ON THE CONTINUITY OF THE
WIENER-HOPF FACTORIZATION OPERATION

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Abstract

The problem of passing from an L_∞ function to a Wiener-Hopf factorization is considered. It is shown that a small L_∞ perturbation which does not change the factorization indices will lead to small L_p $(1 < p < \infty)$ perturbations in the Wiener-Hopf factors, but can lead to large L_∞ perturbations, unless the derivatives are controlled during the perturbation.

1. Introduction

The technique of Wiener-Hopf factorization, which will be defined in Section 3, has a wide variety of applications. These include the classical applications to the prediction theory of stochastic processes [12, 14, 15] and the solution of integral and convolution equations [7, 9]. When applied to non-negative Hermitian matrix functions the technique is termed spectral factorization because of its long association with stochastic theory [12, 15], but also has other applications including the solution of linear quadratic control problems [3]. Another sub-class of Wiener-Hopf factorization is canonical factorization, which includes spectral factorization. This technique has been applied to the determination of system stability via passivity theory [5]. More recently, the general Wiener-Hopf factorization technique has been applied to the identification of errors-in-variables models [10].

A question which naturally arises is the extent to which the operation of Wiener-Hopf factorization is continuous. It is known that if a matrix function A

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has a canonical factorization and another function B is close, in L_∞ norm, to A, then the factors of A and B will be close in L_p norm, $1 < p < \infty$ [7, 2, 1]. However in many applications, particularly in control system theory, it is the L_∞ norm which is most important [6]. It is therefore desirable to know conditions which will ensure that the operation of Wiener-Hopf factorization is continuous in L_∞ norm. This question has been tackled for the special case of spectral factorization in [1], where it is shown that the derivatives of the matrices to be factored must be bounded in L_2 norm. Moreover an example shows that the factors may be very far apart in L_∞ norm when the derivative condition is not satisfied.

It is the aim of this paper to extend these results first to canonical factorization in Section 3 and then to general Wiener-Hopf factorization in Section 4. Further motivation for the results in terms of their application significance will also be given. Section 2 will contain definitions of the various norms and spaces which will be used.

2. Spaces, norms and notation

Denote by $\mathbf{C}^{n \times n}$ the space of complex $n \times n$ matrices and any induced norm on $\mathbf{C}^{n \times n}$ by $|\cdot|$. A *matrix function* on the unit circle is a Lebesgue measurable mapping M from $[0, 2\pi]$ to $\mathbf{C}^{n \times n}$, taking $\omega \in [0, 2\pi]$ to $M(e^{j\omega}) \in \mathbf{C}^{n \times n}$. We define the spaces L_p, $1 \leqslant p \leqslant \infty$ of matrix functions as follows: For $1 \leqslant p < \infty$, let

$$\| M \|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |M(e^{j\omega})|^p \, d\omega \right)^{1/p} \tag{2.1a}$$

$$\| M \|_\infty = \operatorname*{ess\,sup}_\omega |M(e^{j\omega})| \tag{2.1b}$$

L_p is the space of matrix functions whose corresponding norm is finite, i.e. $M \in L_p$ if and only if $\| M \|_p < \infty$. The L_p spaces are Banach spaces, and L_2 is a Hilbert space. Moreover $L_j \subset L_i$ for $i < j$ and for $M \in L_p$, $1 \leqslant p < \infty$ we can write the Fourier series, convergent in $\| \cdot \|_p$

$$M(e^{j\omega}) = \sum_{k=-\infty}^{\infty} M_k e^{j\omega k} \tag{2.2}$$

where $M_k \in \mathbf{C}^{n \times n}$

The space C denotes the space of continuous matrix functions. The *Wiener algebra* W of matrix functions consists of matrix functions of the form (2.2) for

which

$$\|M\| = \sum_{k=-\infty}^{\infty} |M_k| \tag{2.3}$$

is finite. W is an algebra under multiplication of matrix functions. It is easy to verify that $W \subset C \subset L_\infty$.

For any matrix function M of the form (2.2), define the projection operator P_+ by

$$P_+(M)(e^{j\omega}) = \sum_{k=0}^{\infty} M_k e^{j\omega k} \tag{2.4a}$$

and the projection P_- by

$$P_- = I - P_+ \tag{2.4b}$$

If Q is a linear operator on a Banach space $(B, \|\cdot\|_B)$, then the norm of Q, denoted $\|Q\|$ is defined by

$$\|Q\| = \sup_{X \in B} \frac{\|Qx\|_B}{\|x\|_B} \tag{2.5}$$

We shall use the fact later that P_+, P_- are bounded linear operators on W and on L_p, $1 < p < \infty$ [2, 7].

Let S be any of the spaces L_p, W, C and define S^+ (resp. S^-) to be the image space of P_+ (resp. P_-) acting on S. Also define GS, GS^+, GS^- to be the group of elements in S, S^+, $S^- \oplus I$ which have inverses, i.e. $M \in GS$ (GS^+, GS^-) if M and M^{-1} are in S (S^+, $S^- \oplus I$). The classic theorem of [13] states that if $M \in S$, then $M \in GS$ if and only if det $M(e^{j\omega}) \neq 0$ for all ω. Furthermore, if $M \in S^+$ (resp. S^-) we can regard it as the evaluation on $|z| = 1$ of a matrix function of z which is analytic in and continuous up to the boundary of $|z| \leqslant 1$ (resp. $|z| \geqslant 1 \cup \infty$), and then $M \in S^+$ (resp. S^-) is in GS^+ (resp. GS^-) if and only if det $M(z) \neq 0$, $|z| \leqslant 1$ (resp. $|z| \geqslant 1 \cup \infty$).

3. Continuity of canonical factorization

In this section we first briefly review canonical factorization, the material being taken from [5], [7] and [9], and then present two continuity results.

A matrix function $M \in GW$ is said to admit a (left) *canonical factorization* in case

$$M = M_+ M_-, \tag{3.1}$$

where $M_\pm \in GW^\pm$ and $M_-(\infty) = I$.

