gtrsim 1$. In a simple classical picture where the particles are concentrated at a radius R, such wobbling results in a typical distance between the two particles of the order of R/ℓ. The larger the angular momentum the closer is the system to the classical limit and larger will be the probability that the particles are on top of each other. When the relative motion of the particles are coupled to a finite value I of the total angular momentum, aside from the quantal wobbling, there will be a further tilting of the single-particle orbital proportional to I/ℓ. This will add, on average, a distance RI/ℓ between the two particles.

Consider two particles in the same ℓ-orbit coupled to various total angular momenta $L = 0, 1, 2, \ldots$. The radial dependence of the single-particle wavefunctions will, in most cases, describe particles moving in orbits with a radius of the order of the nuclear radius. The main degree of freedom available to the particles corresponds to different possible orientations of the orbital planes. The associated particle correlations are mainly correlations in angle. This is illustrated in Fig. 2.4. Two identical nucleons are assumed to move in time-reversed orbits labelled by the orbital angular momentum ℓ with projections m and $-m$. When the two particles are coupled to an angular momentum $L = 0$ their orbits

wobble within an angle $\theta_{12} \sim 1/\ell$. This is required by the Heisenberg uncertainty relation for conjugate variables $\Delta\ell\,\Delta\theta_{12} \sim 1$. In a simple classical picture where the particles are located at a radius R such a wobbling results in a typical distance between the particles of the order of R/ℓ.

If we now consider a state of the ℓ^2 configuration with $I \neq 0$ (but still $I \ll l$) then the average angle θ_{12} between the particles will be larger. This will increase their average separation by a distance of the order of RI/ℓ, giving a total separation $\sim R(I+1)/\ell$. Consequently, if the range of the force is small compared with R/ℓ, the states with $I = 2$ will have an interaction energy which is a fraction ($\sim 1/3$) of that in the $I = 0$ state.

Let us now expand a general interaction in multipoles (Brink and Satchler (1968))

$$V(r_{12}) = V(|\vec{r_1} - \vec{r_2}|) = \sum_\lambda J_\lambda(r_1, r_2) \sum_\mu Y_{\lambda\mu}(\hat{r_1})Y_{\lambda\mu}^*(\hat{r_2})$$

$$= \sum_\lambda \frac{2\lambda+1}{4\pi} J_\lambda(r_1, r_2) P_\lambda(\cos\theta_{12})$$

$$= \sum_\lambda V_\lambda(r_1, r_2) P_\lambda(\cos\theta_{12}). \tag{2.8}$$

If particles 1 and 2 are in orbitals confined to a fairly restricted radial region, the dependence on r_1, r_2 may be ignored for a particular λ. The function P_λ drops from its maximum at $\theta_{12} = 0$ in an angular distance $1/\lambda$ (Fig. 2.5). Thus 1 and 2 interact through the component λ only if r_{12} in equation (2.8) fulfils $r_{12} < R/\lambda$,
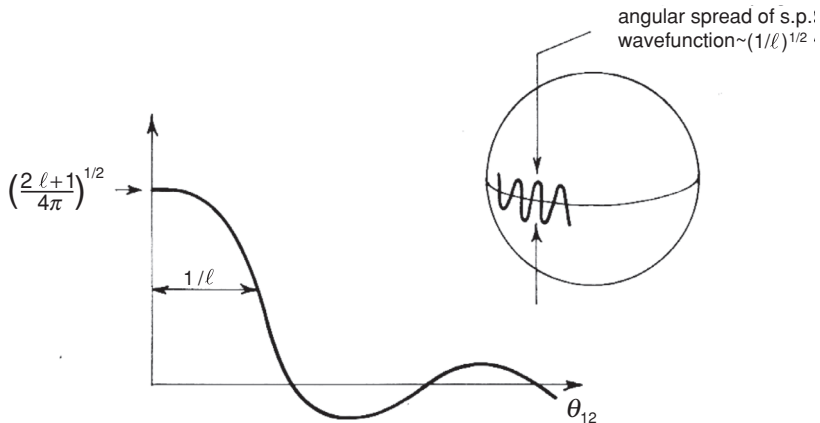
Figure 2.5. Schematic picture indicating the angular spread of the wavefunction of two particles coupled to angular momentum $I = 0$. Particle 1 is moving in an orbital with $m = \ell$ while particle 2 has $m = -\ell$. These one-particle states correspond to wavefunctions concentrated in the equatorial plane, but possessing an angular spreading $\sim (\ell)^{-1/2}$ due to quantal zero point fluctuations (after Mottelson (1962)). Copyright © Società italiana di Fisica.

