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Global existence and large time behaviour for the pressureless Euler–Naver–Stokes system in $\mathbb{R}^3$

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We investigate the global Cauchy problem for a two–phase flow model consisting of the pressureless Euler equations coupled with the isentropic compressible Navier–Stokes equations through a drag forcing term. This model was first derived by Choi–Kwon [J. Differential Equations, 261(1) (2016), pp. 654–711] by taking the hydrodynamic limit of the Vlasov/compressible Navier–Stokes equations. Under the assumption that the initial perturbation is sufficiently small, Choi–Kwon [J. Differential Equations, 261(1) (2016), pp. 654–711] established the global well–posedness and large time behaviour for the three dimensional periodic domain $\mathbb{T}^3$. However, up to now, the global well–posedness and large time behaviour for the three dimensional Cauchy problem still remain unsolved. In this paper, we resolve this problem by proving the global existence and optimal decay rates of classic solutions for the three dimensional Cauchy problem when the initial data is near its equilibrium. One of key observations here is that to overcome the difficulties arising from the absence of pressure in the Euler equations, we make full use of the drag forcing term and the dissipative structure of the Navier–Stokes equations to closure the energy estimates of the variables for the pressureless Euler equations.

Keywords: Pressureless Euler equations; compressible Navier–Stokes equations; Cauchy problem; global existence; large time behaviour

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1. Introduction

In this paper, we are interested in a two–phase flow model consisting of the pressureless Euler equations coupled with the isentropic compressible Navier–Stokes

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equations through a drag forcing term in the whole space $\mathbb{R}^3$. The coupled hydrodynamic system takes the following form (see [**12**]):

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) = -\rho(u - v), \\ \partial_t n + \mathrm{div}\,(nv) = 0, \\ \partial_t(nv) + \mathrm{div}\,(nv \otimes v) + \nabla P(n) - \mu\Delta v - (\mu+\lambda)\nabla\,\mathrm{div}\,v = \rho(u - v). \end{cases} \tag{1.1}$$

Here $\rho = \rho(x, t)$ and $u = u(x, t)$ represent the particle density and velocity for the pressureless flow at a domain $(x, t) \in \mathbb{R}^3 \times \mathbb{R}_+$, and $n = n(x, t)$ and $v = v(x, t)$ represent the fluid density and velocity for the compressible flow. $P(n) = an^\gamma$ $(a > 0,\ \gamma \geqslant 1)$ represents the pressure. The symbol $\otimes$ is the Kronecker tensor product. μ and λ stand for the shear and the bulk viscosity coefficients of the fluid satisfying the following physical conditions:

$$\mu > 0, \quad \text{and} \quad \frac{2}{3}\mu + \lambda \geqslant 0.$$

We consider the initial value problem of (1.1) in the whole space with the initial data

$$(\rho, u, n, v)|_{t=0} = (\rho_0(x), u_0(x), n_0(x), v_0(x)), \quad x \in \mathbb{R}^3, \tag{1.2}$$

satisfying

$$(\rho_0(x), u_0(x), n_0(x), v_0(x)) \longrightarrow (\bar\rho, \overrightarrow{0}, \bar n, \overrightarrow{0}), \quad \text{as } |x| \longrightarrow \infty,$$

where the positive constants $\bar\rho$ and $\bar n$ are the reference densities.

The coupled hydrodynamic system (1.1) is closely related to the kinetic–fluid models, which are used to describe the interactions between particles and fluid. Recently, these types of the kinetic–fluid models have received growing attention due to a very large range of applications, for example, sedimentation, sprays, aerosols, biotechnology, and atmospheric pollution, etc. [**1–6**, **11–13**, **16–22**, **27**, **29–31**, **33**]. More specifically, Choi–Kwon [**12**] first addressed the formal derivation of the coupled hydrodynamic system (1.1) from the Vlasov/compressible Navier–Stokes equations, under the assumption that the particle distribution is mono–kinetic. For the sake of completeness, we recall the details in the process. To begin with, let us denote the distribution of particles at the position–velocity $(x, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3$ and at time $t \in \mathbb{R}_+$ by $f(x, \omega, t)$, and the isentropic compressible fluid density and velocity by $n(x, t)$ and $v(x, t)$, respectively. Then the motion of the particles and fluid is governed by the following kinetic–fluid equations:

$$\begin{cases} f_t + \omega\cdot\nabla_x f + \nabla_\omega\cdot((v-\omega)f) = 0, \\ n_t + \nabla_x\cdot(nv) = 0, \\ (nv)_t + \nabla_x\cdot(nv\otimes v) + \nabla_x P(n) - \mu\Delta_x v - (\mu+\lambda)\nabla_x\nabla_x\cdot v = \displaystyle\int_{\mathbb{R}^3}(\omega - v)f\,\mathrm{d}\omega, \end{cases} \tag{1.3}$$

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for $(x,\, \omega,\, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+$. Next, we define the macroscopic variables of the local mass ρ and momentum ρu for the distribution function f as follows:

$$\rho(x,t) := \int_{\mathbb{R}^3} f(x,\omega,t)\mathrm{d}\omega \quad \text{and}$$
$$(\rho u)(x,t) := \int_{\mathbb{R}^3} \omega f(x,\omega,t)\mathrm{d}\omega \quad \text{for} \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+,$$

and denote the energy–flux $\hat{q}$, the pressure tensor $\hat{\sigma}$, and the temperature θ by the fluctuation terms:

$$\hat{q}(x,t) := \frac{1}{2}\int_{\mathbb{R}^3} |\omega - u(x,t)|^2(\omega - u(x,t))f(x,\omega,t)\mathrm{d}\omega$$
$$\hat{\sigma}(x,t) := \int_{\mathbb{R}^3} (\omega - u(x,t)) \otimes (\omega - u(x,t))f(x,\omega,t)\mathrm{d}\omega$$

and

$$(\rho\theta)(x,t) := \frac{1}{2}\int_{\mathbb{R}^3} |\omega - u(x,t)|^2 f(x,\omega,t)\mathrm{d}\omega$$

First, integrating the equation $(1.3)_1$ with respect to ω over $\mathbb{R}^3$, one can easily get the continuity equation:

$$\frac{d\rho}{dt} + \nabla_x \cdot (\rho u) = 0.$$

Second, multiplying $(1.3)_1$ by ω, and then integrating the resultant equation with respect to ω over $\mathbb{R}^3$, we can deduce the momentum equation:

$$\frac{d(\rho u)}{dt} + \nabla_x \cdot (\rho u \otimes u) + \nabla_x \cdot \hat{\sigma} = -\rho(u - v).$$

Third, multiplying $(1.3)_1$ by $\frac{|\omega|^2}{2}$, and then integrating the resultant equation with respect to ω over $\mathbb{R}^3$, we have from the definitions of the energy–flux $\hat{q}$, the pressure tensor $\hat{\sigma}$, and the temperature θ that

$$\frac{d}{dt}\left(\rho\left(\frac{|u|^2}{2} + \theta\right)\right) + \nabla_x \cdot \left(\left(\rho\left(|u|^2 + \theta\right) + \hat{\sigma}\right)u + \hat{q}\right) = 2\rho\theta - \rho u \cdot (u - v).$$

Finally, by combining all the equations of macroscopic variables with ones of the compressible fluid variables $(n,\, v)$, we deduce that

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t\left(\rho\left(\frac{|u|^2}{2} + \theta\right)\right) + \nabla_x \cdot \left(\left(\rho\left(|u|^2 + \theta\right) + \hat{\sigma}\right)u + \hat{q}\right) = 2\rho\theta - \rho u \cdot (u - v), \\ \partial_t n + \nabla_x \cdot (nv) = 0, \\ \partial_t(nv) + \nabla_x \cdot (nv \otimes v) + \nabla_x p(n) - \mu\Delta_x v - (\mu + \lambda)\nabla_x(\nabla_x \cdot v) = \int_{\mathbb{R}^3} (\omega - v)f\mathrm{d}\omega, \end{cases} \tag{1.4}$$

for $(x,\, t) \in \mathbb{R}^3 \times \mathbb{R}_+$. Noticing that the energy–flux $\hat{q}$ is involved in $(1.4)_2$, the system (1.4) is not closed. In order to close the system (1.4), we make the assumptions

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that the fluctuations are negligible and the velocity distribution is mono–kinetic, i.e., $f(x, \omega, t) = \rho(x, t)\delta(\omega - u(x, t))$, where δ is the standard Dirac delta function. Then, it is clear that the system $(1.4)_2$ reduces to the model (1.1). It should be mentioned that the drag forcing term in the Navier–Stokes equations doesn't involve the Navier–Stokes density n. We remark that this phenomenon is natural. Indeed, if the density n of Navier–Stokes fluid disappears, then it is obvious that there is no particle, i.e., the distribution of particles $f(x, \omega, t) = 0$. Therefore, the density ρ of the Euler equations is zero since $\rho = \int_{R^3} f(x, \omega, t)dw = 0$. Consequently, the drag forcing term $\rho(u - v)$ in the Navier–Stokes equations disappears.

The global existence and large time behaviour of classical solutions to the pressureless Euler equations coupled with the incompressible/compressible Navier–Stokes equations in the periodic domain $\mathbb{T}^3$ were established by [**12**, **20**]. Recently, Choi–Jung [**14**] proved the global well–posedness and large time behaviour for the pressureless Euler equations coupled with the incompressible Navier–Stokes equations in the whole space $\mathbb{R}^3$.

