J. Austral. Math. Soc. (Series A) **55** (1993), 386–402

SADDLE POINT THEOREM AND NONLINEAR SCALAR NEUMANN BOUNDARY VALUE PROBLEMS

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(Received 13 May 1991)

Communicated by E. N. Dancer

Abstract

We are concerned with existence results for nonlinear scalar Neumann boundary value problems $u'' + g(x, u) = 0$, $u'(0) = u'(\pi) = 0$ where $g(x, u)$ satisfies Carathéodory conditions and is (possibly) unbounded. On the one hand we only assume that the function $(\operatorname{sgn} u)g(x, u)$ is bounded either from above or from below in some function space, and we impose conditions which relate the asymptotic behavior of the function $\int_0^\pi G(x, u)dx$ (for $|u|$ large) with the first two eigenvalues of the corresponding linear problem (here $G(x, u) = \int_0^u g(x, s)ds$ is the potential generated by g). On the other hand we consider cases where the function $(\operatorname{sgn} u)g(x, u)$ is unbounded. The potential $G(x, u)$ is not necessarily required to satisfy a convexity condition. Our method of proof is variational, we make use of the Saddle Point Theorem.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): 34 B 15, 34 B 25.

1. Introduction

This paper is devoted to the study of existence results for nonlinear scalar Neumann boundary value problems

$$(1.1)\qquad \begin{aligned} u''(x) + g(x, u(x)) &= 0 \quad \text{a.e. in} \quad I, \\ u'(0) = u'(\pi) &= 0 \end{aligned}$$

where $I = [0, \pi]$, $g : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions (see Section 2) and is (possibly) *unbounded.*

This work was supported in part by US National Science Foundation under grant DMS-9006134.

https://doi.org/10.1017/S144678870003411X Published online by Cambridge University Press

Let $G(x, u)$ be the potential generated by the function g, that is

$$G(x, u) = \int_0^u g(x, s)ds. \tag{1.2}$$

We impose conditions that relate the asymptotic behavior of the function $\int_I G(x, u)dx$ (for $|u|$ large) with the first two eigenvalues of the linear problem

$$\begin{aligned} u''(x) + \lambda u(x) &= 0 \quad \text{in} \quad I, \quad \lambda \in \mathbb{R}, \\ u'(0) = u'(\pi) &= 0. \end{aligned} \tag{1.3}$$

In Section 2, we consider a coercivity condition of Ahmad, Lazer, and Paul's type ([2, 11, 12]) below the first eigenvalue, say zero. In Theorem 2.1 we do not assume that the nonlinearity $g(x, u)$ grows at most linearly. We only assume that the function $(\operatorname{sgn} u)g(x, u)$ is bounded from above in some function space. Theorem 2.2 is devoted to the case when no boundedness condition is imposed on the function $(\operatorname{sgn} u)g(x, u)$. In that case we impose a growth condition on the potential $G(x, u)$. In Theorem 2.3 we consider the case when nonresonance occurs below the first eigenvalue. In all our results the potential $G(x, u)$ is not necessarily required to satisfy a convexity condition. The main results of Section 2 are valid for periodic boundary value problem as well. To conclude Section 2 we provide a counterexample which throws more light on the relevance of our results.

Section 3 is devoted to existence conditions at the first two eigenvalues of the problem (1.3). We consider a coercivity condition of Ahmad, Lazer, and Paul's type with respect to the first eigenvalue, complemented by a nonuniform condition with respect to the second eigenvalue of (1.3). In Theorem 3.1 we assume that the function $g(x, u)$ grows at most linearly and that the function $(\operatorname{sgn} u)g(x, u)$ is only bounded from below in some function space.

Besides the classical real Lebesgue spaces $L^p(I)$ and the spaces $C^p(I)$ of p-times continuously differentiable real valued functions, we shall make use of Sobolev spaces $H^1(I)$ and $W^{2,1}(I)$ (see for example, Brézis [3] for definitions and properties).

For each $u \in H^1(I)$, we shall write

$$u(x) = \overline{u} + \tilde{u}(x)$$

where

$$\overline{u} = \pi^{-1} \int_I u(x)dx \quad \text{and} \quad \tilde{u}(x) = u(x) - \overline{u}.$$

So, with obvious notations,

$$H^1(I) = \overline{H}^1(I) \oplus \tilde{H}^1(I). \tag{1.4}$$

For $u \in L^1(I)$, we define

$$u^+(x) = \max(u(x), 0) \quad \text{and} \quad u^-(x) = \max(-u(x), 0).$$

Hence,

$$u(x) = u^+(x) - u^-(x).$$

2. Existence conditions below the first eigenvalue

In this section we study the solvability of the Neumann boundary value problem (1.1) where g is a Carathéodory function, that is, $g(\cdot, u)$ is measurable for all $u \in \mathbb{R}$, $g(x, \cdot)$ is continuous for a.e. $x \in I$, and for each constant $r > 0$ there exists a function $f_r \in L^1(I)$ such that

$$|g(x, u)| \leq f_r(x) \tag{2.1}$$

for a.e. $x \in I$, and all $u \in \mathbb{R}$ with $|u| \leq r$.