where R is the mean value of the radii $\vec{r}_1$ and $\vec{r}_2$. Thus, as λ increases, the effective forces range decreases. This leads to the expectation that the strength of the λ terms increases as the range of $V(r_{12})$ decreases. For a force of range much greater than the nuclear size, only the $\lambda = 0$ monopole part is important. At the other extreme, a δ-function force has coefficients $V_\lambda(r_1, r_2)$ that increase with λ, as can be seen from the relation

$$V_\lambda = \frac{(2\lambda + 1)}{4\pi r_1^2}\delta(r_1 - r_2). \tag{2.9}$$

Pairing force effects come from all the high λ terms, representing the short-range effects of $V(r_{12})$.

2.3 The δ-function nucleon–nucleon potential

As discussed in Section 2.2, the δ-function potential

$$V(r_{12}) = -4\pi V_0\,\delta(\vec{r}_1 - \vec{r}_2), \tag{2.10}$$

is a simple representation of a short-range attractive effective interaction between identical valence nucleons. Two identical nucleons in a shell model orbit with angular momentum j coupled to a total angular momentum I have a wavefunction $|jjIM\rangle$ and the interaction energy E_I is

$$E_I = \langle jjIM|V|jjIM\rangle.$$

The matrix element can be evaluated to give

$$E_I = -\frac{(2j + 1)}{2}V_0\,I(j)|\langle I\ j\ 0\ \tfrac{1}{2}|\ j\ \tfrac{1}{2}\rangle|^2, \tag{2.11}$$

where

$$I(j) = \int R_j^4 r^2 \mathrm{d}r,$$

is an integral depending on the radial wavefunction R_j of the level j, and $\langle I\ j\ 0\ \tfrac{1}{2}|\ j\ \tfrac{1}{2}\rangle$ is a Clebsch–Gordon coefficient. The details of the derivation of equation (2.11) are given e.g. in Bayman (1960), Brink and Satchler (1968) de-Shalit and Talmi (1963), Lawson (1980) and Heyde (1990). When the total angular momentum $I = 0$, the energy E_0 in equation (2.11) simplifies to

$$E_0 = -\frac{(2j + 1)}{2}\ V_0\ I(j). \tag{2.12}$$

The radial integral $I(j)$ can be estimated by assuming that R_j is constant inside the nuclear radius R_0 and is zero outside. Normalizing the wavefunction gives

$$R_j = \sqrt{3/R_0^3},$$

and

$$I(j) \approx 3/R_0^3 \, .$$

If one corrects this estimate for the spillout of the nucleons (see Appendix D, Section D.2), one has to divide the result by $(1 + a/R)^3 \approx 1.4$, thus leading to

$$I(j) \approx 1.2 \, \text{fm}^{-3}/A, \tag{2.13}$$

if $R_0 = 1.2 A^{\frac{1}{3}} \text{fm}$. In the limit $j \gg I$ the Clebsch–Gordon coefficient in equation (2.11) can be estimated by using its semiclassical limit

$$\lim_{j \gg I} \langle j \tfrac{1}{2} I 0| j \tfrac{1}{2}\rangle \approx P_I(0) = \frac{(-1)^{I/2} \, I!}{2^I \, (I/2)!(I/2)!}. \tag{2.14}$$

Substituting $I = 0, 2, 4$ and 6 into equation (2.14) gives

$$E_2 \sim (1/4) \, E_0, \quad E_4 \sim (9/64) \, E_0, \quad E_6 \sim (25/256) E_0, \tag{2.15}$$

showing that pairing is much stronger for the state with $I = 0$ than for other values of I. The spectrum (2.15) is illustrated in Fig. 2.6.

The pairing force is an approximation to a short-range potential like the δ-function interaction, and is defined so that the energy E_I of a pair is large when $I = 0$ and is zero for $I \neq 0$. It can be expressed in second quantized form by using the pair-creation operator (see Appendix A)

$$P_j^\dagger = \sum_{m>0}(-1)^{j-m} a_{jm}^\dagger a_{j-m}^\dagger, \tag{2.16}$$

and the corresponding pair-annihilation operator
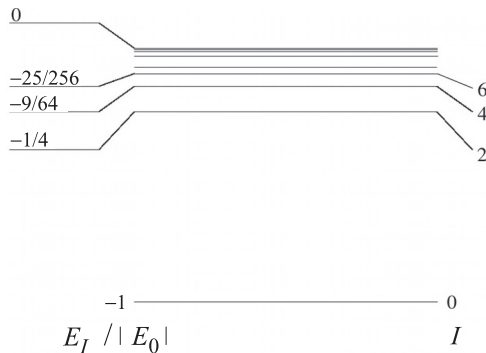
$$P_j = \sum_{m>0}(-1)^{j-m} a_{j-m} a_{jm}. \tag{2.17}$$

Figure 2.6. Spectrum of two particles moving in a single-j orbital coupled to angular momentum I and interacting through a contact nucleon–nucleon potential.