However, up to now, the global well–posedness and large time behaviour for the three dimensional Cauchy problem of the pressureless Euler equations coupled with the compressible Navier–Stokes equations (1.1) still remain unsolved. Due to absence of the pressure term in the Euler equations, the main difficulty lies in the closure of the energy estimates of the particle density ρ. In fact, it is well–known that the pressureless Euler equations may develop the $\delta-$shock in finite-time even with smooth initial data [**7**–**9**, **15**, **20**, **24**]. The main purpose of this paper is to develop a global well–posedness theory for the Cauchy problem of the pressureless Euler system coupled with the compressible Navier–Stokes system (1.1). We first deduce the uniform bound of $(u, n - \bar{n}, v)$ by properly combining the drag forcing effect with the viscous effect in the compressible Navier–Stokes equations under a priori assumption that $\|\varrho(t)\|_{H^2} + \|(u, n - \bar{n}, v)(t)\|_{H^3}$ is sufficiently small. Then, the uniform bound of particle density ρ can be obtained by making a priori decay–in–time estimates on $(u, n - \bar{n}, v)$, which is based on linear decay estimates together with high–order energy estimates. Our methods mainly involve Hodge decomposition, low–frequency and high–frequency decomposition, delicate spectral analysis, and energy methods.

Before stating the main result, let us introduce several notations and conventions used throughout this paper. For $m \geqslant 0$ and $q \geqslant 1$, we use $\|\cdot\|_m$ and $\|\cdot\|_{m,q}$ to denote the norms in the Sobolev spaces $H^m(\mathbb{R}^3)$ and $W^{m,q}(\mathbb{R}^3)$ respectively. For the sake of conciseness, we do not distinguish in functional space names when they are concerned with scalar–valued or vector–valued functions; $\|(f, g)\|_X$ denotes $\|f\|_X + \|g\|_X$. We use $\langle\cdot, \cdot\rangle$ to denote the inner product in $L^2(\mathbb{R}^2)$. We employ the notation $a \lesssim b$ to mean that $a \leqslant Cb$ for a universal constant $C > 0$ which only depends on the parameters coming from the problem. We denote $\nabla = \partial_x = (\partial_1, \partial_2, \partial_3)$, where $\partial_i = \partial_{x_i}$, $\nabla_i = \partial_i$, and put $\partial_x^l f = \nabla^l f = \nabla(\nabla^{l-1} f)$. For $r \in \mathbb{R}$, let Λ^r be the pseudo–differential operator defined by

$$\Lambda^r f = \mathcal{F}^{-1}(|\xi|^r \widehat{f}(\xi)),$$

where $\widehat{f}$ are the Fourier transform of f. Let η be positive constant defined in § 3. For a radial function $\phi \in C_0^\infty(\mathbb{R}^3)$ such that $\phi(\xi) = 1$ where $|\xi| \leqslant \frac{\eta}{2}$ and $\phi(\xi) = 0$

where $|\xi| \geqslant \eta$, we define the low–frequency part of f by

$$f^l = \mathfrak{F}^{-1}[\phi(\xi)\hat{f}]$$

and the high–frequency part of f by

$$f^h = \mathfrak{F}^{-1}[(1-\phi(\xi))\hat{f}].$$

It is direct to check that $f = f^l + f^h$, if the Fourier transform of f exists.

The main novelty of this paper is to establish the global existence and large time behaviour of classical solutions to the Cauchy problem (1.1)–(1.2), and our main results are stated in the following theorem.

THEOREM 1.1. *Assume that* $\rho_0 - \bar{\rho} \in H^2(\mathbb{R}^3)$ *and* $(u_0, n_0 - \bar{n}, v_0) \in H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. *Then there exists a small constant* $\delta_0 > 0$ *such that if*

$$\|\rho_0 - \bar{\rho}\|_{H^2} + \|(u_0, n_0 - \bar{n}, v_0)\|_{H^3 \cap L^1} \leqslant \delta_0, \tag{1.5}$$

the Cauchy problem (1.1)–(1.2) *admits a unique solution* $(\rho, u, n, v)(x, t)$ *such that for any* $t \in [0, \infty)$,

$$\begin{aligned} &\|(u, n-\bar{n}, v)(t)\|_{H^3}^2 + \int_0^t (\|\nabla(n-\bar{n})(\tau)\|_{H^2}^2 + \|(u-v, \nabla v, \nabla u)(\tau)\|_{H^3}^2)\, \mathrm{d}\tau \\ &\quad \leqslant C\|(u_0, n_0-\bar{n}, v_0)\|_{H^3}^2, \end{aligned} \tag{1.6}$$

$$\|\rho(t) - \bar{\rho}\|_{H^2} \leqslant C\left(\|\rho_0 - \bar{\rho}\|_{H^2} + \|(u_0, n_0 - \bar{n}, v_0)\|_{H^3 \cap L^1}\right). \tag{1.7}$$

Moreover, the solution $(\rho - \bar{\rho}, u, n - \bar{n}, v)$ *has the following decay estimates:*

$$\|\nabla(u, n-\bar{n}, v)(t)\|_{H^2} + \|(u-v)\|_{L^2} \leqslant C(1+t)^{-5/4}, \tag{1.8}$$

$$\|(u, n-\bar{n}, v)(t)\|_{L^2} \leqslant C(1+t)^{-3/4}, \tag{1.9}$$

$$\|\partial_t(\rho-\bar{\rho}, u, n-\bar{n}, v)(t)\|_{L^2} \leqslant C(1+t)^{-5/4}. \tag{1.10}$$

REMARK 1.2. Compared to Wu–Zhang-Zou [**32**] where a two–phase model consisting of the isothermal Euler equations coupled with the compressible Navier–Stokes equations through a drag forcing term was investigated, we can not obtain the decay–in–time estimate of the particle density ρ due to the absence of the pressure in the Euler equations. However, all time derivatives $\partial_t(\rho - \bar{\rho}, u, n - \bar{n}, v)$ in L^2–norm decay in time.

REMARK 1.3. It is interesting to make a comparison between Theorem 1.1 and that of Choi–Jung [**14**], where the authors studied the global well–posedness and large time behaviour for the pressureless Euler equations coupled with the incompressible Navier–Stokes equations ($n \equiv 1$ in (1.1)) by combining energy estimates with the standard bootstrapping arguments. The main differences can be outlined as follows: Assume that $\rho_0 \in H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, $u_0 \in H^5(\mathbb{R}^3)$, $v_0 \in H^4(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, and $\|\rho_0\|_{H^3} + \|u_0\|_{H^5} + \|v_0\|_{H^4 \cap L^1}$ is sufficiently small, the authors in [**14**] showed that the pressureless Euler equations coupled with the incompressible Navier–Stokes equations has a small smooth solutions satisfying the following decay estimate:

$$\|u(t)\|_{H^4} + \|v(t)\|_{H^3} \lesssim (1+t)^{-\vartheta}, \text{ for } 0 < \vartheta < \frac{3}{4}. \tag{1.11}$$

In this paper, we only need the smallness assumption on $\|\rho_0 - \bar{\rho}\|_{H^2} + \|(u_0, n_0 - \bar{n}, v_0)\|_{H^3}$, but $\|\rho_0 - \bar{\rho}\|_{H^3} + \|u_0\|_{H^5} + \|v_0\|_{H^4}$ may be arbitrarily large. It should be mentioned that our methods rely on $\bar{\rho} > 0$ heavily, and particularly can not deal with the case $\bar{\rho} = 0$ as in [**14**]. Notice that the dissipation term $-\alpha_2(u - v)$ in the fourth equation of (2.1) will disappear if $\bar{\rho} = 0$. Therefore, it seems impossible for us to make full use of the drag forcing term and the dissipative structure of the Navier–Stokes equations to closure the energy estimates of the variables for the pressureless Euler equations. On the other hand, the decay rates in (1.8)–(1.9) imply that L^2 decay rates of (u, v) and its all–order spatial derivatives are $(1+t)^{-3/4}$ and $(1+t)^{-5/4}$ respectively, which are faster that the L^2 decay rate $(1+t)^{-\vartheta}$ with $0 < \vartheta < \frac{3}{4}$ in (1.11). In addition, the decay rate in (1.8) shows that the optimal L^2 decay rate of the difference $u - v$ between the velocities u and v is $(1+t)^{-5/4}$, which is particularly faster than ones of two velocities themselves, and is totally new as compared to [**14**].

The rest of the paper is organized as follows. In § 2, we reformulate the Cauchy problem (1.1)–(1.2). Then, we derive the linear decay estimates by employing Hodge decomposition technique and making careful spectral analysis. In § 3, by properly combining the drag forcing effect with the smooth effect of the viscosity in the compressible Navier–Stokes equations, we deduce the nonlinear energy estimates to get a key Lyapunov–type energy inequality. Then, this crucial Lyapunov–type energy inequality together with linear decay estimates obtained in § 2 gives the proof of Theorem 1.1.