Let $G(x, u)$ be the potential generated by the function g as defined in (1.2). The following result deals with the case when the function $(\operatorname{sgn} u)g(x, u)$ is bounded from above.

THEOREM 2.1. *Suppose there exist functions $A, B \in L^1(I)$ and a constant $R \in \mathbb{R}$ with $R > 0$ such that*

$$g(x, u) \leq A(x) \tag{2.2}$$

for a.e. $x \in I$ *and all* $u \in \mathbb{R}$ *with* $u \geq R$,

$$g(x, u) \geq B(x) \tag{2.3}$$

for a.e. $x \in I$ *and all* $u \in \mathbb{R}$ *with* $u \leq -R$.

Moreover, assume

$$\lim_{|u| \to \infty} \int_I G(x, u)\,dx = -\infty. \tag{2.4}$$

Then equation (1.1) has at least one solution $u \in W^{2,1}(I)$ that minimizes the functional

$$\phi(u) = \int_I \left[\frac{1}{2}|u'|^2 - G(x,u)\right] dx \tag{2.5}$$

on $H^1(I)$.

Conditions (2.2) and (2.3) are used in the literature in connection with the so called Landesman-Lazer condition (see for example, [8, 1]), here they are used along with the more general condition (2.4) of Ahmad, Lazer and Paul's type [2]. On the other hand, regarding (2.4), we do not assume that g is (necessarily) bounded as is usually required in the literature (see for example, [2, 11]). Also, notice that no convexity assumption is imposed on the potential G. The reader is referred to Theorem 3.1 for similar conditions above the first eigenvalue.

PROOF. By (2.1)–(2.3), it follows that there exists a function $b \in L^1(I)$ such that

$$G(x,u) \leq b(x)|u| \tag{2.6}$$

for a.e. $x \in I$ and all $u \in \mathbb{R}$.

It is easy to verify that the functional ϕ, defined by (2.5), is a C^1-functional on $H^1(I)$ since $H^1(I)$ is compactly imbedded into $C(I)$ (see for example, [11, pp. 90–94]).

We shall show that ϕ is coercive on $H^1(I)$, that is,

$$\phi(u) \to \infty \quad \text{as} \quad |u|_{H^1} \to \infty$$

which would imply that the Palais-Smale condition is satisfied (see for example, [11, p. 94]).

Assuming that this is the case, we deduce, by [11, Theorem 2.7], that ϕ has a minimum at some point $u \in H^1(I)$ (see also [11, p. 25]). Since ϕ is a C^1-functional on $H^1(I)$, necessarily $\phi'(u) = 0$, and u is a weak solution to equation (1.1). Therefore, condition (2.1) and a standard regularity result imply that $u \in W^{2,1}(I)$ (see for example, [3, p. 182]).

Now we are going to prove that the functional ϕ is coercive on $H^1(I)$.

We assume by contradiction that there exists a sequence $(u_n) \subset H^1(I)$ with $|u_n|_{H^1} \to \infty$ as $n \to \infty$ such that

$$\phi(u_n) \leq c_1 \tag{2.7}$$

for some constant c_1 and all $n \in \mathbb{N}$.

Set $v_n = u_n/|u_n|_{H^1}$. Then, one can find a subsequence relabeled (v_n) and a point $v \in H^1(I)$ such that

$$(2.8) \qquad |v_n|_{H^1} = 1, \qquad v_n \to v \quad \text{in } C(I), \qquad v_n \rightharpoonup v \quad \text{in } H^1(I).$$

By (2.7), one has

$$(2.9) \qquad \int_I \frac{1}{2}|v_n'|^2 dx - \int_I \frac{G(x, u_n)}{|u_n|_{H^1}^2} dx \le \frac{c_1}{|u_n|_{H^1}^2}$$

which implies, by (2.6), that

$$(2.10) \qquad \int_I \frac{1}{2}|v_n'|^2 dx - \int_I b(x)\frac{v_n}{|u_n|_{H^1}} dx \le \frac{c_1}{|u_n|_{H^1}^2}.$$

By going to the limit as $n \to \infty$, one obtains

$$v_n' \to 0 \quad \text{in } L^2(I).$$

Therefore, by (2.8), one has $v_n \to v$ in $H^1(I)$, $v'(x) = 0$ for a.e. $x \in I$ which implies that $v \not\equiv 0$, since $v \equiv 0$ would lead to a contradiction with $|v_n|_{H^1} = 1$.