Define the following linear operators:

$$T_M(X) = P_+(XM) + P_-(X), \tag{3.2a}$$

$$R_M(X) = P_+(X) + P_-(MX), \tag{3.2b}$$

$$\hat{T}_M(X) = P_+(M^{-1}X) + P_-(X), \tag{3.2c}$$

$$\hat{R}_M(X) = P_+(X) + P_-(XM^{-1}). \tag{3.2d}$$

THEOREM A. *A matrix function $M \in GW$ admits a left canonical factorization if and only if the operators T_M, R_M are invertible on GW. In this case $\hat{T}_M$, $\hat{R}_M$ are also invertible and the canonical factors $M_\pm$ are given by*

$$M_+ = \hat{T}_M^{-1}(I), \tag{3.3a}$$

$$M_- = \hat{R}_M^{-1}(I), \tag{3.3b}$$

$$M_+^{-1} = T_M^{-1}(I), \tag{3.3c}$$

$$M_-^{-1} = R_M^{-1}(I), \tag{3.3d}$$

where I is the identity matrix function.

Note that the operators used in [1, 5] are $X - P_+(XZ)$, $X - P_-(ZX)$ with $M = I - Z$. These operators are entirely equivalent to T_M, R_M.

We can more generally, and more conveniently from a continuity point of view, consider what is called generalized canonical factorization [4], or *factorization in* L_p. We enlarge the space of matrix functions M which can be factored from GW to GL_∞, and correspondingly enlarge the space in which the factors lie from $GW^\pm$ to $M_+ \in L_q^+$, $M_+^{-1} \in L_p^+$, $M_- \in L_p^-$, $M_-^{-1} \in L_q^-$, where $p^{-1} + q^{-1} = 1$ and $1 < p < \infty$.

THEOREM B. *A matrix function $M \in GL_\infty$ admits a generalized canonical factorization $M = M_+ M_-$ in L_p if and only if T_M, R_M are invertible on L_p, L_q and then equations (3.3) hold. If $M \in GC$ admits a canonical factorization in L_p for some p, $1 < p < \infty$, it admits a canonical factorization in L_r, for all $1 < r < \infty$. That is, $(M_\pm)^{\pm 1} \in L_r$ for all $1 < r < \infty$.*

In light of Theorems A and B we need to examine the continuity of the operators T_M, R_M, $\hat{T}_M$, $\hat{R}_M$.

THEOREM 3.1. *Let $M \in GL_\infty$ admit a canonical factorization in L_p for some $1 < p < \infty$. Then for any $\varepsilon > 0$ there exist a $\delta > 0$ such that $\|M - N\|_\infty < \delta$ implies N has a canonical factorization in L_p and*

$$\|M_+ - N_+\|_q < \varepsilon, \qquad \|M_-^{-1} - N_-^{-1}\|_q < \varepsilon, \tag{3.4a}$$

$$\left\| M_+^{-1} - N_+^{-1} \right\|_p < \varepsilon, \qquad \left\| M_- - N_- \right\|_p < \varepsilon, \qquad (3.4\text{b})$$

where $p^{-1} + q^{-1} = 1$.

This theorem is known [4], but we will give the proof to note some facts about the operators T, R which will be useful later.

LEMMA 3.1. *Let T_1 be an invertible operator and $\|T_1 - T_2\| < \|T_1^{-1}\|^{-1}$. Then T_2 is invertible and*

$$\left\| T_1^{-1} - T_2^{-1} \right\| \leqslant \frac{\left\| T_1^{-1} \right\|^2 \|T_1 - T_2\|}{1 - \left\| T_1^{-1} \right\| \|T_1 - T_2\|}.$$

In particular, if $\|T_1 - T_2\| \leqslant (2\|T_1^{-1}\|)^{-1}$ then

$$\left\| T_1^{-1} - T_2^{-1} \right\| \leqslant 2 \left\| T_1^{-1} \right\|^2 \|T_1 - T_2\|.$$

PROOF. The result is standard; see, for example, [8].

LEMMA 3.2. *Let M, $N \in GL_\infty$ and $1 < p < \infty$. Then $\|T_M - T_N\| \leqslant K_1\|M - N\|_\infty$, $\|R_m - R_N\| \leqslant K_2\|M - N\|_\infty$, and provided $\|M - N\|_\infty$ is sufficiently small, $\|\hat{T}_M - \hat{T}_N\| < K_3\|M - N\|_\infty$, $\|\hat{R}_M - \hat{R}_N\| < K_4\|M - N\|_\infty$, where K_i, $i = 1, \ldots, 4$ are constants, and $\|\cdot\|$ denotes the operator norm induced by L_p or L_q as appropriate.*

PROOF. We give the proof for T, $\hat{T}$ as the argument for R, $\hat{R}$ is almost identical.

$$\begin{aligned}
\left\| T_M(X) - T_N(X) \right\|_p &= \left\| P_+(XM - XN) \right\|_p \\
&\leqslant \|P_+\| \, \|X(M - N)\|_p \\
&\leqslant \|P_+\| \, \|X\|_p \|M - N\|_\infty.
\end{aligned}$$

Just set $K_1 = \|P_+\|$ and recall $K_1 < \infty$ as P_+ is bounded on L_p.

$$\begin{aligned}
\left\| \hat{T}_M(X) - \hat{T}_N(X) \right\|_p &= \left\| P_+(M^{-1} - N^{-1})X \right\|_p \\
&\leqslant \|P_+\| \, \|X\|_p \|M^{-1} - N^{-1}\|_\infty \\
&\leqslant K_3 \|X\|_p \|M - N\|_\infty,
\end{aligned}$$

provided $\|M - N\|_\infty$ is sufficiently small.

PROOF OF THEOREM 3.1. We give details only for $\|M_-^{-1} - N_-^{-1}\|$, the others following almost identical arguments. Fix $\varepsilon > 0$. By Lemma 3.1, R_N is invertible

provided $\|R_M - R_N\| \leqslant \|R_M^{-1}\|^{-1}$. Note R_M is invertible by Theorem B. By (3.3).

$$\begin{aligned}
\|M_-^{-1} - N_-^{-1}\|_q &= \|R_M^{-1}(I) - R_N^{-1}(I)\|_q \\
&\leqslant \|R_M^{-1} - R_N^{-1}\| \|I\|_q \\
&\leqslant 2\|R_M^{-1}\|^2 \|R_M - R_N\| \|I\|_q \quad \text{for } \|R_M - R_N\| \leqslant \tfrac{1}{2}\|R_M^{-1}\|^{-1} \\
&\leqslant 2\|R_M^{-1}\|^2 \|I\|_q K_2 \|M - N\|_\infty.
\end{aligned}$$

So choose

$$\delta_1 = \frac{1}{2K_2\|R_M^{-1}\|\|I\|_q} \min\left(1, \frac{\varepsilon}{\|R_M^{-1}\|}\right).$$

For M_+, N_-, M_+^{-1} we get similarly δ_2, δ_3, δ_4 and set $\delta < \min(\delta_1, \delta_2, \delta_3, \delta_4)$.