2. Reformulated system

Setting

$$\varrho = \ln\rho - \ln\bar{\rho}, \sigma = n - \bar{n}, \quad \alpha_1 = \frac{P'(\bar{n})}{\bar{n}}, \quad \alpha_2 = \frac{\bar{\rho}}{\bar{n}}, \quad \bar{\mu} = \frac{\mu}{\bar{n}}, \quad \text{and} \quad \bar{\lambda} = \frac{\lambda}{\bar{n}},$$

then the Cauchy problem (1.1)–(1.2) can be reformulated as

$$\begin{cases} \partial_t \varrho = -\text{div} u - u\nabla\varrho, \\ \partial_t u + (u - v) = F_1, \\ \partial_t \sigma + \bar{n}\,\text{div}\, v = F_2, \\ \partial_t v + \alpha_1 \nabla\sigma - \bar{\mu}\Delta v - (\bar{\mu} + \bar{\lambda})\nabla\,\text{div}\, v - \alpha_2(u - v) = F_3, \\ (\varrho, u, \sigma, v)|_{t=0} := (\varrho(x), u_0(x), \sigma_0(x), v_0(x)) \to (0, \overrightarrow{0}, 0, \overrightarrow{0}), \text{ as } |x| \to \infty, \end{cases} \tag{2.1}$$

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where

$$
\begin{aligned}
F_1 &= -u\cdot\nabla u,\\
F_2 &= -v\cdot\nabla\sigma - \sigma\,\mathrm{div}\, v,\\
F_3 &= -v\cdot\nabla v + \left(\alpha_1 - \frac{P'(n)}{n}\right)\nabla\sigma + \left(\frac{\mu}{n} - \bar{\mu}\right)\Delta v + \left(\frac{\mu+\lambda}{n} - (\bar{\mu}+\bar{\lambda})\right)\nabla\,\mathrm{div}\, v\\
&\quad + \left(\frac{\rho}{n} - \alpha_2\right)(u-v).
\end{aligned}
$$

The local existence and uniqueness of the classical solution for the Cauchy problem (2.1) can be established by the methods of Kato [**23**] or Majda [**25**].

Proposition 2.1 Local existence. *Assume that the initial data* $(\varrho_0, u_0, \sigma_0, v_0) \in H^2(\mathbb{R}^3)\times H^3(\mathbb{R}^3)\times H^3(\mathbb{R}^3)\times H^3(\mathbb{R}^3)$, *and satisfies*

$$
\min_{x\in\mathbb{R}^3} e^{\varrho_0(x)} > 0, \quad \min_{x\in\mathbb{R}^3}\{\sigma_0(x)+\bar{n}\} > 0. \tag{2.2}
$$

Then there exists a positive constant T_0 *depending only on* $\|\varrho_0\|_{H^2} + \|u_0\|_{H^3} + \|\sigma_0\|_{H^3} + \|v_0\|_{H^3}$ *such that the Cauchy problem* (2.1) *has a unique solution* (ϱ, u, σ, v) *satisfying*

$$
\begin{aligned}
&\varrho \in C^0(0,T_0;H^2(\mathbb{R}^3))\cap C^1(0,T_0;H^1(\mathbb{R}^3)),\\
&u \in C^0(0,T_0;H^3(\mathbb{R}^3))\cap C^1(0,T_0;H^2(\mathbb{R}^3)),\\
&\sigma \in C^0(0,T_0;H^3(\mathbb{R}^3))\cap C^1(0,T_0;H^2(\mathbb{R}^3)) \text{ and}\\
&v \in C^0(0,T_0;H^3(\mathbb{R}^3))\cap C^1(0,T_0;H^1(\mathbb{R}^3)).
\end{aligned}
$$

Moreover, the following estimates hold,

$$
\begin{aligned}
&\|\varrho(t)\|_{H^2}^2 + \|(u,\sigma,v)(t)\|_{H^3}^2 + \int_0^{T_0} (\|\nabla(u,\sigma)(\tau)\|_{H^2}^2 + \|(u-v,\nabla v)(\tau)\|_{H^3}^2)\,\mathrm{d}\tau\\
&\quad \leqslant C(\|\rho_0\|_{H^2}^2 + \|(u_0,\sigma_0,v_0)\|_{H^3}^2),
\end{aligned} \tag{2.3}
$$

and

$$
\min_{x\in\mathbb{R}^3, 0\leqslant t\leqslant T_0} e^{\varrho(x)} > 0, \min_{x\in\mathbb{R}^3, 0\leqslant t\leqslant T_0}\{\sigma(x)+\bar{n}\} > 0. \tag{2.4}
$$

To prove global existence of smooth solutions, it suffices to establish the following a priori estimates.

Proposition 2.2 A priori estimate. *Let* $\varrho_0 \in H^2(\mathbb{R}^3)$, $(u_0, \sigma_0, v_0) \in H^3(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$. *Assume the Cauchy problem* (2.1) *admits a solution* $(\varrho, u, \sigma, v)(x,t)$ *on* $\mathbb{R}^3\times[0, T]$ *for some* $T > 0$ *in the same function class as in Proposition* 2.1. *Then there exist a small constant* $\epsilon > 0$ *and a constant* C, *which are independent of* T, *such that if*

$$\sup_{0\leqslant t\leqslant T}\{\|\varrho(t)\|_{H^2}+\|(u,\sigma,v)(t)\|_{H^3}\}\leqslant\epsilon, \tag{2.5}$$

then for any $t\in[0,\,T]$*, it holds that*

$$\begin{aligned}&\|(u,\sigma,v)(t)\|_{H^3}^2+\int_0^t\|\nabla(u,\sigma)(\tau)\|_{H^2}^2+\|(u-v,\nabla v)(\tau)\|_{H^3}^2\,\mathrm{d}\tau\\&\quad\leqslant C\|(u_0,\sigma_0,v_0)\|_{H^3}^2,\end{aligned} \tag{2.6}$$

$$\|\varrho(t)\|_{H^2}\leqslant C\left(\|\varrho_0\|_{H^2}+\|(u_0,\sigma_0,v_0)\|_{H^3\cap L^1}\right). \tag{2.7}$$

Moreover, $(\varrho,\,u,\,\sigma,\,v)$ *has the following decay propositities*

$$\|\nabla(u,\sigma,v)\|_{H^2}+\|(u-v)\|_{L^2}\leqslant C(1+t)^{-5/4}, \tag{2.8}$$

$$\|(u,\sigma,v)\|_{L^2}\leqslant C(1+t)^{-3/4}, \tag{2.9}$$

$$\|\partial_t(\varrho,u,\sigma,v)\|_{L^2}\leqslant C(1+t)^{-5/4}. \tag{2.10}$$

Theorem 1.1 follows from proposition 2.1 and proposition 2.2 by standard continuity argument.

3. Spectral analysis and linear $\mathbf{L^2}$ estimates

Define $U=(u,\,\sigma,\,v)^t$, by semigroup theory for evolutionary equation, we focus on the following linearized dissipative system for Eq. $(2.1)_2$ to Eq. $(2.1)_4$:

$$\begin{cases}U_t=\mathcal{B}U,\\U|_{t=0}=U_0,\end{cases} \tag{3.1}$$

where the operator $\mathcal{B}$ has the form as

$$\mathcal{B}=\begin{pmatrix}-I_{3\times3}&0&I_{3\times3}\\0&0&-\bar{n}\,\mathrm{div}\\\alpha_2I_{3\times3}&-\alpha_1\nabla&(\bar{\mu}\Delta-\alpha_2)I_{3\times3}+(\bar{\mu}+\bar{\lambda})\nabla\otimes\nabla\end{pmatrix}.$$

Applying the Fourier transform to the system (3.1), we have

$$\begin{cases}\widehat{U}_t=\mathcal{A}(\xi)\widehat{U},\\\widehat{U}|_{t=0}=\widehat{U}_0,\end{cases} \tag{3.2}$$

where $\widehat{U}(\xi,\,t)=\mathfrak{F}(U(x,\,t))$, $\xi=(\xi^1,\,\xi^2,\,\xi^3)^t$, and $\mathcal{A}(\xi)$ can be written as

$$\mathcal{A}=\begin{pmatrix}-I_{3\times3}&0&I_{3\times3}\\0&0&-i\bar{n}\xi^t\\\alpha_2I_{3\times3}&-\alpha_1i\xi&-(\bar{\mu}|\xi|^2+\alpha_2)I_{3\times3}-(\bar{\mu}+\bar{\lambda})\xi\otimes\xi\end{pmatrix}.$$

In order to obtain the linear time-decay estimates for the Cauchy problem (3.1), we need to analysis the properties of the semigroup, as in [**26**]. Unfortunately, it seems untractable, since the system (3.1) has seven equations. To overcome this

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difficulty, we employ the Hodge decomposition of the linear system as in [**32**], and then the system (3.1) can be decoupled into two systems, which enables us to obtain the optimal linear time–decay estimates.