On the other hand, since $v'(x) = 0$ for a.e. $x \in I$, we deduce that

$$(2.11) \qquad v(x) = c$$

on I for some constant $c \neq 0$. By (2.8), it follows that either $u_n \to \infty$ or $u_n \to -\infty$ *uniformly* on I as $n \to \infty$.

Let us assume that $u_n \to \infty$ uniformly on I (the proof for the other case is similar). Setting

$$(2.12) \qquad u_n(x_n) = \min_I u_n(x),$$

we obtain $u_n(x_n) \to \infty$. Therefore, by writing $\phi(u_n)$ as

$$\phi(u_n) = \int_I \left[\frac{1}{2}|u_n'|^2 - G(x, u_n(x_n)) - (G(x, u_n(x)) - G(x, u_n(x_n)))\right] dx$$

and using (2.2), it follows that, for sufficiently large n,

$$\begin{aligned}\phi(u_n) &\ge \int_I \left[\frac{1}{2}|u_n'|^2 - G(x, u_n(x_n))\right] dx - \int_I A(x)(u_n(x) - u_n(x_n)) dx \\ &\ge \frac{1}{2}|u_n'|_{L^2}^2 - \int_I G(x, u_n(x_n)) dx - \int_I A(x)[(u_n(x) - \bar{u}_n) - (u_n(x_n) - \bar{u}_n)] dx;\end{aligned}$$

that is,

$$\phi(u_n) \geq \frac{1}{2}|u_n'|_{L^2}^2 - 2|A|_{L^1}|\tilde{u}_n|_C - \int_I G(x, u_n(x_n))dx.$$

So, by the Sobolev inequality (see for example, [3, p. 129]) one has

$$\phi(u_n) \geq \frac{1}{2}|u_n'|_{L^2}^2 - 2c_2|u_n'|_{L^2} - \int_I G(x, u_n(x_n))dx \tag{2.13}$$

for some constant $c_2 > 0$.

It immediately follows from (2.4) and (2.13) that $\phi(u_n) \to \infty$ as $n \to \infty$, thus contradicting (2.7). The proof is complete.

In the next result we are concerned with the case when no boundedness condition is imposed on the function $(\operatorname{sgn} u)g(x, u)$. In that case we impose a growth condition on the potential $G(x, u)$. This condition includes the case when the potential $G(x, u)$ is *Lipschitz*, that is, when the nonlinearity $g(x, u)$ is bounded (see [2, 11]).

THEOREM 2.2. *Suppose there exist functions $\alpha, \gamma, \Gamma \in L^1(I)$ such that*

$$|G(x, u) - G(x, v)| \leq \Gamma(x)|u - v|^2 + \gamma(x)|u - v| + \alpha(x) \tag{2.14}$$

for a.e. $x \in I$ *and all* $u, v \in \mathbb{R}$, *where*

$$\Gamma(x) \leq 1/2 \tag{2.15}$$

for a.e. $x \in I$ *with strict inequality on a subset of positive measure. Moreover, assume condition (2.4) is fulfilled. Then the conclusion of Theorem 2.1 holds.*

PROOF. Under conditions of Theorem 2.2, we will show that the functional ϕ is coercive on $H^1(I)$.

As in the proof of Theorem 2.1 we assume to the contrary that there exists a sequence $(u_n) \subset H^1(I)$ with $|u_n|_{H^1} \to \infty$ as $n \to \infty$ such that (2.7) is satisfied. By writing $\phi(u_n)$ as

$$\phi(u_n) = \int_I \left[\frac{1}{2}|u_n'|^2 - G(x, \overline{u}_n)\right] dx - \int_I [G(x, u_n(x)) - G(x, \overline{u}_n)]dx$$

and using (2.14), we have

$$\phi(u_n) \geq \int_I \left[\frac{1}{2}|u_n'|^2 - \Gamma(x)|\tilde{u}_n|^2\right] dx - \int_I \gamma(x)|\tilde{u}_n|dx - \int_I \alpha(x)dx - \int_I G(x, \overline{u}_n)dx.$$

https://doi.org/10.1017/S144678870003411X Published online by Cambridge University Press

By Sobolev inequality and Lemma 1 in [9, p. 339], one gets

$$\phi(u_n) \geq \delta|\tilde{u}_n|^2_{H^1} - c_3|\tilde{u}_n|_{H^1} - |\alpha|_{L^1} - \int_I G(x, \bar{u}_n)dx \tag{2.16}$$

for some constants $\delta > 0$ and $c_3 > 0$.