The operator $P_j^\dagger$ creates a pair of identical nucleons (say neutrons) with total angular momentum $I = 0$ and the normalized pair state $|(jj)_0\rangle$ can be written as

$$|(jj)_0\rangle = \frac{1}{\sqrt{j + \frac{1}{2}}} P_j^\dagger |0\rangle. \tag{2.18}$$

The pairing force interaction potential is defined as

$$V = -G\, P_j^\dagger P_j, \tag{2.19}$$

and with this potential the interaction energy of a pair is

$$E_I = \begin{cases} -G(j + \frac{1}{2}) & \text{if } I = 0, \\ 0 & \text{if } I \neq 0. \end{cases} \tag{2.20}$$

The pairing potential can be generalized to the case where the nucleon pair can occupy one of several j-orbits. The generalization is

$$V = -\sum_{jj'} G(jj') P_j^\dagger P_{j'}. \tag{2.21}$$

To understand the physical properties of the pairing potential in another way we write

$$a_{jm}^\dagger = a_\nu^\dagger \quad \text{and} \quad (-1)^{j-m} a_{j-m}^\dagger = a_{\bar{\nu}}^\dagger \tag{2.22}$$

The operator a_ν^+ creates a nucleon in a single-particle state $|\nu\rangle = |jm\rangle$ and $a_{\bar{\nu}}^+$ creates an identical nucleon in the time-reversed state $|\bar{\nu}\rangle = (-1)^{j-m}|j-m\rangle$ (see Appendix A). The generalized pairing potential is

$$V = -\sum_{\nu\nu'>0} G_{\nu\nu'} P_\nu^\dagger P_{\nu'}, \tag{2.23}$$

where $P_\nu^\dagger = a_\nu^\dagger a_{\bar{\nu}}^\dagger$ and $P_\nu = a_{\bar{\nu}} a_\nu$. The sum in equation (2.23) is over $\nu > 0$ and $\nu' > 0$ which corresponds to m and m' positive. The pairing strength $G_{\nu,\nu'}$ is an amplitude for a nucleon pair in the state $|\nu'\rangle$ and the time-reversed state $|\bar{\nu}'\rangle$ to make a transition to the state $|\nu\rangle$ and its time-reversed state $|\bar{\nu}\rangle$. The pairing force potential produces correlations between pairs of nucleons in time-reversed states.

We finish this section by making an estimate of the pairing force strength parameter G. By comparing equations (2.12) and (2.20) we see that

$$G = V_0 I(j) \approx 1.2\,\text{fm}^{-3}\, V_0/A. \tag{2.24}$$

To estimate V_0 we use the δ-function potential to relate V_0 to the strength of the shell model single-particle potential $U(r)$ by writing (see equation (A.28))

$$U(r) = -4\pi \int d^3r' V_0 \delta(\mathbf{r} - \mathbf{r}')\rho(r') = -4\pi V_0 \rho(r), \qquad (2.25)$$

where $\rho(r)$ is the nucleon density inside a nucleus. If $\rho(r) = \rho_0$ is constant then

$$\rho_0 = A/(\frac{4\pi}{3} R_0^3) \simeq 0.17 \, \text{fm}^{-3}.$$

Hence

$$V_0 \approx -\frac{U_0}{4\pi\rho_0} = \frac{50 \, \text{MeV fm}^3}{4\pi \times 0.17} = \frac{294}{4\pi} \, \text{MeV fm}^3, \qquad (2.26)$$

and (see Section 2.5)

$$G \approx \frac{28}{A} \, \text{MeV}. \qquad (2.27)$$

This estimate should not be taken too seriously because the real nucleon–nucleon interaction is much more complicated than that defined in equation (2.10) (see Chapter 8, Section 8.1). We shall, however, see in the following sections that the estimate (2.27) is not unreasonable.