Set

$$\begin{cases} \varphi = \Lambda^{-1}\operatorname{div} u, \\ \psi = \Lambda^{-1}\operatorname{div} v, \\ \Phi = \Lambda^{-1}\operatorname{curl} u, \\ \Psi = \Lambda^{-1}\operatorname{curl} v. \end{cases} \tag{3.3}$$

Then, we can rewrite the system (3.1) as follows:

$$\begin{cases} \partial_t\varphi + \varphi - \psi = 0, \\ \partial_t\sigma + \bar{n}\Lambda\psi = 0, \\ \partial_t\psi - \alpha_1\Lambda\sigma + (2\bar{\mu}+\bar{\lambda})\Lambda^2\psi - \alpha_2(\varphi-\psi) = 0, \\ (\varphi,\sigma,\psi)|_{t=0} = (\Lambda^{-1}\operatorname{div} u_0(x), \sigma_0(x), \Lambda^{-1}\operatorname{div} v_0(x)), \end{cases} \tag{3.4}$$

and

$$\begin{cases} \partial_t\Phi + \Phi - \Psi = 0, \\ \partial_t\Psi + \bar{\mu}\Lambda^2\Psi - \alpha_2(\Phi-\Psi) = 0, \\ (\Phi,\Psi)|_{t=0} = (\Lambda^{-1}\operatorname{curl} u_0(x), \Lambda^{-1}\operatorname{curl} v_0(x)). \end{cases} \tag{3.5}$$

3.1. Spectral analysis for IVP (3.4)

By virtue of the semigroup theory for evolutionary equations, we may express the IVP (3.4) for $\mathcal{U} = (\varphi,\, \sigma,\, \psi)^t$ as

$$\begin{cases} \mathcal{U}_t = \mathcal{B}_1\mathcal{U}, \\ \mathcal{U}|_{t=0} = \mathcal{U}_0, \end{cases} \tag{3.6}$$

where the operator $\mathcal{B}_1$ is given by

$$\mathcal{B}_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & -\bar{n}\Lambda \\ \alpha_2 & \alpha_1\Lambda & -(2\bar{\mu}+\bar{\lambda})\Lambda^2 - \alpha_2 \end{pmatrix}.$$

Taking the Fourier transform to the system (3.6), we have

$$\begin{cases} \widehat{\mathcal{U}}_t = \mathcal{A}_1\widehat{\mathcal{U}}, \\ \widehat{\mathcal{U}}|_{t=0} = \widehat{\mathcal{U}}_0, \end{cases} \tag{3.7}$$

where $\mathcal{A}_1(\xi)$ is defined by

$$\mathcal{A}_1(\xi) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & -\bar{n}|\xi| \\ \alpha_2 & \alpha_1|\xi| & -(2\bar{\mu}+\bar{\lambda})|\xi|^2 - \alpha_2 \end{pmatrix}.$$

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To get the eigenvalues of the matrix $\mathcal{A}_1(\xi)$, we need to compute the determinant

$$\begin{aligned}
&\det(rI - \mathcal{A}_1(\xi)) \\
&\quad = \begin{vmatrix} r+1 & 0 & -1 \\ 0 & r & \bar{n}|\xi| \\ -\alpha_2 & -\alpha_1|\xi| & r + (2\bar{\mu}+\bar{\lambda})|\xi|^2 + \alpha_2 \end{vmatrix} \\
&\quad = (r+1)[r(r+(2\bar{\mu}+\bar{\lambda})|\xi|^2+\alpha_2) + \bar{n}\alpha_1|\xi|^2] - \alpha_2 r \\
&\quad = r^3 + [(2\bar{\mu}+\bar{\lambda})|\xi|^2 + 1 + \alpha_2]r^2 + [(2\bar{\mu}+\bar{\lambda}) + \bar{n}\alpha_1]|\xi|^2 r + \bar{n}\alpha_1|\xi|^2 \\
&\quad = 0,
\end{aligned} \tag{3.8}$$

which implies that the matrix $\mathcal{A}(\xi)$ has three different eigenvalues which can be expressed as

$$r_1 = r_1(|\xi|), \quad r_2 = r_2(|\xi|), \quad r_3 = r_3(|\xi|).$$

By careful computation, we get the following Lemma.

Lemma 3.1. *There exists a positive constant $\eta_1 \ll 1$ such that, for $|\xi| \leqslant \eta_1$, the spectral has the following Taylor series expansion:*

$$\begin{cases}
r_1 = -1 - \alpha_2 + \dfrac{-\alpha_2(\alpha_2+1)(2\bar{\mu}+\bar{\lambda}) + \alpha_1\alpha_2\bar{n}}{(1+\alpha_2)^2}|\xi|^2 + \mathcal{O}(|\xi|^3), \\
r_2 = -\dfrac{(2\bar{\mu}+\bar{\lambda})(1+\alpha_2) + \alpha_1\alpha_2\bar{n}}{2(1+\alpha_2)^2}|\xi|^2 + \mathcal{O}(|\xi|^3) + \left[\sqrt{\dfrac{\bar{n}\alpha_1}{1+\alpha_2}}|\xi| + \mathcal{O}(|\xi|^2)\right] i, \\
r_3 = -\dfrac{(2\bar{\mu}+\bar{\lambda})(1+\alpha_2) + \alpha_1\alpha_2\bar{n}}{2(1+\alpha_2)^2}|\xi|^2 + \mathcal{O}(|\xi|^3) - \left[\sqrt{\dfrac{\bar{n}\alpha_1}{1+\alpha_2}}|\xi| + \mathcal{O}(|\xi|^2)\right] i.
\end{cases} \tag{3.9}$$

Lemma 3.2. *Let*

$$\nu_1 = \frac{(2\bar{\mu}+\bar{\lambda})(1+\alpha_2) + \alpha_1\alpha_2\bar{n}}{2(1+\alpha_2)^2},$$

for any $|\xi| \leqslant \eta_1$, we have

$$|\hat{\varphi}|, |\hat{\sigma}|, |\hat{\psi}| \lesssim e^{-\nu_1|\xi|^2 t}(|\hat{\varphi}_0| + |\hat{\sigma}_0| + |\hat{\psi}_0|). \tag{3.10}$$

Proof. The semigroup $e^{t\mathcal{A}}$ is expressed as

$$e^{t\mathcal{A}_1(\xi)} = \sum_{i=1}^{3} e^{r_i t} P_i(\xi),$$

where the project operators $P_i(\xi)$ can be computed as

$$P_i(\xi) = \prod_{j \neq i} \frac{\mathcal{A}_1(\xi) - r_j I}{r_i - r_j}, \quad i, j = 1, 2, 3.$$

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thus, we have

$$P_1(|\xi|) = \frac{1}{1+\alpha_2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -\alpha_2 & 0 & \alpha_2 \end{pmatrix} + \mathcal{O}(|\xi|), \tag{3.11}$$

$$P_2(|\xi|) = \frac{1}{2(1+\alpha_2)}\begin{pmatrix} \alpha_2 & -i\alpha_1\sqrt{\dfrac{1+\alpha_2}{\bar{n}\alpha_1}} & 1 \\ i\alpha_2\bar{n}\sqrt{\dfrac{1+\alpha_2}{\bar{n}\alpha_1}} & 1+\alpha_2 & i\bar{n}\sqrt{\dfrac{1+\alpha_2}{\bar{n}\alpha_1}} \\ \alpha_2 & -i\alpha_1\sqrt{\dfrac{1+\alpha_2}{\bar{n}\alpha_1}} & 1 \end{pmatrix} + \mathcal{O}(|\xi|), \tag{3.12}$$

$$P_3(|\xi|) = \frac{1}{2(1+\alpha_2)}\begin{pmatrix} \alpha_2 & i\alpha_1\sqrt{\dfrac{1+\alpha_2}{\bar{n}\alpha_1}} & 1 \\ -i\alpha_2\bar{n}\sqrt{\dfrac{1+\alpha_2}{\bar{n}\alpha_1}} & 1+\alpha_2 & -i\bar{n}\sqrt{\dfrac{1+\alpha_2}{\bar{n}\alpha_1}} \\ \alpha_2 & i\alpha_1\sqrt{\dfrac{1+\alpha_2}{\bar{n}\alpha_1}} & 1 \end{pmatrix} + \mathcal{O}(|\xi|), \tag{3.13}$$

for any $|\xi| \leqslant \eta_1$. The solution of IVP (3.4) can be expressed as

$$\widehat{\mathcal{U}}(\xi, t) = e^{\mathcal{A}_1(\xi)t}\widehat{\mathcal{U}}_0(\xi) = \left(\sum_{i=1}^{3} e^{r_i t} P_i(\xi)\right)\widehat{\mathcal{U}}_0(\xi). \tag{3.14}$$

Therefore, by combining lemma 3.1 with (3.11)–(3.14), one has (3.10) immediately. □

With the key estimate (3.10) in hand, we are able to establish the L^2–convergence rate on the low–frequency part of the solution, which is stated in the following proposition.

PROPOSITION 3.3 L^2–theory. *For any $k > -\frac{3}{2}$, there exists a positive constant C which is independent of t such that*

$$\|\nabla^k \mathcal{U}^l\|_{L^2} \leqslant C(1+t)^{-\frac{k}{2}-\frac{3}{4}}\|\widehat{\mathcal{U}^l}(0)\|_{L^\infty}.$$

Proof. Using the Plancherel theorem, together with (3.10), we have

$$\begin{aligned} \|\nabla^k \mathcal{U}^l\|_{L^2}^2 &= \|\widehat{\nabla^k \mathcal{U}^l}\|_{L^2}^2 = \||\xi|^k \widehat{\mathcal{U}^l}\|_{L^2}^2 \\ &= \||\xi|^k e^{\mathcal{A}_1(\xi)t}\widehat{\mathcal{U}^l}(0)\|_{L^2}^2 \\ &\leqslant C(1+t)^{-k-\frac{3}{2}}\|\widehat{\mathcal{U}^l}(0)\|_{L^\infty}^2. \end{aligned} \tag{3.15}$$

□

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3.2. Spectral analysis for IVP (3.5)

Set $\mathcal{V} = (\Phi,\, \Psi)^t$, the IVP (3.5) can be expressed as

$$\begin{cases} \mathcal{V}_t = \mathcal{B}_2\mathcal{V}, \\ \mathcal{V}|_{t=0} = \mathcal{V}_0, \end{cases} \tag{3.16}$$

where

$$\mathcal{B}_2 = \begin{pmatrix} -1 & 1 \\ \alpha_2 & \alpha_2 - \bar{\mu}\Lambda^2 \end{pmatrix}.$$

Similar to the derivation of Lemma (3.1), the spectral of (3.16) has the following Taylor series expansion:

$$\begin{cases} s_1 = -\alpha_2 - 1 - \dfrac{\alpha_2\bar{\mu}}{\alpha_2+1}|\xi|^2 + \mathcal{O}(|\xi|^4), \\ s_2 = -\dfrac{\bar{\mu}}{\alpha_2+1}|\xi|^2 + \mathcal{O}(|\xi|^4), \end{cases}$$

for $|\xi| \leqslant \eta_2$, where $\eta_2 \ll 1$ is a given positive constant.