Since $|u_n|_{H^1} \to \infty$ as $n \to \infty$, it follows that $|\bar{u}_n| \to \infty$ or $|\tilde{u}_n|_{H^1} \to \infty$ as $n \to \infty$. Therefore, condition (2.4) and inequality (2.16) imply that $\phi(u_n) \to \infty$ as $n \to \infty$. This contradicts (2.7), and the proof is complete.

The final main result of this section provides for nonresonance conditions below the first eigenvalue (see [4, 7, 11]). The reader is referred to Theorem 3.1 herein for similar nonresonance conditions at the second eigenvalue.

THEOREM 2.3. *Suppose*

$$\limsup_{u\to\infty} \frac{G(x,u)}{u^2} \leq \beta_+(x), \qquad \limsup_{u\to-\infty} \frac{G(x,u)}{u^2} \leq \beta_-(x) \tag{2.17}$$

uniformly for a.e. $x \in I$, *where* $\beta_+, \beta_- \in L^1(I)$ *are such that*

$$\beta_+(x) \leq 0 \quad \textit{and} \quad \beta_-(x) \leq 0 \tag{2.18}$$

for a.e. $x \in I$ with strict inequalities on subsets of I of positive measure. Then the conclusion of Theorem 2.1 holds.

PROOF. Under conditions of Theorem 2.3, we shall show that the functional ϕ is coercive on $H^1(I)$.

An easy adaptation of the argument used in the proof of [9, Lemma 1, pp. 339–340] and condition (2.18) imply that there exists a constant $\delta = \delta(\beta_+, \beta_-) > 0$ such that, for any $u \in H^1(I)$,

$$\psi(u) \equiv \int_I \left[\frac{1}{2}|u'|^2 - (\beta_+(x)(u^+)^2 + \beta_-(x)(u^-)^2)\right]dx \geq \delta|u|^2_{H^1}. \tag{2.19}$$

On the other hand, conditions (2.1) and (2.17) imply the existence of a function $\beta \in L^1(I)$ such that

$$G(x,u) \leq (\beta_+(x) + \frac{\delta}{4})u^2 + \beta(x) \tag{2.20}$$

https://doi.org/10.1017/S144678870003411X Published online by Cambridge University Press

for a.e. $x \in I$ and all $u \in \mathbb{R}$ with $u \geq 0$,

$$G(x,u) \leq (\beta_-(x) + \frac{\delta}{4})u^2 + \beta(x) \tag{2.21}$$

for a.e. $x \in I$ and all $u \in \mathbb{R}$ with $u < 0$.

Therefore, for $u \in H^1(I)$,

$$\begin{aligned}\phi(u) &= \int_I \frac{1}{2}|u'|^2 dx - \int_{u \geq 0} G(x,u)dx - \int_{u<0} G(x,u)dx \\ &\geq \int_I \left[\frac{1}{2}|u'|^2 - (\beta_+(x)(u^+)^2 + \beta_-(x)(u^-)^2)\right]dx - \frac{\delta}{2}\int_I u^2 dx - 2|\beta|_{L^1}.\end{aligned}$$

Hence, by (2.19),

$$\phi(u) \geq \psi(u) - \frac{\delta}{2}\int_I u^2 dx - 2|\beta|_{L^1} \geq \frac{\delta}{2}|u|^2_{H^1} - 2|\beta|_{L^1}$$

which implies that ϕ is coercive on $H^1(I)$. The proof is complete.

EXAMPLE 2.1. (A counterexample)

For $c \in \mathbb{R}$ with $|c| > 1$ the equation

$$u''(x) + \frac{\cos x}{c + \cos x}u(x) = 0, \quad u'(0) = u'(\pi) = 0 \tag{2.22}$$

has nontrivial solutions of the form

$$u(x) = A[c + \cos x] \tag{2.23}$$

for any $A \in \mathbb{R}$ with $A \neq 0$. Therefore, by the Fredholm alternative, the equation

$$u''(x) + \frac{\cos x}{c + \cos x}u(x) = \cos x, \quad u'(0) = u'(\pi) = 0 \tag{2.24}$$

(with $|c| > 1$), has *no* solution. (Note that $\int_0^\pi (c + \cos x)\cos x \, dx \neq 0$.) Moreover, by easy computations and change of variables, one has

$$\int_0^\pi \frac{\cos x}{c + \cos x}dx = -2\int_0^{\pi/2} \frac{\cos^2 x}{c^2 - \cos^2 x}dx < 0 \tag{2.25}$$

since $|c| > 1$. We deduce two facts:

(1) The coercivity condition (2.4) alone does not guarantee the existence of solution to equation (1.1).