2.4 The degenerate model and quasi-spin

A simple shell model Hamiltonian for a number of identical nucleons outside a closed shell and interacting by a pairing force residual interaction can be written as

$$H = \sum_j N_j \varepsilon_j - \sum_{jj'} G_{jj'} P_j^\dagger P_{j'}, \qquad (2.28)$$

where ε_j is the energy of the single-particle orbit j while $N_j = a_j^\dagger a_j$. There is a simple limiting case of this Hamiltonian for which the eigenvalues and eigenvectors can be found analytically. This is the degenerate model (see Appendix H), where all the single-particle energies are the same and all the pairing strengths are equal

$$\varepsilon_j = \varepsilon; \qquad G_{jj'} = G. \qquad (2.29)$$

In this case several j-levels are degenerate and the total degeneracy is

$$\sum_j (2j + 1) = 2\Omega.$$

The Hamiltonian (2.28) can be written as

$$H = \varepsilon N - G S_+ S_- = \varepsilon N - G P^\dagger P, \qquad (2.30)$$

where

$$S_+ = P^\dagger = \sum_{\nu=1}^{\Omega} a_\nu^\dagger a_{\bar{\nu}}^\dagger \quad \text{and} \quad S_- = P = \sum_{\nu=1}^{\Omega} a_{\bar{\nu}} a_\nu, \qquad (2.31)$$

and $\hat{N}$ is the number operator for the total number of nucleons outside the closed shells

$$\hat{N} = \sum_{\nu=1}^{\Omega} (a_\nu^\dagger a_\nu + a_{\bar{\nu}}^\dagger a_{\bar{\nu}}). \qquad (2.32)$$

The operators S_+, S_- and $S_z = \frac{1}{2}(\hat{N} - \Omega)$ satisfy the commutation relation

$$[S_+, S_-] = -2S_z = \Omega - \hat{N},$$
$$[S_+, S_z] = S_z, \qquad [S_-, S_z] = -S_z. \qquad (2.33)$$

These are the same as the commutation relation for the angular momentum operators J_+, J_- and J_z and for this reason they are called quasi-spin operators. The quasi-spin method was introduced by Anderson (1958) and used by Ichimura (1964), Lawson(1980) and others. Angular momentum methods can be used to find the eigenvalues and eigenvector of the simplified Hamiltonian (2.28). The operator

$$\mathbf{S}^2 = S_+ S_- + S_z(S_z - 1), \qquad (2.34)$$

which is the analogue of the square of the total angular momentum, commutes with S_+, S_- and also with H.

The Hamiltonian (2.28) can be written as

$$H = \varepsilon \hat{N} - G\left(\mathbf{S}^2 - S_z(S_z - 1)\right), \qquad (2.35)$$

and the eigenvalues are

$$E = \varepsilon \hat{N} - G\left(S(S + 1) - S_z(S_z - 1)\right)$$
$$= \varepsilon \hat{N} - G(S + S_z)(S - S_z + 1). \qquad (2.36)$$

The standard convention is to write

$$S = \frac{1}{2}(\Omega - \upsilon), \qquad (2.37)$$

where the quantum number υ is called the seniority, a concept which was introduced by Racah (1942, 1943) in papers on the group theoretical classification of atomic wavefunctions (see de Shalit and Talmi (1963)). One can then write

$$(S + S_z) = \frac{1}{2}(\hat{N} - \upsilon) \quad \text{and} \quad (S - S_z) = \Omega - \frac{1}{2}(\hat{N} + \upsilon), \qquad (2.38)$$

and

$$E = \varepsilon N - E(N, \upsilon),$$

with

$$E(N, \upsilon) = -\tfrac{1}{4} G(N - \upsilon)(2\Omega - N - \upsilon + 2). \tag{2.39}$$

The combination $(S + S_z)$ must be an integer, hence $N - \upsilon$ is an even integer. It is useful to consider the cases where N is even and N is odd.

(i) *N-even*: The ground state has $\upsilon = 0$ and has energy

$$E(N, 0) = -\tfrac{1}{4} G(2N\Omega - N^2 + 2N). \tag{2.40}$$

The first excited state has $\upsilon = 2$ with excitation energy

$$\Delta E = G\Omega = 2\Delta. \tag{2.41}$$

High excited states have $\upsilon = 4, 6, \ldots$

(ii) *N-odd*: The ground state has $\upsilon = 1$ and energy

$$E(N, 1) = -\frac{G}{4}(N - 1)(2\Omega - N + 1), \tag{2.42}$$

and the first excited state has $\upsilon = 3$ again with excitation energy

$$\Delta E = G\Omega = 2\Delta. \tag{2.43}$$

This excitation energy is the pairing energy and the gap parameter Δ is analogous to the gap parameter appearing in the BCS theory of pairing to be discussed in Chapter 3 (see also Appendix G, in particular equation (G.10); see also Appendix H, equation (H.4)).