From the results of Wu–Zhang–Zou [**32**], the L^2–converge rate on the low–frequency part of the solution of $\mathcal{V}$ can be given by following proposition.

Proposition 3.4. *For any $k > -\frac{3}{2}$, there exists a positive constant C which is independent of t such that*

$$\|\nabla^k \mathcal{V}^l\|_{L^2} \leqslant C(1+t)^{-\frac{k}{2}-\frac{3}{4}}\|\widehat{\mathcal{V}}^l(0)\|_{L^\infty}.$$

Combining the definitions of ϕ, ψ, Φ and Ψ, with the relations

$$u = -\Lambda^{-1}\nabla\varphi - \Lambda^{-1}\operatorname{div}\Phi, \text{ and } v = -\Lambda^{-1}\nabla\psi - \Lambda^{-1}\operatorname{div}\Psi,$$

the estimates in space $H^k(\mathbb{R}^3)$ for $(u,\, v)$ are the same as $(\phi,\, \psi,\, \Phi,\, \Psi)$.

Proposition 3.5. *For any $k > -\frac{3}{2}$, $2 \leqslant r < \infty$, and any $t \geqslant 0$, assume the initial data $U_0 \in L^1(\mathbb{R}^3)$, then the global solution $U = (u,\, \sigma,\, v)^t$ of the IVP (3.1) satisfies*

$$\|\nabla^k U^l\|_{L^2} \leqslant C(1+t)^{-\frac{k}{2}-\frac{3}{4}}\|\widehat{U}^l(0)\|_{L^\infty} \leqslant C(1+t)^{-\frac{k}{2}-\frac{3}{4}}\|U(0)\|_{L^1}.$$

In the following two lemmas, we recall Sobolev's inequality and the Galiardo–Nirenberg inequality.

Lemma 3.6. *Let $f \in H^2(\mathbb{R}^3)$. Then it holds that*

$$(i) \quad \|f\|_{L^\infty} \leqslant C\|\nabla f\|_{L^2}^{1/2}\|\nabla f\|_{H^1}^{1/2} \leqslant C\|\nabla f\|_{H^1};$$
$$(ii) \quad \|f\|_{L^6} \leqslant C\|\nabla f\|_{L^2};$$
$$(iii) \quad \|f\|_{L^p} \leqslant C\|f\|_{H^1}, 2 \leqslant p \leqslant 6.$$

LEMMA 3.7. *For $0 \leqslant i,\ j \leqslant k$, if*

$$a \in [\frac{i}{k}, 1] \text{ and } \frac{i}{3} - \frac{1}{p} = \left(\frac{j}{3} - \frac{1}{q}\right)(1-a) + \left(\frac{k}{3} - \frac{1}{r}\right)a$$

hold, then we have

$$\|\nabla^i f\|_{L^p} \leqslant C\|\nabla^i f\|_{L^q}^{1-a}\|\nabla^k f\|_{L^r}^{a}.$$

Especially, when $p = q = r = 2$, it holds that

$$\|\nabla^i f\|_{L^2} \leqslant C\|\nabla^i f\|_{L^2}^{\frac{k-j}{k-i}}\|\nabla^k f\|_{L^2}^{\frac{i-j}{k-j}}.$$

Proof. This is a special case of [28]. □

We also record the following lemma, which is used to deal with the L^2-norm of the spatial derivatives of the product of two functions.

LEMMA 3.8. *If $f,\ g \in H^k(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ for any integer $k \geqslant 1$, then we have*

$$\|\nabla^k(fg)\|_{L^2} \leqslant C(\|f\|_{L^\infty}\|\nabla^k g\|_{L^2} + \|g\|_{L^\infty}\|\nabla^k f\|_{L^2})$$

and

$$\|\nabla^k(fg)\|_{L^1} \leqslant C(\|f\|_{L^2}\|\nabla^k g\|_{L^2} + \|g\|_{L^2}\|\nabla^k f\|_{L^2}).$$

Proof. See [10] □

4. A priori estimates

We suppose that the inequality (2.5) in proposition 2.2 holds throughout this section and the next section. We will deduce a series of lemmas about the energy estimates in what follows. The first lemma is concerned with the lower order energy estimate of $(u,\ \sigma,\ v)$.

LEMMA 4.1. *There exists a suitably large constant $D_1 > 0$ which is independent of ϵ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{D_1\|(u,\sigma,v)(t)\|_{L^2}^2 + \langle\nabla\sigma, v\rangle(t)\right\} + C\big(\|\nabla(\sigma,v)\|_{L^2}^2 + \|u-v\|_{L^2}^2\big) \lesssim \epsilon(\|\nabla u\|_{L^2}^2 + \|\nabla^2 v\|_{L^2}^2), \tag{4.1}$$

for any $0 \leqslant t \leqslant T$.

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Proof. Multiplying $(2.1)_2$ – $(2.1)_4$ by u, σ, v respectively, and then integrating the resultant equations over $\mathbb{R}^3$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + \|u-v\|_{L^2}^2 + \langle u-v, v\rangle = \langle F_1, u\rangle\,, \tag{4.2}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma(t)\|_{L^2}^2 + \bar{n}\,\langle \operatorname{div} v, \sigma\rangle = \langle F_2, \sigma\rangle\,, \tag{4.3}$$

and

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|v(t)\|_{L^2}^2 - \alpha_1\langle \operatorname{div} v, \sigma\rangle + \bar{\mu}\|\nabla v\|_{L^2}^2\\ &+ (\bar{\mu}+\bar{\lambda})\|\operatorname{div} v\|_{L^2}^2 - \alpha_2\langle u-v, v\rangle = \langle F_3, v\rangle\,.\end{aligned} \tag{4.4}$$

Multiplying (4.2) by $\alpha_2\bar{n}$, (4.3) by α_1, (4.4) by $\bar{n}$, and adding them together, it follows that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\left(\alpha_2\bar{n}\|u(t)\|_{L^2}^2 + \alpha_1\|\sigma(t)\|_{L^2}^2 + \bar{n}\|v(t)\|_{L^2}^2\right) + \alpha_2\bar{n}\|u-v\|_{L^2}^2 + \bar{\mu}\bar{n}\|\nabla v\|_{L^2}^2\\ &+ (\bar{\mu}+\bar{\lambda})\bar{n}\|\operatorname{div} v\|_{L^2}^2\\ &\lesssim \big|\langle F_1, u\rangle\big| + \big|\langle F_2, \sigma\rangle\big| + \big|\langle F_3, v\rangle\big|.\end{aligned} \tag{4.5}$$

The three terms on the right hand side of the above inequality can be estimated as follows.

Firstly, for the first term, by virtue of (2.5), lemma 3.6 and Hölder inequality, we obtain

$$|\langle F_1, u\rangle| = |\langle -u\cdot\nabla u, u\rangle| \lesssim \|u\|_{L^3}\|u\|_{L^6}\|\nabla u\|_{L^2} \lesssim \|u\|_{H^1}\|\nabla u\|_{L^2}^2 \lesssim \epsilon\|\nabla u\|_{L^2}^2. \tag{4.6}$$

For the second term, by using integration by parts and performing the similar way to the proof of (4.6), one has

$$\begin{aligned}|\langle F_2, \sigma\rangle| &= |\langle -v\cdot\nabla\sigma, \sigma\rangle + \langle -\sigma\operatorname{div} v, \sigma\rangle|\\ &\lesssim |\langle \sigma\operatorname{div} v, \sigma\rangle|\\ &\lesssim \|\sigma\|_{L^3}\|\sigma\|_{L^6}\|\nabla v\|_{L^2}\\ &\lesssim \epsilon(\|\nabla\sigma\|_{L^2}^2 + \|\nabla v\|_{L^2}^2).\end{aligned} \tag{4.7}$$

Using the fact that

$$\alpha_1 - \frac{P'(n)}{n} \sim \sigma, \quad \frac{\mu}{n} - \bar{\mu} \sim \sigma, \quad \text{and} \quad \frac{\mu+\lambda}{n} - (\bar{\mu}+\bar{\lambda}) \sim \sigma, \tag{4.8}$$

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the third term can be estimated as

$$\begin{aligned}
|\langle F_3, v\rangle| &\lesssim |\langle -v\cdot\nabla v, v\rangle| + \left|\left\langle \left(\alpha_1 - \frac{P'(n)}{n}\right)\nabla\sigma, v\right\rangle\right| + \left|\left\langle \left(\frac{\mu}{n} - \bar{\mu}\right)\Delta v, v\right\rangle\right| \\
&\quad + \left|\left\langle \left(\frac{\mu+\lambda}{n} - (\bar{\mu}+\bar{\lambda})\right)\nabla \operatorname{div} v, v\right\rangle\right| + \left|\left\langle \left(\frac{\rho}{n} - \alpha_2\right)(u-v), v\right\rangle\right| \\
&\lesssim \|v\|_{L^3}\|v\|_{L^6}\|\nabla v\|_{L^2} + \|\sigma\|_{L^3}\|\nabla\sigma\|_{L^2}\|v\|_{L^6} \\
&\quad + \|(\sigma, v)\|_{H^1}\|\nabla v\|_{L^2}\|\nabla(\sigma, v)\|_{L^2} \\
&\quad + \|(\rho,\sigma)\|_{H^1}\|u-v\|_{L^2}\|\nabla v\|_{L^2} \\
&\lesssim \epsilon(\|\nabla v\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2 + \|u-v\|_{L^2}^2).
\end{aligned} \tag{4.9}$$