From a control systems viewpoint, Theorem 3.1 is of limited value. Consider, for example, the problem of controller design using linear-quadratic-gaussian-methods [3]. There is given the transfer-function matrix of a plant, and from this one calculates a controller transfer-function matrix. The controller is implemented, perhaps on a computer, and connected to the plant. Among the various requirements on the interconnected (closed loop) system is stability (freedom from unbounded signals). The calculation procedure leading to the controller uses two spectral factorizations, the "spectra" being determined by the plant transfer matrix. However, the transfer function is never known exactly, since it is only in some way an approximation to the physical plant. It follows that the controller can be in error.

The stability requirement for the plant-controller interconnection is one which, at least in the scalar case (i.e. 1×1 case) has a graphical interpretation. One defines from the plant and controller a loop-gain transfer function and studies, for the purpose of assessing stability, the number of encirclements of the point -1 in the complex plane of the graph of this transfer function, viewed as a mapping from the unit circle to the complex plane. If the overall mapping from plant to loop gain transfer function is such that a small L_∞ variation in the plant can lead to a small L_p, $p \neq \infty$, variation in the loop gain (but sizeable L_∞ variation), then a small perturbation in the plant can change a stable situation to an unstable one, through a change in the number of encirclements of the -1 point. Because the loop gain in part depends on two spectral factorizations, Theorem 3.1 leaves open this possibility.

This motivates the study of L_∞ continuity, which is the subject of Theorem 3.2. The example of [1] however shows that Theorem 3.1 is tight. That is, one cannot extend to the case $p = \infty$ without assumptions on M and N in addition to those

imposed in Theorem 3.1. The condition we impose is absolute continuity, or equivalently that M and N are indefinite integrals (see e.g. [11]).

Let S_a be the "phase shift by a" operator on the space of matrix functions. That is, if M is a matrix function,

$$(S_aM)(e^{j\omega}) = M(e^{j(\omega+a)}).$$ (3.5)

Now let dM be the matrix function defined by the pointwise limit, where it exists;

$$dM(e^{j\omega}) = \lim_{\Delta\omega \to 0} \frac{(S_{\Delta\omega}M)(e^{j\omega}) - M(e^{j\omega})}{\Delta\omega}.$$ (3.6)

If M is absolutely continuous, then dM exists everywhere, is in L_1 and the fundamental theorem of calculus holds [11].

THEOREM 3.2. *Let $M \in GC \subset GL_\infty$ and suppose M has a canonical factorization in L_p, $1 < p < \infty$. Further suppose M is absolutely continuous and $dM \in L_r$, for some $r > 1$. Then for any $\varepsilon > 0$ and any $K < \infty$, there exists a $\delta = \delta(K, \varepsilon, M) > 0$ such that $N \in L_\infty$ and absolutely continuous, $\|M - N\|_\infty < \delta$, $dN \in L_r$ and $\|dM - dN\|_r < K$ imply $M_\pm$, $N_\pm$, $M_\pm^{-1}$, $N_\pm^{-1} \in GC$. Furthermore*

$$\|M_\pm - N_\pm\|_\infty < \varepsilon, \qquad \|M_\pm^{-1} - N_\pm^{-1}\|_\infty < \varepsilon.$$ (3.7)

The appendix contains a straightforward proof for the scalar spectral factorization case (i.e. M scalar and positive definite). Before proceeding to the more "modern" and independent proof of the full result, we derive a result about the differentiability of the factors $M_\pm$, $N_\pm$ which we will use in the proof.

LEMMA 3.3. *Let $A \in GC$ admit a canonical factorization in L_p, $1 < p < \infty$ and suppose $dA \in L_r$, $r > 1$. Then*

$$dA_+, dA_-^{-1}, dA_+^{-1}, dA_- \in L_r.$$ (3.8)

Furthermore

$$dA_+^{-1} = -T_A^{-1}P_+(A_+^{-1}dA)$$ (3.9a)

$$dA_-^{-1} = -R_A^{-1}P_-(dAA_-^{-1})$$ (3.9b)

$$dA_+ = -\hat{T}_A^{-1}P_+(dA^{-1}A_+)$$ (3.9c)

$$dA_- = -\hat{R}_A^{-1}P_-(A_-dA^{-1})$$ (3.9d)

PROOF. The four cases are almost identical, so we give the argument only for A_+^{-1}, i.e. for (3.9a).

Note that (3.8) follows from (3.9), since by minor variation on Hölder's inequality $A_+^{-1}dA \in L_t$, $1 < t < r$, provided $dA \in L_r$ and $A_+^{-1} \in L_p$, where $p = rt(r-t)^{-1}$. By Theorem B, $A_+^{-1} \in L_p$, for all $1 < p < \infty$, P_+ is bounded on L_t and T_A is invertible on L_t. Thus, with dA_+^{-1} given by (3.9a), $dA_+^{-1} \in L_t$ for any $1 < t < r$. However, this implies $A_+^{-1} \in L_\infty$. Repeating the above argument allowing $p = \infty$ gives $dA_+^{-1} \in L_r$. We now prove (3.9a).