2.5 Pairing binding energy formula

The lowest eigenvalue of the pairing Hamiltonian (2.25) can be written as

$$E_{\mathrm{g}} = \varepsilon N + \tfrac{1}{4} GN(N - 1) - \tfrac{1}{4} G(2\Omega + 1)[N], \tag{2.44}$$

where

$$[N] = N \qquad \text{if} \quad N \text{ is even,}$$
$$[N] = N - 1 \quad \text{if} \quad N \text{ is odd.} \tag{2.45}$$

An accurate binding energy formula (Talmi (1972)) follows from (2.44) by adding an average interaction $\overline{E}$ between all pairs of nucleons. The result is

$$E_{\mathrm{g}} = \varepsilon N + \tfrac{1}{2}\alpha N(N - 1) + \tfrac{1}{2}\beta[N], \tag{2.46}$$

where

$$\alpha = \tfrac{1}{2}G + \overline{E} \quad \text{and} \quad \beta = -\tfrac{1}{2}G(2\Omega + 1). \tag{2.47}$$

The last term in equation (2.47) is the pairing energy term found in the systematics of nuclear binding energies which depends on the evenness or oddness of the number of protons and neutrons. The large observed odd–even effect is a consequence of the pair correlations induced by the pairing force. Using the expression given in equation (1.28) to extract a pairing energy from equation (2.46), the terms proportional to ε and α cancel and there is a contribution only from the third term and we obtain

$$\Delta = -\tfrac{1}{2}\beta = \tfrac{1}{4}G(2\Omega + 1). \tag{2.48}$$

The empirical fit to Δ given in equation (1.30) can be used to obtain an estimate of the pairing force parameter G.

When Ω is large the value (2.48) for Δ obtained from the binding energy formula given in equation (2.46) is almost equal to the gap parameter (2.43) from excitation energies. They differ by a factor $(1 + 1/2\Omega)$ (see also equation (H.4)).

Lawson (1980) has fitted Talmi's formula to the binding energies of Ca isotopes and obtained

$$\beta = -3.23 \,\text{MeV}, \quad \Delta = 1.62 \,\text{MeV}. \tag{2.49}$$

Equation (1.30) gives $\Delta = 1.81 \,\text{MeV}$ for $A = 44$, which is close to Lawson's number. Using $\Omega = j + \tfrac{1}{2} = 4$ for the 7/2 shell, (2.48) and (2.49) yield a value of $G = 0.72 \,\text{MeV}$, which corresponds to a relation $G \approx 31/A \,\text{MeV}$ (see equation (2.27)).

2.6 Quasi-spin wavefunctions

States in a quasi-spin multiplet have the same value of S but different values of S_z. In other words, they have the same seniority but different particle number. The operator S_+ increases S_z by one unit and the particle number by two units without changing S or υ. In a similar way S_- conserves S and υ but reduces the particle number by 2 units.

We consider some special cases. When $\upsilon = 0$ the quasi-spin quantum number $S = \tfrac{1}{2}\Omega$. This case gives the ground state of even nuclei. The state $|0\rangle$ with $N = 0$ corresponds to $S_z = -S = -\tfrac{1}{2}\Omega$. The ground state with N nucleons and $\upsilon = 0$ is obtained by creating $N/2$ pairs with the pair-creation operator S_+,

$$|N, 0\rangle = A(N, 0)S_+^{N/2}|0\rangle, \tag{2.50}$$

where $A(N, 0)$ is a normalization constant. Next we consider the case where $\upsilon = 2$ and $S = \tfrac{1}{2}\Omega - 1$. A state with $S_z = -S$ has nucleon number $N = 2$ and

the property

$$S_-|N = 2, \upsilon = 2\rangle = 0. \tag{2.51}$$

This state is highly degenerate, because all two-nucleon states where the nucleons are not paired have this property. Each of the unpaired nucleon states leads to a sequence of states with the same seniority and different particle number. Thus the general state with seniority υ should be written as $|N, \upsilon, \alpha\rangle$ where the quantum number α distinguishes between states with the same particle number and the same seniority. The energies do not depend on α and the pair operators S_+ and S_- do not change α. The state with lowest seniority for any particular N and α has $N = \upsilon$ unpaired nucleons and

$$S_-|\upsilon, \upsilon, \alpha\rangle = 0. \tag{2.52}$$

The state $|N, \upsilon, \alpha\rangle$ can be obtained from it by adding $(N - \upsilon)/2$ pairs

$$|N, \upsilon, \alpha\rangle = A(N, \upsilon)S_+^{(N-\upsilon)/2}|\upsilon, \upsilon, \alpha\rangle. \tag{2.53}$$

