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Multilinear Proofs for Convolution Estimates for Degenerate Plane Curves

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Abstract. Suppose that $\gamma \in C^2\big([0,\infty)\big)$ is a real-valued function such that $\gamma(0) = \gamma'(0) = 0$, and $\gamma''(t) \approx t^{m-2}$, for some integer $m \geq 2$. Let $\Gamma(t) = \big(t, \gamma(t)\big)$, $t > 0$, be a curve in the plane, and let $d\lambda = dt$ be a measure on this curve. For a function f on $\mathbf{R}^2$, let

$$Tf(x) = (\lambda * f)(x) = \int_0^\infty f\big(x - \Gamma(t)\big)\,dt, \quad x \in \mathbf{R}^2.$$

An elementary proof is given for the optimal L^p-L^q mapping properties of T.

Fix an integer $m \geq 2$. Suppose that $\gamma \in C^2\big([0,\infty)\big)$ is a real-valued function such that $\gamma(0) = \gamma'(0) = 0$, and $\gamma''(t) \approx t^{m-2}$. That is, there exist constants c_1, $c_2 > 0$ such that $c_1 \leq \gamma''(t)/t^{m-2} \leq c_2$ for $t > 0$. Let Γ be a curve in the plane given by $\Gamma(t) = \big(t, \gamma(t)\big)$, $t > 0$, and let λ denote the measure $d\lambda\big(\Gamma(t)\big) = dt$ on Γ. Define a singular convolution operator T by

$$(Tf)(x) = (\lambda * f)(x) = \int_0^\infty f\big(x - \Gamma(t)\big)\,dt, \quad x \in \mathbf{R}^2,$$

for suitably nice functions f, say continuous functions with compact support. The problem is to determine all pairs (p,q) such that T is bounded from $L^p(\mathbf{R}^2)$ to $L^q(\mathbf{R}^2)$. Recently a lot of work has been done on this type of problems (see *e.g.* [RS], [O1], [O3] and the references given there).

Let $A = \big(2/(m+1), 1/(m+1)\big)$, $B = \big(m/(m+1), (m-1)/(m+1)\big)$ be points in the plane. It is well known that for T to be bounded from $L^p(\mathbf{R}^2)$ to $L^q(\mathbf{R}^2)$, it is necessary that $(1/p, 1/q)$ is on the closed line segment AB. (In fact, this may be shown as follows. Assume that T is bounded from $L^p(\mathbf{R}^2)$ to $L^q(\mathbf{R}^2)$. Taking f to be the characteristic function of the square $[0,\delta] \times [0,\delta]$ for small $\delta > 0$ shows that $\delta^{1+1/q} \leq C\delta^{2/p}$. Thus $1 + 1/q \geq 2/p$, and so by duality $(1/p, 1/q)$ is in the closed triangle with vertices $(0,0)$, $(1,1)$, $(2/3, 1/3)$. Now taking f to be the characteristic function of the rectangle $[0,a] \times [0,\gamma(a)]$ shows that $a^{1+(m+1)/q} \leq Ca^{(m+1)/p}$ for $a > 0$, which implies that $1 + (m+1)/q = (m+1)/p$. Therefore, it follows that $(1/p, 1/q)$ is on AB. See *e.g.* [RS], [BMO], [O3].)

It is possible to prove the converse statement—that T is bounded from $L^p(\mathbf{R}^2)$ to $L^q(\mathbf{R}^2)$, if $(1/p, 1/q)$ is on the closed segment AB—by using the methods in [C2] based on the

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Littlewood-Paley theory (see also [Se]). Thus the following theorem holds. The purpose of this note is to give an elementary proof of this result.

Theorem 1 *There exists a constant $C = C(m, c_1, c_2)$, independent of f, such that*

$$\|\lambda * f\|_{L^q(\mathbf{R}^2)} \le C\|f\|_{L^p(\mathbf{R}^2)} \tag{1}$$

if and only if $(\frac{1}{p}, \frac{1}{q})$ is on the closed line segment AB.

The proof is an adaptation of the multilinear proof of (1) given by Oberlin [O2] in the case that λ is the arc length measure on the unit circle. (See [B] for a proof of (1) on the open segment AB. The latter proof also applies to some curves and surfaces which contain a point where the curvature vanishes to infinite order.) In what follows, the symbol C denotes a positive constant which may not be the same at each occurrence.

Proof By duality and interpolation it is enough to prove (1) when $(1/p, 1/q) = A = \big(2/(m+1), 1/(m+1)\big)$, or to prove the equivalent multilinear estimate

$$\left|\int_{\mathbf{R}^2} \prod_{j=1}^{m+1} Tf_j(x)\,dx\right| \le C \prod_{j=1}^{m+1} \|f_j\|_{\frac{m+1}{2}}. \tag{2}$$

By the multilinear trick of Christ (see [C1], [D1]), (2) follows from

$$\left|\int_{\mathbf{R}^2} \prod_{j=1}^{m+1} Tf_j(x)\,dx\right| \le C\|f_1\|_1 \prod_{j=2}^{m+1} \|f_j\|_{m,1}, \tag{3}$$

where $\|\cdot\|_{p,q}$ stands for the Lorentz space norm on $\mathbf{R}^2$. It is enough to show this when $f_j \ge 0$ and f_1 is the point mass at the origin, in which case (3) becomes

$$\int_0^\infty \prod_{j=2}^{m+1} Tf_j\big(\Gamma(t)\big)\,dt \le C \prod_{j=2}^{m+1} \|f_j\|_{m,1}. \tag{4}$$

(To see that (4) actually implies (3), replace each f_j in (4) by its translate $f_{j,x}(y) = f_j(x+y)$, and integrate in x after multiplying both sides by $f_1(x)$.)