By Theorem B, $T_A(A_+^{-1}) = I$. Also, since $S_{\Delta\omega}$ is a rigid rotation of the unit circle,

$$T_{S_{\Delta\omega}A}\left(S_{\Delta\omega}A_+^{-1}\right) = I$$

Subtracting and using (3.2a) we obtain, after a trivial manipulation

$$P_+\left[\left(S_{\Delta\omega}A_+^{-1} - A_+^{-1}\right)S_{\Delta\omega}A\right] + P_-\left[S_{\Delta\omega}A_+^{-1} - A_+^{-1}\right] = -P_+\left[A_+^{-1}(S_{\Delta\omega}A - A)\right],$$

or equivalently

$$\left(T_{S_{\Delta\omega}A}\right)\left[S_{\Delta\omega}A_+^{-1} - A_+^{-1}\right] = -P_+\left[A_+^{-1}(S_{\Delta\omega}A - A)\right].$$

Hence

$$\frac{S_{\Delta\omega}A_+^{-1} - A_+^{-1}}{\Delta\omega} = -\left(T_{S_{\Delta\omega}A}\right)^{-1}P_+\left[A_+^{-1}\left(\frac{S_{\Delta\omega}A - A}{\Delta\omega}\right)\right], \qquad (3.10)$$

provided $T_{S_{\Delta\omega}A}$ is invertible. However, by Lemma 3.1 we know that for $\|T_{S_{\Delta\omega}A} - T_A\|$ sufficiently small, $T_{S_{\Delta\omega}A}$ is invertible, and

$$\left\|\left(T_{S_{\Delta\omega}A}\right)^{-1} - T_A^{-1}\right\| \cdot \leq K\|T_{S_{\Delta\omega}A} - T_A\|.$$

Now by Lemma 3.2, $\|T_{S_{\Delta\omega}A} - T_A\| \leq C\|S_{\Delta\omega}A - A\|_\infty \to 0$ as $\Delta\omega \to 0$ by the continuity of A (recall $A \in GC$). Thus $\|(T_{S_{\Delta\omega}A})^{-1} - (T_A)^{-1}\| \to 0$ as $\Delta\omega \to 0$, and letting $\Delta\omega \to 0$ in (3.10) we obtain (3.9a).

REMARKS.
1. If A_+^{-1} is differentiable, we can obtain (3.9a) by differentiating the equation $T_A(A_+^{-1}) = I$. The main task of the proof is to show A_+^{-1} is differentiable. This amounts to showing $\|(T_{S_{\Delta\omega}A})^{-1} - T_A^{-1}\| \to 0$ as $\Delta\omega \to 0$.
2. The restriction $r > 1$, rather than $r \geq 1$, is because P_+ is not bounded on L_1, and T_A is not invertible on L_1.

PROOF OF THEOREM 3.2. By Theorem 3.1, we know that for δ sufficiently small, N has a generalized canonical factorization in L_p, for all $1 < p < \infty$. Applying Lemma 3.3 to M and N we find that their canonical factors are in GL_∞ (indeed GC).

Again, the argument is almost identical for M_+, M_-, etc., so we give the details only for $\|M_+^{-1} - N_+^{-1}\|_\infty$.

Let $W = M_+^{-1} - N_+^{-1}$. By the equivalence of norms on $\mathbf{C}^{n \times n}$, there are constants K_1, K_2 such that

$$(1/K_2)\,\mathrm{tr}\big[W(e^{j\omega})W(e^{j\omega})^*\big] \leqslant \big|W(e^{j\omega})\big|^2 \leqslant K_1\,\mathrm{tr}\big[W(e^{j\omega})W(e^{j\omega})^*\big]. \quad (3.11)$$

Hence

$$\big|W(e^{j\omega})\big|^2 \leqslant K_1\,\mathrm{tr}\big[W(e^{j\omega})W(e^{j\omega})^*\big]$$

$$\leqslant K_1\,\mathrm{tr}\int_\theta^\omega \frac{d}{d\tau}\big[W(e^{j\tau})W(e^{j\tau})^*\big]\,d\tau + K_1\,\mathrm{tr}\,W\big((e^{j\theta})We^{j\theta}\big)^*,$$

since W is absolutely continuous. Now let $t \in (1, r)$ and $p^{-1} = 1 - t^{-1}$. Choose θ such that

$$\mathrm{tr}\big[W(e^{j\theta})W(e^{j\theta})^*\big] \leqslant K_2\big|W(e^{j\theta})\big|^2 \leqslant K_2\|W\|_p^2.$$

It is trivial to verify by contradiction that such a θ exists. Hence we obtain

$$\big|W(e^{j\omega})\big|^2 \leqslant K_1\int_\theta^\omega \mathrm{tr}\big[\big(dW(e^{j\tau})We^{j\tau}\big)^*\big]\,d\tau$$

$$+ K_1\int_\theta^\omega \mathrm{tr}\big[W(e^{j\tau})dW(e^{j\tau})^*\big]\,d\tau + K_1K_2\|W\|_p^2. \quad (3.12)$$

Now dW, $dW^* \in L_r$, hence $\in L_t$ and $W \in L_p$, $\forall p > 1$, so by Hölder's inequality applied to (3.12) we obtain, with $p^{-1} = 1 - t^{-1}$,

$$\big|W(e^{j\omega})\big|^2 \leqslant 2K_1\bigg(\int_\theta^\omega \mathrm{tr}\big[dW(e^{j\tau})dW(e^{j\tau})^*\big]^{t/2}\,d\tau\bigg)^{1/t}$$

$$\times \bigg(\int_\theta^\omega \mathrm{tr}\big[W(e^{j\tau})W(e^{j\tau})^*\big]^{p/2}\,d\tau\bigg)^{1/p}$$

$$+ K_1K_2\|W\|_p^2$$

$$\leqslant 2K_1K_2\bigg(\int_\theta^\omega \big|dW(e^{j\tau})\big|^t\,d\tau\bigg)^{1/t}\bigg(\int_\theta^\omega \big|W(e^{j\tau})\big|^p\,d\tau\bigg)^{1/p}$$

$$+ K_1K_2\|W\|_p^2$$

$$\leqslant K_1K_2\big(2\|dW\|_t + \|W\|_p\big)\|W\|_p. \quad (3.13)$$

Provided $\|dW\|_t$ is bounded, the result will follow from Theorem 3.1 by choosing δ such that $\|W\|_p$ is small enough to make the right hand side of (3.13) less than