Substituting (4.6), (4.7), and (4.9) into (4.5) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u,\sigma,v)(t)\|_{L^2}^2 + C\left(\|u-v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\operatorname{div} v\|_{L^2}^2\right) \lesssim \epsilon(\|\nabla\sigma\|_{L^2}^2 + \|\nabla u\|_{L^2}^2). \tag{4.10}$$

Next, we shall derive the energy dissipation for $\|\nabla\sigma\|_{L^2}^2$. Multiplying $(2.1)_4$ by $\nabla\sigma$, integrating them over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
\alpha_1\|\nabla\sigma\|_{L^2}^2 &= \langle -v_t, \nabla\sigma\rangle + \left\langle \bar{\mu}\Delta v + (\bar{\mu}+\bar{\lambda})\nabla\operatorname{div} v, \nabla\sigma\right\rangle \\
&\quad + \alpha_2\langle u-v, \nabla\sigma\rangle + \langle F_3, \nabla\sigma\rangle.
\end{aligned} \tag{4.11}$$

From $(2.1)_3$, the first term on the right hand side can be written as

$$\begin{aligned}
\langle -v_t, \nabla\sigma\rangle &= -\frac{\mathrm{d}}{\mathrm{d}t}\langle v, \nabla\sigma\rangle + \langle \nabla\sigma_t, u\rangle \\
&= -\frac{\mathrm{d}}{\mathrm{d}t}\langle v, \nabla\sigma\rangle + \bar{n}\|\operatorname{div} v\|_{L^2}^2 + \langle -F_2, \operatorname{div} v\rangle.
\end{aligned} \tag{4.12}$$

By the definition of F_2, we obtain

$$\begin{aligned}
|\langle -F_2, \operatorname{div} v\rangle| &\lesssim |\langle v\cdot\nabla\sigma, \operatorname{div} v\rangle| + |\langle \sigma \operatorname{div} v, \operatorname{div} v\rangle| \\
&\lesssim \|\nabla\sigma\|_{L^3}\|v\|_{L^6}\|\operatorname{div} v\|_{L^2} + \|\sigma\|_{L^\infty}\|\operatorname{div} v\|_{L^2}^2 \\
&\lesssim \|\nabla\sigma\|_{H^1}\|\nabla v\|_{L^2}^2 + \|\nabla\sigma\|_{H^1}\|\operatorname{div} v\|_{L^2}^2 \lesssim K_0\|\nabla v\|_{L^2}^2.
\end{aligned} \tag{4.13}$$

Taking the same argument to the term $\langle F_3, \nabla\sigma\rangle$, it is easy to get

$$\begin{aligned}
|\langle F_3, \nabla\sigma\rangle| &\leqslant |\langle -v\cdot\nabla v, \nabla\sigma\rangle| + \left|\left\langle \left(\alpha_1 - \frac{P'(n)}{n}\right)\nabla\sigma, \nabla\sigma\right\rangle\right| \\
&\quad + \left|\left\langle \left(\frac{\mu}{n} - \bar\mu\right) K_0 v, \nabla\sigma\right\rangle\right| \\
&\quad + \left|\left\langle \left(\frac{\mu+\lambda}{n} - (\bar\mu + \bar\lambda)\right)\nabla \operatorname{div} v, \nabla\sigma\right\rangle\right| + \left|\left\langle \left(\frac{\rho}{n} - \alpha_2\right)(u-v), \nabla\sigma\right\rangle\right| \\
&\lesssim \|\nabla v\|_{L^3}\|v\|_{L^6}\|\nabla\sigma\|_{L^2} + \|\sigma\|_{L^\infty}\|\nabla\sigma\|^2 + \|\sigma\|_{L^\infty}\|\nabla^2 v\|_{L^2}\|\nabla\sigma\|_{L^2} \\
&\quad + \|(\rho,\sigma)\|_{H^1}\|u-v\|_{L^2}\|\nabla\sigma\|_{L^2} \\
&\lesssim \|\nabla v\|_{H^1}^2 + \|\nabla\sigma\|_{L^2}^2 + \|u-v\|_{L^2}^2.
\end{aligned} \tag{4.14}$$

Substituting (4.12)–(4.14) into (4.11) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v, \nabla\sigma\rangle + C\|\nabla\sigma\|_{L^2}^2 \lesssim \|\nabla v\|_{H^1}^2 + \|u-v\|_{L^2}^2. \tag{4.15}$$

Multiplying (4.10) by D_1 suitably large and adding it to (4.15), one has (4.1) since $\epsilon > 0$ is sufficiently small. This completes the proof of lemma 4.1 □

For the higher order energy estimate for (u, σ, v), we have following lemma.

LEMMA 4.2. *For any $0 \leqslant t \leqslant T$, there exists a suitably large constant $D_2 > 0$ which is independent of ϵ such that*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{ D_2 H_1(u(t),\sigma(t),v(t)) + \sum_{1\leqslant|k|\leqslant 2}\left\langle \nabla^k v, \nabla\nabla^k\sigma\right\rangle(t)\right\} + C\big(\|\nabla^2\sigma\|_{H^1}^2 \\
&\quad + \|(\nabla(u-v), \nabla^2 v)\|_{H^2}^2 + \|\nabla^2 u\|_{H^1}^2\big) \lesssim \epsilon\|\nabla(u,\sigma,v)\|_{L^2}^2,
\end{aligned} \tag{4.16}$$

where $H_1(u, \sigma, v)$ is equivalent to $\|\nabla(u, \sigma, v)\|_{H^2}^2$.

Proof. For each multi–index k with $1 \leqslant |k| \leqslant 3$, by applying the operator ∇^k to $(2.1)_2$–$(2.1)_4$ and multiplying them by $\nabla^k u$, $\nabla^k\sigma$, $\nabla^k v$ respectively, and then integrating them over $\mathbb{R}^3$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k u(t)\|_{L^2}^2 + \|\nabla^k(u-v)\|_{L^2}^2 + \left\langle \nabla^k(u-v), \nabla^k u\right\rangle = \left\langle \nabla^k F_1, \nabla^k u\right\rangle, \tag{4.17}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k\sigma(t)\|_{L^2}^2 + \bar n\left\langle \nabla^k \operatorname{div} v, \nabla^k\sigma\right\rangle = \left\langle \nabla^k F_2, \nabla^k\sigma\right\rangle, \tag{4.18}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k v(t)\|_{L^2}^2 - \alpha_1\left\langle \nabla^k \operatorname{div} v, \nabla^k\sigma\right\rangle + \bar\mu\|\nabla^{k+1}v\|_{L^2}^2 + (\bar\mu+\bar\lambda)\|\nabla^k \operatorname{div} v\|_{L^2}^2 \\
&\qquad - \alpha_2\left\langle \nabla^k(u-v), \nabla^k u\right\rangle = \left\langle \nabla^k F_3, \nabla^k v\right\rangle.
\end{aligned} \tag{4.19}$$

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Computing

$$\sum_{1\leqslant k\leqslant 3}(\alpha_2\bar{n}\times(4.17)+\alpha_1\times(4.18)+\bar{n}4.19),$$

we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\sum_{1\leqslant k\leqslant 3}(\alpha_2\bar{n}\|\nabla^k u(t)\|_{L^2}^2+\alpha_1\|\nabla^k\sigma(t)\|_{L^2}^2+\bar{n}\|\nabla^k v(t)\|_{L^2}^2)\\ &+C\left(\sum_{1\leqslant k\leqslant 3}\|\nabla^k(u-v)\|_{L^2}^2+\sum_{1\leqslant k\leqslant 3}\|\nabla^{k+1}v\|_{L^2}^2\right)\\ &\lesssim\sum_{1\leqslant k\leqslant 3}\left|\left\langle\nabla^k F_1,\nabla^k u\right\rangle+\left\langle\nabla^k F_2,\nabla^k\sigma\right\rangle+\left\langle\nabla^k F_3,\nabla^k v\right\rangle\right|.\end{aligned}\tag{4.20}$$

In what follows, we shall give the estimates of the three terms on the right hand side of the above equation one by one.