The following physical picture emerges from the arguments in this section. An eigenstate $|N, \upsilon, \alpha\rangle$ of the quasi-spin Hamiltonian has υ unpaired nucleons. The state of these nucleons is labelled by the quantum numbers α. The remaining $N - \upsilon$ nucleons form coherent pairs, with properties contained in the pair-creation operator S_+. The ground state in any nucleus is the state with the maximum number of pairs or alternatively the smallest number υ of unpaired nucleons.

From angular momentum theory (Brink and Satchler (1968)) the matrix elements of the ladder operator S_+ and S_- between normalized states are

$$\langle S, S_z + 1|S_+|S, S_z\rangle = \sqrt{(S - S_z)(S + S_z - 1)}, \tag{2.54}$$

$$\langle S, S_z - 1|S_-|S, S_z\rangle = \sqrt{(S + S_z)(S - S_z + 1)}.$$

In the following we write the matrix elements in terms of the particle number N and the seniority υ and replace the quasi-spin raising and lowering operators by the pair-creation and pair-annihilation operators $P^+ = S_+$ and $P = S_-$ so that they become

$$\langle N + 2, \upsilon, \alpha|P^\dagger|N, \upsilon, \alpha\rangle = \tfrac{1}{2}\sqrt{(2\Omega - N - \upsilon)(N - \upsilon + 2)}, \tag{2.55}$$

$$\langle N - 2, \upsilon, \alpha|P|N, \upsilon, \alpha\rangle = \tfrac{1}{2}\sqrt{(N - \upsilon)(2\Omega - N - \upsilon + 2)}.$$

The matrix elements (2.55) are called pair-transfer matrix elements and involve the addition or removal of a correlated pair from the initial state. Transitions with large neutron pair-transfer matrix elements have large cross-sections in two-neutron transfer reactions, for example in (t, p) or (p, t) reactions. The cross-section is proportional to the square of the magnitude of the pair-transfer matrix element.

Pairing correlations enhance pair-transfer processes (Broglia *et al.* (1973)). Consider, for example, the pair-addition matrix element between states of seniority $\upsilon = 0$. The basic matrix element is between an initial state with $N = 0$ and a final state with $N = 2$. Its value from equation (2.55) is

$$\langle 2, 0|P^\dagger|0, 0\rangle = \sqrt{\Omega}. \tag{2.56}$$

The corresponding matrix element for adding a pair to the state with $N/2$ correlated pairs is

$$\langle N + 2, 0|P^\dagger|N, 0\rangle = \tfrac{1}{2}\sqrt{(2\Omega - N)(N + 2)}. \tag{2.57}$$

The enhancement of the transfer cross-section is given by

$$\left|\frac{\langle N + 2, 0|P^\dagger|N, 0\rangle}{\langle 2, 0|P^\dagger|0, 0\rangle}\right|^2 = \frac{(2\Omega - N)(N + 2)}{2\Omega} \approx \frac{N + 2}{2}, \tag{2.58}$$

when $2\Omega \gg N$. In this limit the enhancement is proportional to the number of pairs in the final state. The pair-transfer operators S_+ and S_- do not change the seniority. Hence pair-transfer cross-sections which involve a change in seniority should be small (see Chapter, 4, equation (4.52), Fig. 4.2).

2.7 Pairing rotations

The expression (2.39) for the energy in the degenerate pairing model can also be written in terms of the seniority and the number π of pairs missing or in excess of the middle of the shell. The energy eigenvalues in this representation are

$$E(\upsilon, \pi) = -\tfrac{1}{4}G(\Omega - \upsilon)(\Omega + 2 - \upsilon) + G\pi\,(\pi + 1)\,. \tag{2.59}$$

The dependence of the energy and of the transfer matrix element on υ and π exhibits a natural grouping of levels. States with the same seniority and different number of particles can be interpreted as members of a collective band. Their energy displays a smooth dependence on π and they are connected by enhanced and fairly constant matrix elements of the two-nucleon transfer operator. States belonging to different bands are widely separated in energy and are not connected by the pairing operator. In fact the bands resemble those of a rotor and we can call them pairing rotational bands (Bohr (1968), Bes and Broglia (1966), Belyaev (1972)). Later we shall see that they can be interpreted as rotational bands in gauge space (Chapter 4).