(2) In the nonresonance case, conditions (2.17)–(2.18) cannot be replaced by a weaker assumption of the type

$$\limsup_{|u|\to\infty} \frac{G(x,u)}{|u|^2} \le \Gamma(x) \tag{2.26}$$

uniformly for a.e. $x \in I$, where $\Gamma \in L^1(I)$ is such that

$$\int_0^\pi \Gamma(x)dx < 0. \tag{2.27}$$

(Also, clearly condition (2.14) of Theorem 2.2 is not satisfied.)

3. Existence conditions at the first two eigenvalues

We shall be concerned with existence results for equation (1.1) under resonance and nonresonance conditions between the first two eigenvalues of (1.3). Throughout this section, we shall assume that the function g satisfies Carathéodory's conditions (see Section 2), and grows at most linearly, that is, there exist a constant $d \ge 0$ and a function $e \in L^1(I)$ such that

$$|g(x,u)| \le d|u| + e(x) \tag{3.1}$$

for a.e. $x \in I$ and all $u \in \mathbb{R}$.

THEOREM 3.1. *Suppose*

$$\limsup_{u\to\infty} \frac{g(x,u)}{u} \le \Gamma_+(x) \quad \textit{and} \quad \limsup_{u\to-\infty} \frac{g(x,u)}{u} \le \Gamma_-(x) \tag{3.2}$$

for a.e. $x \in I$ *where*

$$0 \le \Gamma_+(x) \le 1, \quad 0 \le \Gamma_-(x) \le 1 \tag{3.3}$$

for a.e. $x \in I$ *with*

$$\int_{w>0} (1-\Gamma_+)w^2dx + \int_{w<0} (1-\Gamma_-)w^2dx > 0 \tag{3.4}$$

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for all $w \in \operatorname{Span}\{\cos x\}$ *with* $w \not\equiv 0$. *Moreover, assume there exist functions* $A, B \in L^1(I)$ *and a constant* $R \in \mathbb{R}$ *with* $R > 0$ *such that*

$$g(x, u) \geq A(x) \tag{3.5}$$

for a.e. $x \in I$ *and all* $u \in \mathbb{R}$ *with* $u \geq R$,

$$g(x, u) \leq B(x) \tag{3.6}$$

for a.e. $x \in I$ *and all* $u \in \mathbb{R}$ *with* $u \leq -R$. *Finally, suppose*

$$\lim_{|u|\to\infty} \int_I G(x, u)dx = \infty \tag{3.7}$$

where G *is the potential defined in (1.2). Then equation (1.1) has at least one solution* $u \in W^{2,1}(I)$.

PROOF. Under conditions of Theorem 3.1 we shall prove that the conditions of the Rabinowitz Saddle Point Theorem [11, pp. 24–25] are fulfilled where the Banach space is $H^1(I)$ as given in (1.4) and the functional ϕ on $H^1(I)$ is defined in (2.5). Let

$$V = \overline{H}^1(I), \qquad X = \tilde{H}^1(I) \quad \text{and} \quad D = \{\overline{u} \in V : |\overline{u}| \leq \rho\}. \tag{3.8}$$

We will prove that there exists a constant $\rho > 0$ such that

$$\sup_{\partial D} \phi < \inf_X \phi. \tag{3.9}$$

where ∂D is the boundary of D.

On the one hand an elementary adaptation of the argument used in the proof of Lemma 1 of [9, pp. 339–340] and conditions (3.3)–(3.4) imply that there exists a constant $\delta = \delta(\Gamma_+, \Gamma_-) > 0$ such that, for any $\tilde{u} \in X$,

$$\psi(\tilde{u}) \equiv \frac{1}{2}\int_I [|\tilde{u}'|^2 - (\Gamma_+(x)(\tilde{u}^+)^2 + \Gamma_-(x)(\tilde{u}^-)^2)]dx \geq \delta|\tilde{u}|^2_{H^1}. \tag{3.10}$$

On the other hand conditions (3.1)–(3.2) imply the existence of a function $\Gamma \in L^1(I)$ such that

$$G(x, u) \leq \frac{1}{2}(\Gamma_+(x) + \frac{\delta}{2})u^2 + \Gamma(x) \tag{3.11}$$

for a.e. $x \in I$ and all $u \in \mathbb{R}$ with $u \geq 0$,

$$G(x, u) \leq \frac{1}{2}(\Gamma_-(x) + \frac{\delta}{2})u^2 + \Gamma(x) \tag{3.12}$$

for a.e. $x \in I$ and all $u \in \mathbb{R}$ with $u < 0$.