Firstly, for the term $\langle\nabla^k F_1,\,\nabla^k u\rangle$, making use of integration by parts, (2.5), lemma 3.6, we obtain

$$\begin{aligned}&\left|\left\langle\nabla^k F_1,\nabla^k u\right\rangle\right|\\ &\quad\lesssim\left|\int_{\mathbb{R}^3}\operatorname{div}u|\nabla^k u|^2\,\mathrm{d}x\right|\\ &\qquad+\left|\int_{\mathbb{R}^3}\nabla u|\nabla^k u|^2\,\mathrm{d}x\right|+\mathcal{H}(k-2)\sum_{m=2}^{k-1}\left|\int_{\mathbb{R}^3}\nabla^{k-m+1}u\nabla^m u\nabla^k u\,\mathrm{d}x\right|\\ &\quad\lesssim\|\nabla u\|_{L^\infty}\|\nabla^k u\|_{L^2}^2+\mathcal{H}(k-2)\sum_{m=2}^{k-1}\|\nabla^m u\|_{L^4}\|\nabla^{k-m+1}u\|_{L^4}\|\nabla^k u\|_{L^2}\\ &\quad\lesssim\epsilon\|\nabla^k u\|_{L^2}^2,\end{aligned}\tag{4.21}$$

where $\mathcal{H}=\mathcal{X}(0,\,\infty)$ is the Heaviside function, and in the last inequality, we have used lemma 3.7 to get

$$\begin{aligned}\|\nabla^m u\|_{L^4}&\leqslant C\|\nabla^{5/2}u\|_{L^2}^{\frac{4(k-m)-3}{4k-10}}\|\nabla^k u\|_{L^2}^{\frac{4m-7}{4k-10}},\\ \|\nabla^{k-m+1}u\|_{L^4}&\leqslant C\|\nabla^{5/2}u\|_{L^2}^{\frac{4m-7}{4k-10}}\|\nabla^k u\|_{L^2}^{\frac{4(k-m)-3}{4k-10}}.\end{aligned}$$

Using the similar argument as (4.21), $\langle\nabla^k F_2,\,\nabla^k\sigma\rangle$ can be estimated as

$$\begin{aligned}\left|\left\langle\nabla^k F_2,\nabla^k\sigma\right\rangle\right|&\lesssim\left|\left\langle\nabla^k(v\cdot\nabla\sigma),\nabla^k\sigma\right\rangle\right|+\left|\left\langle\nabla^k(\sigma\operatorname{div}v),\nabla^k\sigma\right\rangle\right|\\ &\lesssim\epsilon(\|\nabla^k\sigma\|_{L^2}^2+\|\nabla^k v\|_{L^2}^2+\|\nabla^{k+1}v\|_{L^2}^2).\end{aligned}\tag{4.22}$$

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From the definition of F_3, we have from a direct computation that

$$\begin{aligned}
&\big|\big\langle \nabla^k F_3, \nabla^k v\big\rangle\big| \\
&\lesssim \big|\big\langle \nabla^k (v\cdot\nabla v), \nabla^k v\big\rangle\big| \\
&\quad + \left|\left\langle \nabla^k[(\alpha_1 - \frac{P'(n)}{n})\nabla\sigma], \nabla^k v\right\rangle\right| + \left|\left\langle \nabla^k[(\frac{\mu}{n}-\bar{\mu})\Delta v], \nabla^k v\right\rangle\right| \\
&\quad + \left|\left\langle \nabla^k[(\frac{\mu+\lambda}{n} - (\bar{\mu}+\bar{\lambda}))\nabla \operatorname{div} v], \nabla^k v\right\rangle\right| + \left|\left\langle \nabla^k((\frac{\rho}{n}-\alpha_2)(u-v)), \nabla^k v\right\rangle\right| \\
&= I_{11} + I_{12} + I_{13} + I_{14} + I_{15}.
\end{aligned} \tag{4.23}$$

Similar to the proof of (4.21), we have from (4.8) that

$$I_{11} + I_{12} + I_{13} + I_{14} \lesssim \epsilon(\|\nabla^k\sigma\|_{L^2}^2 + \|\nabla^k v\|_{L^2}^2 + \|\nabla^{k+1} v\|_{L^2}^2). \tag{4.24}$$

For the term I_{15}, it follows from (2.5) and lemma 3.7 that

$$\begin{aligned}
I_{15} &= \left|\left\langle \nabla^{k-1}\left(\left(\frac{\rho}{n}-\alpha_2\right)(u-v)\right), \nabla^{k+1} v\right\rangle\right| \\
&\lesssim \|(\rho,\sigma)\|_{L^\infty}\|\nabla^k(u-v)\|_{L^2}\|\nabla^k v\|_{L^2} \\
&\quad + \mathcal{H}(k-1)\sum_{m=1}^{k-1}\|\nabla^m(u-v)\|_{L^4}\|\nabla^{k-1-m}(\rho,\sigma)\|_{L^4}\|\nabla^{k+1}v\|_{L^2} \\
&\lesssim \epsilon(\|\nabla^k(u-v)\|_{L^2}^2 + \|\nabla^{k-1}\sigma\|_{L^2}^2 + \|\nabla^{k+1}v\|_{L^2}^2 + \|\nabla^k v\|_{L^2}^2),
\end{aligned} \tag{4.25}$$

where in the last inequality, we have used the fact that

$$\|\nabla^m(u-v)\|_{L^4} \leqslant C\|\nabla^{3/2}(u-v)\|_{L^2}^{\frac{4(k-m)-3}{4k-6}}\|\nabla^k(u-v)\|_{L^2}^{\frac{4m-3}{4k-6}}.$$

Putting (4.24) and (4.25) into (4.23), one has

$$\big|\big\langle \nabla^k F_3, \nabla^k v\big\rangle\big| \lesssim \epsilon(\|\nabla^k(u-v)\|_{L^2}^2 + \|\nabla^k\sigma\|_{L^2}^2 + \|\nabla^k v\|_{L^2}^2 + \|\nabla^{k+1}v\|_{L^2}^2). \tag{4.26}$$

Substituting (4.21), (4.22) and (4.26) into (4.20) gives

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\leqslant k\leqslant 3}\big(\alpha_2\bar{n}\|\nabla^k u(t)\|_{L^2}^2 + \alpha_1\|\nabla^k\sigma(t)\|_{L^2}^2 \\
&\quad + \bar{n}\|\nabla^k v(t)\|_{L^2}^2\big) + C\sum_{1\leqslant k\leqslant 3}\big(\|\nabla^k(u-v)\|_{L^2}^2 \\
&\quad + \sum_{1\leqslant k\leqslant 3}\|\nabla^{k+1}v\|_{L^2}^2\big) \lesssim \epsilon\sum_{1\leqslant k\leqslant 3}\|\nabla^k(u,v,\sigma)\|_{L^2}^2.
\end{aligned} \tag{4.27}$$

On the other hand, for the estimates of $\nabla^k u\,(1\leqslant k\leqslant 3)$, we also have the form as

$$\|\nabla^k u\|_{L^2}^2 \leqslant \|\nabla^k(u-v)\|_{L^2}^2 + \|\nabla^k v\|_{L^2}^2. \tag{4.28}$$

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Combining (4.27) with (4.28), we find that there exists a function $H_1(u, \sigma, v)$ which is equivalent to $\|\nabla(u, \sigma, v)\|_{H^2}^2$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}H_1(u(t),\sigma(t),v(t)) + C\left(\|(\nabla(u-v),\nabla^2 v)\|_{H^2}^2 + \|\nabla^2 u\|_{H^1}^2\right) \lesssim \epsilon(\|\nabla\sigma\|_{H^2}^2 + \|\nabla(u,v)\|_{L^2}^2). \tag{4.29}$$

Next, we shall derive the energy dissipation for $\|\nabla^k\nabla\sigma\|_{L^2}^2$ for $1 \leqslant |k| \leqslant 2$. Applying the operator ∇^k to $(2.1)_4$, multiplying the resulting equation by $\nabla\nabla^k\sigma$, summing up and integrating it over $\mathbb{R}^3$, we get

$$\begin{aligned}\alpha_1\|\nabla^2\sigma\|_{H^1}^2 = \sum_{1\leqslant|k|\leqslant 2}\Big(&-\left\langle\nabla^k v_t, \nabla\nabla^k\sigma\right\rangle + \bar\mu\left\langle\nabla^k(\Delta v), \nabla\nabla^k\sigma\right\rangle \\ &+ (\bar\mu+\bar\lambda)\left\langle\nabla^k\nabla\operatorname{div} v, \nabla\nabla^k\sigma\right\rangle \\ &+ \alpha_2\left\langle\nabla^k(u-v), \nabla\nabla^k\sigma\right\rangle + \left\langle\nabla^k F_3, \nabla\nabla^k\sigma\right\rangle\Big) \\ = {}& I_{21}+I_{22}+I_{23}+I_{24}+I_{25}.\end{aligned} \tag{4.30}$$

For the term I_{21}, by virtue of $(2.1)_3$, we can apply integration by parts, Hölder inequality, Lemma (3.7) and lemma 3.2 to deduce that

$$\begin{aligned}I_{21} &= -\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\leqslant|k|\leqslant 2}\left\langle\nabla^k v, \nabla\nabla^k\sigma\right\rangle(t) - \left\langle\nabla^k\operatorname{div} v, \nabla^k\sigma_t\right\rangle \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\leqslant|k|\leqslant 2}\left\langle\nabla^k v, \nabla\nabla^k\sigma\right\rangle(t) + \bar n\|\nabla^k\operatorname{div} v\|_{L^2}^2 + \left\langle\nabla^k(v\cdot\nabla\sigma), \nabla^k\operatorname{div} v\right\rangle \\ &\quad + \left\langle\nabla^k(\sigma\cdot\operatorname{div} v), \nabla^k\operatorname{div} v\right\rangle \\ &\leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\leqslant|k|\leqslant 2}\left\langle\nabla^k v, \nabla\nabla^k\sigma\right\rangle(t) + C\left(\epsilon\|\nabla^{k+1}\sigma\|_{L^2}^2 + \|\nabla^{k+1}v\|_{L^2}^2\right).\end{aligned} \tag{4.31}$$

For the terms I_{22}, I_{23} and I_{24}, by the similar argument, we have

$$|I_{22}|+|I_{23}|+|I_{24}| \lesssim \epsilon(\|\nabla^{k+1}\sigma\|_{L^2}^2 + \|\nabla^{k+2}v\|_{L^2}^2 + \|\nabla^k(u-v)\|_{L^2}^2). \tag{4.32}$$