ε. The boundedness of $\|dW\|_t$ follows from Lemma 3.2 as follows:

$$
\begin{aligned}
\|dW\|_t &= \|dM_+^{-1} - dN_+^{-1}\|_t \leqslant \|dM_+^{-1}\|_t + \|dN_+^{-1}\|_t \\
&\leqslant \|T_M^{-1} P_+ (M_+^{-1} dM)\|_t + \|T_N^{-1} P_+ (N_+^{-1} dN)\|_t \\
&\leqslant \|T_M^{-1}\| \|P_+\| \|M_+^{-1} dM\|_t + \|T_N^{-1}\| \|P_+\| \|N_+^{-1} dN\|_t \\
&\leqslant \|T_M^{-1}\| \|P_+\| \|M_+^{-1}\|_q \|dM\|_r + \|T_N^{-1}\| \|P_+\| \|N_+^{-1}\|_q \|dN\|_r, \\
& \hspace{8cm} q = rt(r-t)^{-1} \\
&\leqslant C,
\end{aligned}
$$

by Theorem 3.1. Note C depends on $\|M_+^{-1}\|_q$, $\|M_+^{-1} - N_+^{-1}\|_q$ (bounded by Theorem 3.1), $\|dM\|_r$ and $\|dM - dN\|_r$ ($< K$ by hypothesis).

REMARKS.

1. The Theorem does not impose a smallness assumption on $\|dM - dN\|_r$. That is, K is not dependent on ε. Naturally a smaller K will lead to a more generous δ. In other words, for a given ε, $\delta(\varepsilon, K)$ is a decreasing function of K.

2. If M, N are positive definite Hermitian, it is easy to show [9] that $M_+ = M_-^*$, and then Theorems 3.1 and 3.2 specialize to the corresponding spectral factorization results in [1].

3. Any rational matrix function with no poles on the unit circle is absolutely continuous on the unit circle and has L_r derivatives for all $1 \leqslant r \leqslant \infty$.

4. Continuity of Wiener-Hopf factorization

In this section we generalize, as far as is possible, Theorem 3.2 to Wiener-Hopf (W-H) factorization. We first however, review briefly this factorization and note some properties which will limit the generalization. The review material comes from [4, 9].

A matrix function $M \in GW$ is said to admit a (left) Wiener-Hopf (W-H) factorization in case

$$
M = M_+ D M_-, \tag{4.1}
$$

where $M_\pm \in GW^\pm$, $M_-(\infty) = I$ and either

$$
D(z) = \mathrm{diag}\left(\left(\frac{z - z_+}{z - z_-} \right)^{k_i} \quad i = 1, \ldots, n \right), \qquad k_i \geqslant k_{i+1}, \tag{4.2a}
$$

with $|z_+| < 1$, $|z_-| > 1$ arbitrary and k_i integers, called the *partial indices* of M, or

$$
D(z) = \mathrm{diag}(z^{k_i}, \, i = 1, \ldots, n), \tag{4.2b}
$$

in which case the factorization is called *standard*.

THEOREM D. *Every $M \in GW$ admits a W-H factorization, and the partial indices are unique.*

We can also discuss generalized W-H factorization in L_p of matrix functions in GL_∞ or GC, where D is as in (4.2) and $M_\pm$ satisfy the same conditions as for canonical factorization in L_p.

THEOREM E. *Every $M \in GC$ admits a generalized W-H factorization in L_p for every $1 < p < \infty$.*

A matrix function $M \in GC$ will be said to have *stable indices* if there is a $\delta > 0$ such that for any $N \in GC$ satisfying $\|M - N\|_\infty < \delta$, M and N have the same partial indices.

The major difficulty in extending the theorems of Section 3 to W-H factorization in L_p is the following result [4, 9].

Let S_n denote the set of ordered sequences $\{k_i\}$ of n integers with $k_i \geqslant k_{i+1}$, $i = 1, \ldots, n$. Suppose $\{k_i\}$ and $\{k_i'\} \in S_n$. Define a partial order $\succ$ by

$$\{k_i\} \succ \{k_i'\} \quad \text{if} \quad \sum_{j=1}^{i} k_j \geqslant \sum_{j=1}^{i} k_j', \ i = 1, \ldots, n - 1 \qquad (4.3a)$$

and

$$\sum_{j=1}^{n} k_j = \sum_{j=1}^{n} k_j'. \qquad (4.3b)$$

Further, define the "average" of an element $\{k_i\}$ of S_n, denoted $\{\hat{k}_i\}$

$$\hat{k}_i = q + 1, \qquad i = 1, \ldots, r, \qquad (4.4a)$$

$$\hat{k}_i = q, \qquad i = r + 1, \ldots, n, \qquad (4.4b)$$

where

$$\sum_{j=1}^{n} k_j = nq + r, \qquad 0 \leqslant r < n. \qquad (4.4c)$$

The average has the "minimality" property that if $\{k_i'\} \in S_n$ and $\{k_i\} \succ \{k_i'\}$, then $\{k_i'\} \succ \{\hat{k}_i\}$.

THEOREM F. *Let $M \in GL_\infty$ (or GC) admit a generalized W-H factorization in L_p, some $1 < p < \infty$. Then there exists a $\delta > 0$ such that for any $N \in GL_\infty$, $\|M - N\|_\infty < \delta$ the following "inequality" holds:*

$$\{k_i(M)\} \succ \{k_i(N)\} \succ \{\hat{k}_i(M)\}, \qquad (4.5)$$

where $\{k_i(A)\}$ *denotes the partial indices of a matrix function A. Conversely, let* $\{\tilde{k}_i\}$ *be arbitrary in* S_n *with* $\{k_i(M)\} \succ \{\tilde{k}_i\}$. *Then, for any* $\delta > 0$, *there exists an* $N \in GL_\infty(GC)$, *such that* $\|M - N\| < \delta$ *and* $k_i(N) = \tilde{k}_i$, $i = 1, \ldots, n$.

The difficulty in extending the theorems of Section 3 to the general W-H factorization is because of the extremely undesirable properties of partial indices —c.f. the converse statement in Theorem F.

It is very easy to deduce the following corollary.

COROLLARY G. $M \in GL_\infty$ *has stable indices if and only if* $k_i(M) = \hat{k}_i(M)$, $i = 1, \ldots, n$. *Equivalently,* $k_1 \leqslant k_n + 1$, *i.e. there is a difference of* 1 *or* 0 *between the largest and smallest indices.*

REMARKS. If all the indices of M are zero (i.e. M has a canonical factorization) then obviously M has stable indices.