Therefore, for $\tilde{u} \in X$,

$$\begin{aligned}\phi(\tilde{u}) &= \frac{1}{2}\int_I |\tilde{u}'|^2 dx - \int_{\tilde{u}\geq 0} G(x, \tilde{u})dx - \int_{\tilde{u}<0} G(x, \tilde{u})dx \\ &\geq \frac{1}{2}\int_I [|\tilde{u}'|^2 - (\Gamma_+(x)(\tilde{u}^+)^2 + \Gamma_-(x)(\tilde{u}^-)^2)]dx - \frac{\delta}{2}\int_I \tilde{u}^2 dx - 2|\Gamma|_{L^1}.\end{aligned}$$

Hence, by (3.10),

$$\phi(\tilde{u}) \geq \psi(\tilde{u}) - \frac{\delta}{2}|\tilde{u}|^2_{L^2} - 2|\Gamma|_{L^1},$$

that is,

$$\phi(\tilde{u}) \geq \frac{\delta}{2}|\tilde{u}|^2_{H^1} - 2|\Gamma|_{L^1}, \tag{3.13}$$

which implies that ϕ is bounded below on X by $-2|\Gamma|_{L^1}$. Moreover, for $\overline{u} \in V$,

$$\phi(\overline{u}) = -\int_I G(x, \overline{u})dx. \tag{3.14}$$

Therefore, by using condition (3.7) and inequality (3.13), we obtain the assertion (3.9) for some constant $\rho > 0$.

It remains to prove that the functional ϕ satisfies the Palais-Smale condition. For this purpose, it suffices, as is easily seen from [11, pp. 94–95], to show that for any sequence $(u_n) \subset H^1(I)$ such that $\phi(u_n)$ is bounded and $\phi'(u_n) \to 0$ as $n \to \infty$, it follows that (u_n) is bounded.

Suppose there exist a sequence $(u_n) \subset H^1(I)$ and a constant $c > 0$ such that

$$\begin{cases} |u_n|_{H^1} \to \infty & \text{as} \quad n \to \infty, \\ |\phi(u_n)| \leq c & \text{for all} \quad n \in \mathbb{N}, \\ \phi'(u_n) \to 0 & \text{as} \quad n \to \infty. \end{cases} \tag{3.15}$$

Define the sequence $(v_n) \subset H^1(I)$ by

$$v_n = u_n/|u_n|_{H^1}. \tag{3.16}$$

The last assertion in (3.15) implies that

$$(3.17) \quad \int_I [u_n' w' - g(x, u_n)w]dx \to 0 \quad \text{as } n \to \infty \text{ for all } w \in H^1(I).$$

Therefore

$$(3.18) \int_I [v_n' w' - (g(x, u_n)/|u_n|_{H^1})w]dx \to 0 \quad \text{as } n \to \infty \text{ for all } w \in H^1(I).$$

Since $|v_n|_{H^1} = 1$, by the compact imbedding of $H^1(I)$ into $C(I)$ we deduce, passing if necessary to a subsequence relabelled (v_n), that there exists $v \in H^1(I)$ such that

$$(3.19) \quad v_n \to v \quad \text{in} \quad C(I), \quad v_n \rightharpoonup v \quad \text{in} \quad H^1(I), \quad \text{as} \quad n \to \infty.$$

On the other hand, by the growth condition (3.1), one has that the sequence $(g(x, u_n)/|u_n|_{H^1})$ is such that

$$(3.20) \qquad |g(x, u_n)|/|u_n|_{H^1} \le c_0 + e(x)$$

for a.e. $x \in I$ and all $n \in \mathbb{N}$ with n sufficiently large, where c_0 is some constant.

Hence, by the Dunford-Pettis Theorem (see [3]), the sequence $(g(x, u_n)/|u_n|_{H^1})$ converges weakly in $L^1(I)$. So, by (3.18), we get

$$(3.21) \qquad \int_I [v'w' - K(x)w]dx = 0 \quad \text{for all } w \in H^1(I)$$

where K is the weak limit (in $L^1(I)$) of the sequence $(g(x, u_n)/|u_n|_{H^1})$.