Because of nuclear shell structure, the j-shells are bunched together within a major shell, and a general tendency towards the degenerate pairing model is realized in some nuclei, especially in single closed-shell nuclei.

The main term in nuclear binding energies is linear in the number of particles and this must be subtracted before a comparison with pairing rotations can be

made. The experimental binding energies of some Sn-isotopes with this subtraction are displayed in Fig. 4.2 (see Bes and Broglia (1977)). They follow the rotational parabola closely. The available data on two-neutron transfer cross-sections is also given in the same figure. These should be proportional to the squares of the matrix elements of P^+. Two important features of the band description are well satisfied, namely (i) the cross-sections between ground states are much stronger than those linking ground and excited states and (ii) the ground state cross-sections are rather constant (see Appendix H, Section H.3).

2.8 Exact solution of the pairing Hamiltonian

Exact solutions of the pairing problem have been studied by a number of authors (Kerman *et al.* (1960), Lipkin (1960), Nogami (1963, 1964)). In what follows we will discuss an exact solution of the pairing force problem for a non-degenerate set of single-particle levels ε_j and a constant pairing strength $G_{jj'} = G$ (Richardson (1963, 1965, 1977), Richardson and Sherman (1964)). The method did not involve diagonalizing the pairing Hamiltonian, but instead led to a set of non-linear equations for parameters in the pairing wavefunctions. Recently there has been renewed interest in Richardson's method both for condensed matter and nuclear physics applications (Sierra *et al.* (2000), Dukelsky *et al.* (2002)). Also new and efficient algorithms for solving Richardson's equations have been developed. This section introduces Richardson's method and draws attention to some recent developments. Here we quote some of the important equations and refer to the original papers for details.

The Hamiltonian in question is

$$H = \sum_j N_j \varepsilon_j - G \sum_{jj'} P_j^\dagger P_j, \tag{2.60}$$

the exact eigenstates with n pairs being

$$\psi = \prod_\alpha^n \left[\sum_j \frac{1}{2\varepsilon_j - e_\alpha} P_j^\dagger \right] |0\rangle, \tag{2.61}$$

where $|0\rangle$ is a state without any paired particles. This wavefunction has a very interesting structure. It depends on the single-particle energies and on parameters e_α. For example, in a system with 8 pairs distributed among 16 pair levels Richardson's wavefunction for the ground state depends on only 8 parameters. On the other hand, the dimension of the shell model space for the seniority zero levels is about 12 000.

The quantities e_α in the above wavefunction are solutions of a set of n non-linear equations which can be written as

$$\sum_j \frac{d_j}{2\varepsilon_j - e_\alpha} + \sum_{\beta \neq \alpha}^n \frac{1}{e_\beta - e_\alpha} + \frac{1}{2G} = 0. \qquad (2.62)$$

The parameters $d_j = (\nu_j - \Omega_j)/2$ depend on the pair degeneracies $\Omega_j = (2j + 1)/2$. The seniority ν_j is the number of unpaired particles in the level j. This condition allows for the blocking of single-particle levels by unpaired particles. The notation here is the one used by Dukelsky *et al.* (2002). The total energy of the state ψ is

$$E(e_\alpha) = \langle 0|H|0\rangle + \sum_\alpha e_\alpha. \qquad (2.63)$$

In the ground state, pairs fill up the lowest available levels up to the Fermi level when the interaction strength G is zero. When the interaction strength is small the pair-occupation numbers for pair states below the Fermi level are almost unity and the occupation numbers of states above the Fermi level are small. When the interaction strength increases, the occupation numbers change smoothly from unity to zero as ε_j increases through the Fermi energy. The pair-occupation numbers are given by

$$n_j = \frac{\partial E(e_\alpha)}{\partial \varepsilon_j} = \sum_\alpha \frac{\partial e_\alpha}{\partial \varepsilon_j}. \qquad (2.64)$$

Differentiating equation (2.62) with respect to ε_j yields a set of linear equations for the derivatives $\partial e_\alpha/\partial \varepsilon_j$ (Richardson (1977)).