In the light of Theorem F, it is unreasonable to expect the factorization operation to be continuous (in any norm) unless one has a matrix with stable indices. Note that by Theorem F, there is a matrix function with stable indices arbitrarily close (in L_∞ norm) to any factorizable matrix function. Stable indices are the generic case, indeed matrix functions with stable indices are dense in GL_∞, with L_∞ norm. Thus we generalize below Theorem 3 to matrix function $M \in GC$ with stable indices.

By way of motivation, let us note an application in system theory. Errors-in-variables identification [2] is the name given to a class of procedures for passing from noisy measurements of the input and output signals of a linear system characterized by an unknown convolution operator, to a description of that system (convolution operator or transfer function). The procedure of [10] uses W-H factorization of a matrix function, which is a sub-matrix of the power spectrum matrix associated with the measured signals, to calculate the convolution operator. The determination of the power spectrum matrix can never be totally error free, and so the factors resulting from the W-H factorization will be in error. Since these factors are used to determine the unknown convolution operator the calculated convolution operator will inherit errors. It is important that these errors be controlled in an L_∞ sense, particularly if a controller is subsequently to be connected to the identified system, as noted previously.

The next theorem provides conditions for the L_∞ continuity.

THEOREM 4.1. *Let* $M \in GC$ *be absolutely continuous and suppose* $dM \in L_r$, *for some* $r > 1$. *Further suppose M has stable partial indices. Then for any* $\varepsilon > 0$ *and any* $K < \infty$, *there exists a* $\delta = \delta(K, \varepsilon, M)$ *such that if N is absolutely continuous,* $dN \in L_r$, $\|M - N\|_\infty < \delta$, *and* $\|dM - dN\|_r < K$, *then* $M_\pm$, $N_\pm$, $M_\pm^{-1} \in GC$.

Furthermore

$$\left\| M_{\pm} - N_{\pm} \right\|_{\infty} < \varepsilon, \qquad \left\| M_{\pm}^{-1} - N_{\pm}^{-1} \right\|_{\infty} < \varepsilon. \tag{4.6}$$

To prove Theorem 4.1, we shall make use of the following result of [4].

THEOREM H. *Every element $A \in GC$ can be factored in the form $A = D_0 A_0$, where $A_0 \in GC$ admits a left canonical factorization in L_p, $1 < p < \infty$, and $D_0(t) = diag(z^{\tau}i, \; i = 1, \ldots, n)$ with τ, a permutation of the set $\{k_i(A)\}$ of partial indices of A. Moreover if $A = A_+ D A_-$, A_0 may be factored as $A_0 = \tilde{A}_+ A_-$.*

PROOF OF THEOREM 4.1. That N has a factorization for δ sufficiently small follows from Theorem E, since $N \in C$ and $\|M - N\|_{\infty}$ sufficiently small imply $N \in GC$. Also, since M has stable partial indices, we can choose δ small enough for N to have the same indices as M, by Theorem F. Then

$$M = M_+ D M_- = D_0 \tilde{M}_+ M_- = D_0 M_0,$$
$$N = N_+ D N_- = D_1 \tilde{N}_+ N_- = D_1 N_0.$$

We must now show that for δ sufficiently small, $D_1 = D_0$. This depends on an analysis of the proof of Theorem H (see (see [4], pp. 128–131). Briefly, the indices are reordered in going from D to D_1 (or D_0) on the basis of certain minors of $N_+(\infty)$ (resp. $M_+(\infty)$) being non-zero, and on the solution of linear equations derived from $M_+(\infty)$ (resp. N_+)). Thus if δ is sufficiently small, the reordering can be chosen to be the same for N and M, i.e. we may choose $D_0 = D_1$ provided δ is sufficiently small. Then

$$\left\| M_0 - N_0 \right\|_{\infty} = \left\| D_0^{-1}(M - N) \right\|_{\infty}$$
$$\leqslant \left\| D_0^{-1} \right\|_{\infty} \left\| M - N \right\|_{\infty}$$
$$\leqslant K_1 \left\| M - N \right\|_{\infty}, \text{ since } \left\| D^{-1} \right\|_{\infty} = K_1 < \infty.$$

Thus we can make $\|M_0 - N_0\|$ as small as we like. Moreover, M_0 and N_0 are clearly absolutely continuous, since M, N and D_0 are, and their derivatives are in L_r. Thus by Theorem 3.2 applied to M_0, N_0, for δ sufficiently small,

$$\left\| M_- - N_- \right\|_{\infty} < \varepsilon, \qquad \left\| M_-^{-1} - N_-^{-1} \right\|_{\infty} < \varepsilon.$$

Furthermore,

$$\left\| M_+ - N_+ \right\|_{\infty} = \left\| \left(M M_-^{-1} - N N_-^{-1} \right) D^{-1} \right\|_{\infty}$$
$$\leqslant \left\| \left(M M_-^{-1} - N N_-^{-1} \right) \right\|_{\infty} \left\| D^{-1} \right\|_{\infty}$$
$$\leqslant \left[\left\| M \left(M_-^{-1} - N_-^{-1} \right) \right\|_{\infty} + \left\| (M - N) N_-^{-1} \right\|_{\infty} \right] \left\| D^{-1} \right\|_{\infty}$$
$$\leqslant \left[\left\| M \right\|_{\infty} \left\| M_-^{-1} - N_-^{-1} \right\|_{\infty} + \left\| M - N \right\|_{\infty} \left\| N_-^{-1} \right\|_{\infty} \right] \left\| D^{-1} \right\|_{\infty}$$

and it follows that $\|M_+ - N_+\|_\infty < \varepsilon$ for δ sufficiently small. Also

$$\begin{aligned}
\left\|M_+^{-1} - N_+^{-1}\right\| &= \left\|DM_- M^{-1} - DN_- M^{-1}\right\|_\infty \\
&= \left\|D\left[(M_- - N_-)M^{-1} + N_-(M^{-1} - N^{-1})\right]\right\|_\infty \\
&\leqslant \|D\|_\infty \left[\|M_- - N_-\|_\infty \|M^{-1}\|_\infty + \|N_-\|_\infty \|M^{-1} - N^{-1}\|_\circ\right],
\end{aligned}$$

so $\|M_+^{-1} - N_+^{-1}\|_\infty < \varepsilon$ for δ sufficiently small. The theorem is proved.