We claim that $v \not\equiv 0$. Indeed, by conditions (3.1)–(3.6), there exist functions $a, b, h \in L^1(I)$ such that

$$a(x)|u| - b(x) \le G(x, u) \le 2|u|^2 + h(x)$$

for a.e. $x \in I$ and all $u \in \mathbb{R}$. Hence,

$$(3.22) \qquad \frac{a(x)}{|u_n|_{H^1}} v_n - \frac{b(x)}{|u_n|^2_{H^1}} \le \frac{G(x, u_n)}{|u_n|^2_{H^1}} \le 2v_n^2 + \frac{h(x)}{|u_n|^2_{H^1}}.$$

Now, if $v \equiv 0$, then by the first assertion in (3.19), one deduces $v_n \to 0$ uniformly. So, by (3.22),

$$\int_I \frac{G(x, u_n)}{|u_n|^2_{H^1}} dx \to 0 \quad \text{as} \quad n \to \infty,$$

which would imply, by the second assertion in (3.15), that $v_n' \to 0$ in $L^2(I)$. Thus, $v_n \to 0$ in $H^1(I)$ as $n \to \infty$. This contradicts the fact that $|v_n|_{H^1} = 1$. Therefore $v \not\equiv 0$.

Let us define the function k_v by

$$k_v(x) = \begin{cases} K(x)/v(x) & \text{if } v(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then (3.21) becomes

$$\int_I [v'w' - k_v(x)vw]dx = 0 \quad \text{for all} \quad w \in H^1(I),$$

that is, v is a weak solution to the problem

$$\begin{aligned} z'' + k_v(x)z &= 0 \\ z'(0) = z'(\pi) &= 0. \end{aligned}$$

By standard regularity results [3, pp. 139–140], it follows that $v \in W^{2,1}(I)$, and that

$$\begin{aligned} v''(x) + k_v(x)v(x) &= 0 \quad \text{a.e. in} \quad I, \\ v'(0) = v'(\pi) &= 0. \end{aligned} \tag{3.23}$$

Define the functions k_v^+ and k_v^+ by

$$k_v^+(x) = \begin{cases} k_v(x) & \text{if} \quad v(x) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$k_v^-(x) = \begin{cases} k_v(x) & \text{if} \quad v(x) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Equation (3.23) becomes

$$\begin{aligned} v'' + k_v^+(x)v^+ - k_v^-(x)v^- &= 0 \quad \text{a.e. in} \quad I, \\ v'(0) = v'(\pi) &= 0. \end{aligned} \tag{3.24}$$

By using the definition of the functions k_v^+, k_v^-, inequalities (3.2), (3.5)–(3.6), and properties of liminf and limsup, it follows that for a.e. $x \in I$,

$$\begin{aligned} 0 &\leq k_v^+(x) \leq \Gamma_+(x), \\ 0 &\leq k_v^-(x) \leq \Gamma_-(x). \end{aligned} \tag{3.25}$$

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By multiplying (3.23) with $\overline{v} - \tilde{v}$, integrating over I, using inequalities (3.25) and (3.10), and the property of the eigenfunction associated with the first eigenvalue of equation (1.3), it follows that $v(x) = c_1$ for some constant $c_1 \neq 0$. We shall assume $c_1 > 0$. (The proof for the case $c_1 < 0$ is similar.)

Since $v_n \to c_1$ uniformly on I as $n \to \infty$, it immediately follows from (3.16) that

$$u_n \to \infty \quad \text{uniformly on} \quad I \quad \text{as} \quad n \to \infty, \tag{3.26}$$

which implies that there exists $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$,

$$u_n(x) \geq R \quad \text{for all} \quad x \in I. \tag{3.27}$$

So, by (3.5), $g(x, u_n(x)) \geq A(x)$ for a.e. $x \in I$ when $n \geq n_0$.

Therefore

$$\int_{g \leq 0} g(x, u_n(x))dx \geq c_2 \tag{3.28}$$

for some constant c_2.

On the other hand, by (3.17) with $w = 1$, one has

$$\left| \int_I g(x, u_n(x))dx \right| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.29}$$

Hence, by combining (3.28) and (3.29), one gets

$$\int_I |g(x, u_n(x))|dx \leq c_3 \tag{3.30}$$

for some constant c_3.

