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A NOTE ON THE IMPLICATIONAL CLASS GENERATED BY A CLASS OF STRUCTURES

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We use the notations of [2], particularly for operators on classes of structures; in addition, $\mathbf{P}_r^*(K)$ (respectively, $\mathbf{P}_r(K)$) denotes the class of all reduced products of families of structures in K (those with respect to a proper dual ideal, respectively). We prove:

THEOREM. *Let K be a class of structures. The universal Horn class generated by K is $\mathbf{ISPP}_p(K)$ and the implicational class*[2] *generated by K is $\mathbf{ISP^*P}_p(K)$.*

The proof of the theorem is based on Lemma 1, due to A. I. Mal'cev [6], and Lemma 2.

LEMMA 1 (A. I. Mal'cev). *The universal Horn class generated by the class K is $\mathbf{ISP}_r(K)$, and the implicational class generated by K is $\mathbf{ISP}_r^*(K)$.*

LEMMA 2. *For any class K of structures $\mathbf{P}_r(K) \subseteq \mathbf{IP}_s\mathbf{P}_p(K)$ and $\mathbf{P}_r^*(K) \subseteq \mathbf{IP}_s^*\mathbf{P}_p(K)$.*

Proof. Let $(\mathfrak{A}_i \mid i \in I)$ be a family of structures in K, let $\mathcal{Q}$ be a dual ideal in the lattice of subsets of I, and let $\prod_{\mathcal{Q}} (\mathfrak{A}_i \mid i \in I)$ be the reduced product with respect to $\mathcal{Q}$. Now

$$\prod\nolimits_{\mathcal{Q}} (\mathfrak{A}_i \mid i \in I) = \prod (\mathfrak{A}_i \mid i \in I)/\Theta(\mathcal{Q}),$$

where $\Theta(\mathcal{Q})$ is the congruence on $\prod (\mathfrak{A}_i \mid i \in I)$ determined by requiring that $a \equiv b(\Theta(\mathcal{Q}))$ iff $\{i \mid a(i)=b(i)\} \in \mathcal{Q}$. Let D be the set of all prime dual ideals containing $\mathcal{Q}$. Then $\mathcal{Q} = \cap (\mathcal{D} \mid \mathcal{D} \in D)$ and it follows immediately that $\Theta(\mathcal{Q}) = \bigwedge (\Theta(\mathcal{D}) \mid \mathcal{D} \in D)$. Thus $\prod_{\mathcal{Q}} (\mathfrak{A}_i \mid i \in I)$ is isomorphic to a subdirect product of the family $(\prod_{\mathcal{D}} (\mathfrak{A}_i \mid i \in I) \mid \mathcal{D} \in D)$. Observing that D is nonvoid iff $\mathcal{Q}$ is proper completes the proof.

The theorem now follows by noting that

$$\mathbf{ISPP}_p(K) \subseteq \mathbf{ISP}_r(K) \subseteq \mathbf{ISIP}_s\mathbf{P}_p(K) \subseteq \mathbf{ISPP}_p(K),$$

and similarly for $\mathbf{ISP^*P}_p(K)$.

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(2) An implicational class, also called a quasivariety, is a class determined by sentences which are the universal closures of formulas of the form $(\Phi_2 \wedge \cdots \wedge \Phi_n) \rightarrow \Phi_1$, $n \geq 1$, where all Φ_i are atomic formulas.

Two corollaries follow directly:

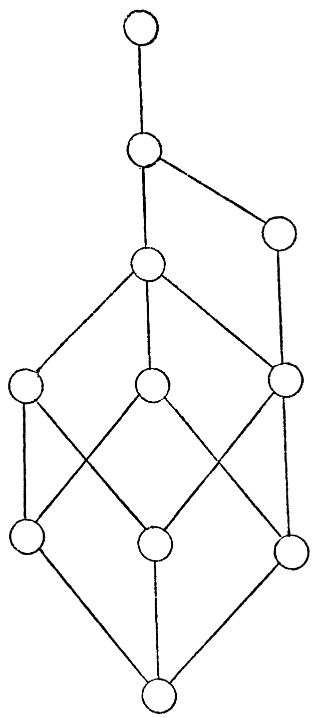
COROLLARY 1 (Fujiwara [1]). *The universal Horn class generated by K is* $\mathbf{I}\underrightarrow{\mathbf{L}}\mathbf{SP}(K)$ *and the implicational class generated by K is* $\mathbf{I}\underrightarrow{\mathbf{L}}\mathbf{SP}^*(K)$.

Proof. We need only recall that $\mathbf{P}_p(K) \subseteq \mathbf{I}\underrightarrow{\mathbf{L}}\mathbf{P}(K)$ ([2, p. 160, Exercise 100]) and that universal classes are closed under $\underrightarrow{\mathbf{L}}$.

COROLLARY 2. *Let K be a finite set of finite structures. Then the universal Horn class generated by K is* $\mathbf{ISP}(K)$, *and the implicational class generated by K is* $\mathbf{ISP}^*(K)$.

Proof. Since K consists of a finite number of finite structures, $\mathbf{P}_p(K) \subseteq \mathbf{I}(K)$.

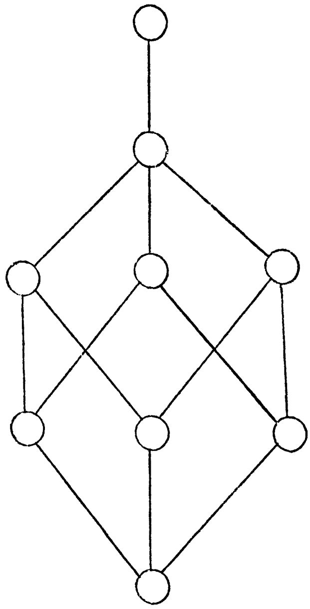
Corollary 2 is in a very convenient form for computation. For example, it provides a counterexample to a claim of Shafaat [7]. Specifically, we construct an



$\mathfrak{L}_1$

Figure 1

implicational class of pseudocomplemented distributive lattices that is not equational. Let $\mathfrak{L}_1$ be the pseudocomplemented distributive lattice depicted in Figure 1 and let $\mathfrak{L}_2$ be that depicted in Figure 2. Then, since $\mathfrak{L}_2$ cannot be embedded in $\mathfrak{L}_1$ so as to preserve pseudocomplementation and since $\mathfrak{L}_2$ is subdirectly irreducible (see [5], also [3]), $\mathfrak{L}_2 \notin \mathbf{ISP}^*(\mathfrak{L}_1)$. Since $\mathfrak{L}_2$ is a homomorphic image of $\mathfrak{L}_1$, we conclude that $\mathbf{ISP}^*(\mathfrak{L}_1)$ is not an equational class.



$\mathfrak{L}_2$

Figure 2

We remark in closing that by using Lemma 2 we can give a very short proof of Lemma 1. The fundamental result of Horn [4] states that a universal class is a Horn class iff it is closed under **P**. Now a class is a universal axiomatic class iff it is closed under **I**, **S** and $\mathbf{P}_p$; thus a class is universal Horn iff it is closed under **I**, **S**, **P** and $\mathbf{P}_p$. Consequently, $\mathbf{ISP}_r(K)$ is a universal Horn class and the consequence $\mathbf{ISP}_r(K) \subseteq \mathbf{ISPP}_p(K)$ of Lemma 2 shows that $\mathbf{ISP}_r(K)$ is the least universal Horn class containing K. An analogous proof holds for implicational classes.

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