REMARK. Let S be an algebra of functions. If every $F \in GS$ has a W – H factorization, with the spaces containing the factors $F_\pm$ subsets of S, the factorization is called *proper* (this is called canonical factorization in [7]). Factorization in W is proper, since $W^\pm \subset W$. Generalized factorization of GL_∞ or GC functions in L_p is not proper ($L_{p,q} \not\subset L_\infty$, $q^{-1} + p^{-1} = 1$, $1 < p < \infty$). If we define $C' = \{A \in C$ such that A is absolutely continuous and $dA \in L_r$, $r > 1\}$, then Lemma 3.3 and Theorem H give the following:

COROLLARY 4.2. *Factorization of matrix functions in GC' is proper.*

PROOF. Reduce to the canonical case using Theorem H and the result follows from Lemma 3.3 and elementary manipulations. Remember $D^{\pm 1} \in C'$.

It is known that if $A \subset C$ is a Banach algebra, then factorization in GA is proper if and only if A is decomposing, or equivalently if P_+ is bounded on A [4, 7]. W is decomposing, since $W = W^+ \oplus W^-$, but C and L_∞ are not (P_+ is not bounded in L_∞). In a decomposing algebra, the continuity of the factorization follows directly from the boundedness (hence continuity) of P_+. However C' is not decomposing and P_+ is not bounded on C' (in L_∞ norm), even though factorization is proper in C'. The problem is that C', with L_∞ norm, is not a Banach algebra. In fact it is a *non-closed* sub-algebra of the Banach algebra C. The bound $\|dM - dN\|_r < K$ in Theorems 3.2, and 4.1 is to counter the non-closed nature of C'. It prevents the situation of a sequence $\{N_n\}$ in C' such that $\|M - N_n\|_\infty < \delta$ for all n, but $\|dM - dN_n\|_r \to \infty$ as $n \to \infty$ from being considered by Theorems 3.2 and 4.1. In this case $\{N_n\}$ is escaping from C': it has a limit point in L_∞ but not in C'. This is the situation in the example in [1]. If C' is equipped with the norm $\|M\| = \|M\|_\infty + \|dM\|_r$, it becomes a Banach algebra. Hence it is decomposing and P_+ is bounded, since factorization is proper in C_r. This leads to a different continuity result, namely:

COROLLARY 4.3. *Let $M \in GC'$, some $r > 1$, have a W-H factorization with stable partial indices. Then for any $\varepsilon > 0$ there exist a $\delta > 0$ such that if $N \in C'$*

and $\|M - N\| < \delta$ *then*

$$\|M_{\pm} - N_{\pm}\| < \varepsilon \quad \text{and} \quad \|M_{\pm}^{-1} - N_{\pm}^{-1}\| < \varepsilon,$$

where for $A \in C'$, $\|A\| = \|A\|_{\infty} + \|dA\|_{r}$.

PROOF. $(C', \|\cdot\|)$ is a Banach sub-algebra of C. Factorization is proper in C'. Thus P_{+} is continuous on $(C', \|\cdot\|)$ and consequently so are the operators T, R, $\hat{T}$, $\hat{R}$ defined in Section 3. The result follows.

REMARKS.
1. This result requires the derivatives dM and dN to be close, and implies that dM_{+} and dN_{+} are close. It thus has more restrictive assumptions and a stronger conclusion than Theorems 3.2 and 4.1.
2. The continuity of P_{+} on C' can be proved without reference to factorization ideas. However, the essential ideas for an independent proof are contained in the proofs of Lemma 3.3 and Theorem 3.2.

5. Conclusion

The key conclusion of this paper is that small L_{∞} perturbations in a matrix function will lead to small L_{∞} perturbations in its Wiener-Hopf factors provided the factorization indices are stable and the matrix functions (i.e. the original and perturbed) have derivatives in L_{r}, for some $r > 1$. This condition is seen to be tight by consideration of the example in [1].
The result is seen to be important in assessing the continuity (error containment) properties of the many procedures in system and control theory which use Wiener-Hopf factorization. These include spectral factorization, LQG controller design [3], multiplier theory [5], and errors-in-variables identification [10].
A more desirable property for the above applications would be a "local L_{∞} continuity property", or "localization principle" for Wiener-Hopf factorization. That is, that small perturbations of a matrix function "mostly" on a subinterval of $[0, 2\pi]$ will produce perturbations of the Wiener-Hopf factors "mostly" in that subinterval. This property is conjectured on the basis of scalar spectral factorization evidence (see appendix). Such a result would be extremely desirable for system theoretic applications since it would mean, for example, that the often significant high frequency errors of a measured power spectrum matrix, arising because of sensor limitations, would not produce large low frequency errors in the calculated Wiener-Hopf (or spectral) factors.

Appendix

The purpose of this appendix is to provide what may be a more intuitive proof of Theorem 3.2 for an important special case. The special case is scalar spectral factorization, where one can use the Bode gain/phase relationships to construct the spectral factors. The conditions imposed in Theorem 3.2 are related to properties of these gain/phase relationships which we will derive. This method of proof also leads to the conjecture of a localization principle for scalar spectral factorization, as was mentioned in the conclusion.

THEOREM A1. *Let* $\phi_1(e^{j\omega}) = \sum_{-\infty}^{\infty} a_k e^{jk\omega}$, $a_k = a_{-k}$, *real*, $k = 0, 1, \ldots$ *be a positive definite scalar spectrum which is absolutely continuous with derivative* $d\phi_1$ *in* $L_r[-\pi, \pi]$, *some* $1 < r \leq \infty$. *Now let* $V_1(e^{j\omega})$ *be the stable minimum phase spectral factor of* ϕ_1, *that is* $V_1(e^{j\omega})$ *is the evaluation on* $|z| = 1$ *of a function* $V_1(z)$ *which is analytic and non-zero in* $|z| \leq 1$ *and satisfies*

$$V_1(e^{j\omega})V_1(e^{j\omega})^* = \phi_1(e^{j\omega}).$$

For any $\varepsilon > 0$ *and any* $K < \infty$, *there exists a* $\delta = \delta(\phi_1, K, \varepsilon) > 0$ *such that for any real spectrum* ϕ_2 *such that*

$$\|\phi_1 - \phi_2\|_\infty < \delta \quad and \quad \|d\phi_1 - d\phi_2\|_r < K,$$

then

$$\|V_1 - V_2\|_\infty < \varepsilon,$$

where V_2 *is the spectral factor of* ϕ_2.