The estimate (4), in turn, follows by the multiple Hölder inequality from

$$\left(\int_0^\infty \big[Tf\big(\Gamma(t)\big)\big]^m\,dt\right)^{1/m} \le C\|f\|_{m,1},$$

which is equivalent to the estimate

$$I \equiv \int_0^\infty Tf\big(\Gamma(t)\big)g(t)\,dt \le C\|f\|_{m,1}\|g\|_{L^{\frac{m}{m-1}}(P)},$$

for nonnegative functions f on $\mathbf{R}^2$ and g on $P = [0, \infty)$.

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The transformation $x_1 = t - s$, $x_2 = \gamma(t) - \gamma(s)$ of P^2 into $\mathbf{R}^2$ is one-to-one off the line $s = t$, and the absolute value J of the Jacobian is given by $J = |\gamma'(t) - \gamma'(s)|$. So

$$I = \int_0^\infty \int_0^\infty f\big(\Gamma(t) - \Gamma(s)\big) g(t)\, ds\, dt = \int f(x)\tilde{g}(x)\, dx,$$

where $\tilde{g}(x) = g(t)J^{-1}$. Hence, by Hölder's inequality for Lorentz spaces,

$$I \le C\|f\|_{m,1}\|\tilde{g}\|_{\frac{m}{m-1},\infty}.$$

It remains to show that

$$\|\tilde{g}\|_{\frac{m}{m-1},\infty} \le C\|g\|_{L^{\frac{m}{m-1}}(P)}.$$

That is, we need to prove

$$(5) \qquad |\{x \in \mathbf{R}^2 : \tilde{g}(x) > \alpha\}| \le C \int_0^\infty \left(\frac{g(t)}{\alpha}\right)^{m/(m-1)} dt.$$

The left-hand side of (5) is equal to the integral $\int_G J\, ds\, dt$, where

$$G = \{(s,t) \in P^2 : g(t)J^{-1} > \alpha\}.$$

We split the integral into the part with $t > s$ and the part with $s > t$. Since

$$J = \left|\int_s^t \gamma''(u)\, du\right| \approx |t^{m-1} - s^{m-1}|,$$

we have $J \approx t^{m-2}(t-s)$ when $t > s > 0$, and $J \approx s^{m-2}(s-t)$ when $s > t > 0$. So

$$\int_{G\cap\{t>s\}} J\, ds\, dt \le C \int_{\{(s,t)\in P^2 : 0 < t^{m-2}(t-s) < Cg(t)/\alpha\}} t^{m-2}(t-s)\, ds\, dt.$$

For each fixed $t > s$, the substitution $u = t^{m-2}(t-s)$ shows that the last integral is bounded by

$$C \int_0^\infty \int_0^{Cg(t)/\alpha} u^{1/(m-1)}\, du\, dt \le C \int_0^\infty \left(\frac{g(t)}{\alpha}\right)^{m/(m-1)} dt,$$

because $|\partial u/\partial s| = t^{m-2} \ge u^{(m-2)/(m-1)}$. The term $\int_{G\cap\{s>t\}} J\, ds\, dt$ is estimated similarly. Thus we have shown (5), and the proof is complete. ■

Next, fix a real number $m \ge 2$, and let $\Gamma(t) = (t, t^m)$, $t > 0$. Then $d\mu = t^{(m-2)/3}\, dt$ is (a constant multiple of) the affine arc length measure on this curve. Consider the convolution operator

$$(\mu * f)(x) = \int_0^\infty f\big(x - \Gamma(t)\big) t^{(m-2)/3}\, dt, \quad x \in \mathbf{R}^2.$$

A multilinear argument also gives an easy proof of the following result, which was proved originally by using complex interpolation (see *e.g.* [D2]).

Theorem 2 *There is a constant C such that*

$$\|\mu * f\|_{L^q(\mathbf{R}^2)} \le C\|f\|_{L^p(\mathbf{R}^2)}$$

if and only if $(\frac{1}{p}, \frac{1}{q}) = (\frac{2}{3}, \frac{1}{3})$.

Proof The change of variables $t = s^{3/(m+1)}$ gives

$$(\mu * f)(x) = C \int_0^\infty f\big(x - (s^b, s^{3-b})\big)\, ds,$$

where $0 < b = 3/(m+1) \leq 1$. A reduction as above shows that the inequality $\|\mu * f\|_3 \leq C\|f\|_{3/2}$ follows from

$$\int_G J\, ds\, dt \leq C \int_0^\infty \left(\frac{g(t)}{\alpha}\right)^2 dt, \tag{6}$$

where $J = C(st)^{b-1}|s^{3-2b} - t^{3-2b}|$ and $G = \{(s,t) \in P^2 : J < g(t)/\alpha\}$. For each fixed t, put $u = (st)^{b-1}|s^{3-2b} - t^{3-2b}|$. Since $0 < b \leq 1$, we have $|\partial u/\partial s| \geq c(t/s)^{2-b} \geq c > 0$ when $t > s > 0$, and $|\partial u/\partial s| \geq c(s/t)^{1-b} \geq c > 0$ when $s > t > 0$. Therefore, we obtain

$$\int_G J\, ds\, dt \leq C \int_0^\infty \int_0^{Cg(t)/\alpha} u\, du\, dt,$$

which implies (6). ∎

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