The solutions e_α may be real or complex. Complex solutions occur in complex conjugate pairs. Until recently most numerical applications have focused on problems with doubly degenerate single-particle levels and have used Richardson's (1977) technique for solving the equations of the theory, but recently problems of more direct relevance to nuclear structure have been studied. Dukelsky *et al.* (2002) have solved the pairing force problem for ^{114}Sn and ^{116}Sn in a large basis of single-particle states ($d_{5/2}$, $g_{7/2}$, $s_{1/2}$, $d_{3/2}$ and $h_{11/2}$) and calculated the occupation numbers of the single-particle states as a function of the pairing strength G. Their calculations illustrate how the e_α move in the complex plane as G changes. As Richardson's method is exact the energies and occupation numbers vary smoothly with G. More recently Rombouts *et al.* (2004) have found a new method for solving Richardson's equations which is especially convenient for shell model applications where the single-particle levels are degenerate. The studies in these last two references provide interesting insights into the variation of the e_α with the coupling strength G. One disadvantage of Richardson's method is that the parameters e_α do not seem to have a

simple physical interpretation. Another is that, although physical quantities like energies and occupation numbers vary smoothly with the interaction strength G, the e_α can have a singular cusp-like behaviour for certain values of G. Ways of avoiding this problem have been developed by Rombouts *et al.* (2004).

In 1977 Richardson was able to show that his theory was equivalent to the BCS theory in a suitable large N limit by using an analogue with a two-dimensional electrostatic problem. The energy levels j were represented by a system of fixed negative charges d_j and positions ε_j on the y-axis and the e_α were represented by n movable positive charges with positions x_α, y_α in the x–y plane equal to the real and imaginary parts of e_α. The attractive coupling strength G was represented by a uniform electric field $1/2G$ acting in the negative y-direction. The real and imaginary parts of the e_α correspond to the x- and y-coordinates of the charges. Equation (2.62) is the equilibrium equation for the forces acting on the positive charge α. The electrostatic potential energy of the movable charges corresponding to the force equation (2.62) is

$$U = \sum_j d_j \ln |2\varepsilon_j - e_\alpha| - \sum_{\alpha \neq \beta} \ln |e_\alpha - e_\beta| + \sum_\alpha e_\alpha/2G. \tag{2.65}$$

Stationary points of the electrostatic energy U are solutions of Richardson's equations. This electrostatic analogy was exploited by Dukelsky *et al.* (2002) in the solution of the pairing problem for ^{114}Sn and ^{116}Sn. It allows one to get a physical picture of the solutions of Richardson's equations. It also points to possible instabilties because the stationary points of U are saddle points rather than minima. Recently Volya *et al.* (2001) have developed a method based on quasi-spin for diagonalizing the Hamiltonian of a system with a constant pairing interaction. It is an alternative to solving Richardson's equations and can be extended to include other terms in the nuclear Hamiltonian (see Volya *et al.* (2002)).

2.8.1 The degenerate case

The pairing force problem for a set of degenerate single-particle levels was solved with the quasi-spin method in Section 2.4. Richardson's equations (2.62) also have a simple solution in this case. The solution gives the energy eigenvalues $E(e_\alpha)$ with seniority υ when there are n pairs in a level with degeneracy Ω. The total number of particles N is related to the number of pairs by $n = (N - \upsilon)/2$. If one takes the energy of the degenerate single-particle state to be $\varepsilon_0 = 0$, equation (2.62) reduces to

$$\frac{\Omega - \upsilon}{2e_\alpha} + \sum_{\beta \neq \alpha} \frac{1}{e_\beta - e_\alpha} + \frac{1}{2G} = 0. \tag{2.66}$$

Multiplying by e_α and summing over α gives

$$E(e_\alpha) = \sum_1^n e_\alpha = -2G \left(\frac{n(\Omega - \upsilon)}{2} - \sum_{\beta \neq \alpha} \frac{e_\alpha}{e_\alpha - e_\beta} \right) \qquad (2.67)$$

$$= -G \left(n(\Omega - \upsilon) - n(n-1) \right) \qquad (2.68)$$

$$= -\frac{G}{4}(N - \upsilon)(2\Omega - N - \upsilon + 2), \qquad (2.69)$$

which is identical to equation (2.39) in Section 2.4. In this example the Richardson wavefunction (2.61) does not depend on the individual e_α and reduces to the simple pairing force wavefunction in equation (2.50). When there is more than one single-particle level, as in the case of ^{116}Sn, the interplay between the different e_α and single-particle energies determines the structure of the pairing wavefunction.