Note that this is a very special case of Theorem 3.2. Since the proof by "classical" methods is lengthy in detail, but straightforward, we will give an outline of the main ideas.

OUTLINE OF PROOF. Let $V_i(e^{j\omega}) = R_i(e^{j\omega})\exp\{jI_i(e^{j\omega})\}$ where R_i, I_i are real valued and

$$0 < R_i(e^{j\omega}) = R_i(e^{-j\omega}), \quad I_i(e^{j\omega}) = -I_i(e^{-j\omega}), \qquad i = 1, 2$$

It is easy to verify that

$$\|V_1 - V_2\|_\infty \leq \|R_1 - R_2\|_\infty + \|R_1\|_\infty \|I_1 - I_2\|_\infty.$$

So we want to choose δ such that $\|R_1 - R_2\|_\infty < \varepsilon/2$ and $\|R_1\| \|I_1 - I_2\|_\infty < \varepsilon/2$.

Observe the formula $V_iV_i^* = \phi_i$ implies $\phi_i = R_i^2$. Hence

$$\|\phi_1 - \phi_2\|_\infty < \delta \Rightarrow \|R_1 - R_2\|_\infty < \delta^{1/2},$$

since

$$\| R_1 - R_2 \|_\infty^2 = \left\| (R_1 - R_2)^2 \right\|_\infty$$
$$\leqslant \left\| (R_1 - R_2)(R_1 + R_2) \right\|_\infty \quad \text{as } R_1, R_2 > 0$$
$$= \| \phi_1 - \phi_2 \|_\infty.$$

The problem is thus the $\| I_1 - I_2 \|_\infty$ term. Observe $\ln V_i = \ln R_i + j I_i$. Now we use the Bode gain/phase relation to construct I_i from $\ln R_i = 1/2 \ln \phi_i$.

$$I_i(e^{j\omega}) \cdot \frac{1}{2} \int_{-\pi}^{\pi} \frac{\ln \phi_i(e^{j\omega}) - \ln \phi_i(e^{j\theta})}{\cos \omega - \cos \theta} d\omega, \qquad (A.1)$$

or, since ϕ_i is differentiable with integrable derivative, we may integrate by parts to obtain

$$I_i(e^{j\theta}) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\phi_i(e^{j\omega})}{\phi_i(e^{j\omega})} \ln \left(\left| \frac{\sin(\theta + \omega)/2}{\sin(\theta - \omega)/2} \right| \right) d\omega. \qquad (A.2)$$

Formula (A.1) has the advantage of not requiring derivatives, but the serious disadvantage that it is not a bounded operator from L_∞ to L_∞, i.e. small ϕ_i will not necessarily give small I_i. An example is given in [1]. Fortunately, (A.2) is a bounded operator from L_r to L_∞ (for any $r > 1$). To prove this, one must show that

$$\sup_{\theta \in [-\pi, \pi)} \int_{-\pi}^{\pi} \left| \ln \left| \frac{\sin(\theta + \omega)/2}{\sin(\theta - \omega)/2} \right| \right|^r d\omega \leqslant M_r < \infty. \qquad (A.3)$$

This is not completely trivial, but is straightforward using the fact that

$$\left| \ln \left| \frac{\sin(\theta + \omega)/2}{\sin(\theta - \omega)/2} \right| \right|^r |\theta - \omega|^q |\theta + \omega|^q \to 0 \quad \text{as } \omega \to \pm\theta \text{ for any } r > 1, q > 0.$$

Then use Hölder's inequality to show $\int_{-\pi}^{\pi} d\omega / [|\theta - \omega|^q |\theta + \omega|^q]$ is bounded for some $0 < q < 1$ (take $q = \frac{1}{4}$). This proves (A.3).

We now break the integral in (A.1) up into three parts: one integral over $[\theta - \delta_1, \theta + \delta_1]$, one over $[-\theta - \delta_1, -\theta + \delta_1]$, and the rest. The rest is easy, since so long as ω is bounded away from θ (as we have just made sure) the integrand in (A.1) is bounded, so we can choose δ small enough so that this term is small (say less than $\varepsilon(6\|R_1\|_\infty)^{-1}$).

For the small intervals near $\pm\theta$, we integrate by parts to get the integral in (A.2), but over the small intervals $[\pm\theta - \delta_1, \pm\theta + \delta_1]$, plus the end point evaluation term. The bound (A.3) means that we can choose δ_1 small enough for the integral term to be small (say less than $\varepsilon(6\|R_1\|)^{-1}$), since by hypothesis $\|d\phi_1 - d\phi_2\|_r < K$. Finally, we need to consider the integrated term (from the integration by parts). This causes no problem except when θ is near zero (in fact,

when $\theta \in [-\delta_1, \delta_1]$), or π or $-\pi$. This is because the singularity $\pm \theta$ can coincide with an end point. However, we now simply increase the interval of integration to $[-3\delta_1, 3\delta_1]$ to avoid the problem.

Notice that because we can use formula (A.1) except over the small intervals, we do not need the derivative bound K to be small. It can be any finite number, and need not tend to zero as ε tends to zero.

The conjecture of a "local L_∞ continuity" result, or "localization principle" derives from the observation that since the integrands of (A.1), (A.2) have singularities at θ, the value of the spectrum ϕ at θ will have a much greater effect on the value of V at θ than will values of ϕ away from θ. That is, the value of ϕ at θ is "overweighted" in (A.1), (A.2), and consequently in $V(\theta)$.

Consequently "localized errors" in ϕ should lead to "localized errors" in V. This is what we mean by "local continuity".

From an applications viewpoint the desirability of such a result is obvious. In fact it is required if many practical applications of spectral or other Wiener-Hopf factorizations are to be justified—otherwise (for example) high frequency model errors could lead to completely erroneous low frequency LQG controller behaviour. The fact that this does not appear to happen provides circumstantial evidence for the localization principle.

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