Since $\phi'(u_n) \to 0$ as $n \to \infty$, it follows that, for some constant $c_4 > 0$,

$$|\phi'(u_n)w| \leq c_4 |w|_{H^1} \quad \text{for all} \quad w \in H^1(I). \tag{3.31}$$

By inequalities (3.30)–(3.31), one has

$$\begin{aligned} c_4 |\tilde{u}_n|_{H^1} \geq |\phi'(u_n)\tilde{u}_n| &= \int_I [|\tilde{u}_n'|^2 - g(x, u_n(x))\tilde{u}_n(x)]dx \\ &\geq \int_I |\tilde{u}_n'|^2 dx - c_3 |\tilde{u}_n|_C. \end{aligned}$$

Using the continuous imbedding of $H^1(I)$ into $C(I)$, one gets

$$c_4 |\tilde{u}_n|_{H^1} \geq |\tilde{u}_n'|^2_{L^2} - c_5 |\tilde{u}_n|_{H^1}$$

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for some constant $c_5 > 0$. So, by the Poincaré inequality, it follows that

$$|\tilde{u}_n|_{H^1} \leq c_6 \tag{3.32}$$

for some constant $c_6 > 0$.

Set

$$u_n(x_n) = \min_I u_n(x). \tag{3.33}$$

Then, by (3.26),

$$u_n(x_n) \to \infty \quad \text{as} \quad n \to \infty. \tag{3.34}$$

which also shows that (for n sufficiently large)

$$u_n(x_n) \geq R. \tag{3.35}$$

On the other hand,

$$\begin{aligned}\int_I G(x, u_n(x))dx &= \int_I G(x, u_n(x_n))dx + \int_I [G(x, u_n(x)) - G(x, u_n(x_n))]dx \\ &= \int_I G(x, u_n(x_n))dx + \int_I \left[\int_{u_n(x_n)}^{u_n(x)} g(x, s)ds\right] dx.\end{aligned}$$

So, by assumption (3.5), one has

$$\int_I G(x, u_n(x))dx \geq \int_I G(x, u_n(x_n))dx + \int_I (u_n(x) - u_n(x_n))A(x)dx.$$

By using the fact that $u_n(x) - u_n(x_n) = u_n(x) - \overline{u}_n + \overline{u}_n - u_n(x_n)$, and the imbedding of $H^1(I)$ into $C(I)$, one deduces that

$$\int_I G(x, u_n(x))dx \geq \int_I G(x, u_n(x_n))dx - c_7 |\tilde{u}_n|_{H^1} |A|_{L^1}$$

for some constant $c_7 > 0$. This implies, by (3.32), that

$$\int_I G(x, u_n(x))dx \geq \int_I G(x, u_n(x_n))dx - c_8 \tag{3.36}$$

for some constant $c_8 > 0$. Therefore, by (3.34), (3.36), and assumption (3.7), it follows that

$$\int_I G(x, u_n(x))dx \to \infty \quad \text{as} \quad n \to \infty. \tag{3.37}$$

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Moreover, by (3.32),

$$\phi(u_n) = \int_I \left[\frac{1}{2}|u_n'(x)|^2 - G(x, u_n(x))\right]dx \le c_6^2 - \int_I G(x, u_n(x))dx.$$

So, by (3.37),

$$\phi(u_n) \to -\infty \quad \text{as} \quad n \to \infty, \tag{3.38}$$

contradicting the second assertion in (3.15). The proof is complete.

EXAMPLE 3.1. Let

$$g(x, u) = p(x, u)u \sin^2 u + a \cos u + h(x)$$

where $a \in \mathbb{R}$, $h \in L^1(I)$, and $p : I \times \mathbb{R} \to \mathbb{R}$ is defined by

$$p(x, u) = \begin{cases} 1 & \text{for } x \in I \text{ and } u \in \mathbb{R} \text{ with } u \ge 0, \\ \Gamma_-(x) & \text{for } x \in I \text{ and } u \in \mathbb{R} \text{ with } u < 0. \end{cases}$$

We suppose that $\Gamma_-(x)$ is such that

$$0 \le \Gamma_-(x) \le \frac{1}{2} \quad \text{for a.e. } x \in I$$

with $\Gamma_-(x) > 0$ on a subset of I of positive measure.

It is easily checked that the potential generated by g is given by

$$G(x, u) = \frac{p(x, u)}{2}u^2 - \frac{p(x, u)}{2}u \sin 2u - \frac{p(x, u)}{4}\cos 2u + a \sin u + h(x)u.$$

Therefore, by Theorem 3.1 herein, equation (1.1) has at least one solution for *every* $h \in L^1(I)$.

REMARK 3.1. Theorem 3.1 may be related to a result in [6] where the periodic problem is considered. Both results (as others in the literature [1, 6, 11, 12] and references therein) rely upon the Saddle Point Theorem [11]. However, in verifying the Palais-Smale condition, it is important to note that the approach used in [6] does not seem to work for Neumann boundary value problems; while the one developed herein does apply, in a natural way, to the periodic boundary value problem and provides for a more general result as illustrated by Example 3.1 above.

Acknowledgement

The author wishes to thank the referee for his careful reading of the manuscript.

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