For the term I_{25}, we write it as

$$\begin{aligned}I_{25} &= -\left\langle\nabla^k(v\cdot\nabla v), \nabla\nabla^k\sigma\right\rangle + \left\langle\nabla^k[(\alpha_1 - \frac{P'(n)}{n})\nabla\sigma], \nabla\nabla^k\sigma\right\rangle \\ &\quad + \left\langle\nabla^k[(\frac{\mu}{n}-\bar\mu)K_0 v], \nabla\nabla^k\sigma\right\rangle + \left\langle\nabla^k[\frac{\mu+\lambda}{n} - (\bar\mu+\bar\lambda)\nabla\operatorname{div} v], \nabla\nabla^k\sigma\right\rangle \\ &\quad + \left\langle\nabla^k[(\frac{\rho}{n}-\alpha_2)(u-v)], \nabla\nabla^k\sigma\right\rangle \\ &= I_{251}+I_{252}+I_{253}+I_{254}+I_{255}.\end{aligned} \tag{4.33}$$

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For the terms I_{251}–I_{254}, using (4.8) and making a direct computation, we have

$$|I_{251}| + |I_{252}| + |I_{253}| + |I_{254}| \lesssim \epsilon(\|\nabla^{k+1}\sigma\|_{L^2}^2 + \|\nabla^{k+1}v\|_{L^2}^2 + \|\nabla^{k+2}v\|_{L^2}^2). \tag{4.34}$$

For the term I_{255}, employing the similar argument used in the proof of (4.25), we get that

$$\begin{aligned} |I_{255}| &\lesssim \|\frac{\rho}{n} - \alpha_2\|_{L^\infty}\|\nabla^k(u-v)\|_{L^2}\|\nabla\nabla^k\sigma\|_{L^2} \\ &\quad + \mathcal{H}(k-1)\sum_{m=1}^{k-1}\|\nabla^m(u-v)\|_{L^4}\|\nabla^{k-1-m}(\rho,\sigma)\|_{L^4}\|\nabla\nabla^k\sigma\|_{L^2} \\ &\lesssim \epsilon(\|\nabla^k(u-v)\|_{L^2}^2 + \|\nabla^{k+1}\sigma\|_{L^2}^2 + \|\nabla^{k-1}\sigma\|_{L^2}^2). \end{aligned} \tag{4.35}$$

Substituting (4.35) and (4.34) into (4.33) yields

$$|I_{25}| \lesssim K_0(\|\nabla^k(u-v)\|_{L^2}^2 + \|\nabla^{k+1}\sigma\|_{L^2}^2 + \|\nabla^{k+1}v\|_{L^2}^2 + \|\nabla^{k+2}v\|_{L^2}^2 + \|\nabla^k\sigma\|_{L^2}^2). \tag{4.36}$$

Adding (4.31), (4.32) and (4.36) into (4.30), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\leqslant|k|\leqslant 2}\left\langle\nabla^k v, \nabla\nabla^k\sigma\right\rangle(t) + C\sum_{1\leqslant k\leqslant 2}\|\nabla^k\nabla\sigma\|_{L^2}^2 \\ &\quad\lesssim \epsilon(\|\nabla\sigma\|_{L^2}^2 + \|\nabla(u-v)\|_{H^1}^2 + \|\nabla^2 v\|_{H^2}^2). \end{aligned} \tag{4.37}$$

Since K_0 is sufficiently small, multiplying (4.29) by D_2 suitably large and adding it to (4.37), we have (4.16). Thus, we complete the proof of the lemma. □

5. The proof of global well–posedness

In this section, we are devoted to proving proposition 2.2. We will do it by three steps.

Step 1: Combining lemma 4.1 with lemma 4.2, there exists a function $H_2(u, \sigma, v)$ which is equivalent to $\|(u, \sigma, v)\|_{H^3}$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}H_2(u,\sigma,v) + \|\nabla(u,\sigma)\|_{H^2}^2 + \|(u-v,\nabla v)\|_{H^3}^2 \lesssim 0, \tag{5.1}$$

for any $0 \leqslant t \leqslant T$, which implies (2.6).

Step 2: From $(2.1)_2$ and $(2.1)_4$, we see that

$$\partial_t(u-v) + (1+\alpha_2)(u-v) = F_1 + F_3 - \alpha_1\nabla\sigma + \bar{\mu}\Delta v - (\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}v.$$

Performing the similar procedure as in lemma 4.1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u-v)(t)\|_{L^2}^2 + (1+\alpha_2)\|(u-v)\|_{L^2}^2 \lesssim \|\nabla(u,\sigma,v)\|_{H^1}^2. \tag{5.2}$$

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Now we define the temporal energy functional

$$H_3(t) = D_3 H_1(u(t), \sigma(t), v(t)) + \sum_{1\leqslant k\leqslant 2} \left\langle \nabla^k v, \nabla\nabla^k \sigma \right\rangle + \|(u-v)(t)\|_{L^2}^2,$$

for any $0 \leqslant t \leqslant T$, where it is noticed that $H_3(t)$ is equivalent to $\|\nabla(u,\, \sigma,\, v)\|_{H^2}^2$ since D_3 is large enough.

Using lemma 4.2, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H_3(t) &+ C(\|\nabla^2\sigma\|_{H^1}^2 + \|\nabla^2 v\|_{H^2}^2 + \|\nabla^2 u\|_{H^1}^2) \\ &\lesssim \epsilon \|\nabla(u,\sigma,v)\|_{L^2}^2 + \|\nabla(u^l,\sigma^l,v^l)\|_{L^2}^2. \end{aligned} \tag{5.3}$$

Adding $\|\nabla(u^l,\, \sigma^l,\, v^l)\|^2$ to both side of (5.3), we deduce that there exists a suitably large constant $D_4 > 0$ which is independent of ϵ, such that

$$\frac{\mathrm{d}}{\mathrm{d}t} H_3(t) + \frac{1}{D_4} H_3(t) \lesssim \|\nabla(u^l,\sigma^l,v^l)\|_{L^2}^2, \tag{5.4}$$

where we have used the fact that $\|\nabla(u^h,\, \sigma^h,\, v^h)\|_{L^2} \leqslant \|\nabla^2(u,\, \sigma,\, v)\|_{L^2}$. If we define

$$M(t) = \sup_{0\leqslant\tau\leqslant t} (1+\tau)^{\frac{5}{2}} H_3(\tau), \tag{5.5}$$

then

$$\|\nabla(u,\sigma,v)\|_{H^2} \leqslant C\sqrt{H_3(t)} \leqslant C(1+\tau)^{-5/4}\sqrt{M(t)}, \quad 0 \leqslant \tau \leqslant t \leqslant T. \tag{5.6}$$

To close the estimate (5.4), we will derive the time–decay estimate of $\|\nabla(u,\, \sigma,\, v)\|_{L^2}^2$.

From Duhamel's principle, the solutions of system (2.1) have the form as

$$U = e^{-t\mathcal{B}} U(0) + \int_0^t e^{-(t-\tau)B} \mathcal{F}(\tau)\,\mathrm{d}\tau. \tag{5.7}$$

By virtue of proposition 3.5, Plancherel theorem, Hölder inequality, and the Hausdorff-Young inequality, we have

$$\begin{aligned} &\|\nabla(u^l(t),\sigma^l(t),v^l(t))\|_{L^2} \\ &\quad\leqslant C(1+t)^{-\frac{5}{4}} \|(u,\sigma,v)(0)\|_{L^1} + \int_0^t (1+t-\tau)^{-\frac{5}{4}} \|(F_1,F_2,F_3)(\tau)\|_{L^1\cap H^1}\,\mathrm{d}\tau \\ &\quad\leqslant C\left(\delta_0(1+t)^{-5/4} + \epsilon\int_0^t (1+t-\tau)^{-5/4}(1+\tau)^{-5/4}\sqrt{M(t)}\mathrm{d}\tau\right) \\ &\quad\leqslant C(1+t)^{-5/4}(\delta_0 + \epsilon\sqrt{M(t)}), \end{aligned} \tag{5.8}$$

where we have used the fact that

$$\|(F_1,F_2,F_3)\|_{L^1} \leqslant C\epsilon\|\nabla(u,\sigma,v)\|_{H^1} \leqslant C\epsilon\|\nabla(u,\sigma,v)\|_{H^2}. \tag{5.9}$$

$$\|(F_1,F_2,F_3)\|_{H^1} \leqslant C\epsilon\|\nabla(u,\sigma,v)\|_{H^2}. \tag{5.10}$$

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Hence, by using Gronwall's inequality and putting (5.8) into (5.4), we have

$$\begin{aligned} H_3(t) &\leqslant e^{-\frac{1}{D_4}t} H_3(0) + C\int_0^t e^{-\frac{1}{D_4}(t-\tau)} \|\nabla(u(\tau),\sigma(\tau),v(\tau))\|_{L^2}^2 \,\mathrm{d}\tau \\ &\leqslant e^{-\frac{1}{D_4}t} H_3(0) + C\int_0^t e^{-\frac{1}{D_4}(t-\tau)} (1+\tau)^{-\frac{5}{2}} (\delta_0^2 + \epsilon^2 M(t)) \,\mathrm{d}\tau \\ &\leqslant C(1+t)^{-\frac{5}{2}} (\delta_0^2 + \epsilon^2 M(t)). \end{aligned} \tag{5.11}$$

Since $M(t)$ is non–decreasing, we have from (5.5) and (5.11) that

$$M(t) \leqslant C(N_0^2 + K_0^2 M(t)),$$

for any $0 \leqslant t \leqslant T$, which implies that

$$M(t) \leqslant CN_0^2,$$

since $K_0 > 0$ is small enough. Thus we obtain (2.8).

Next, by making use of proposition 3.5, (5.9) and (5.10), from Duhamel's principle, we obtain

$$\begin{aligned} &\|(u^l,\sigma^l,v^l)(t)\|_{L^2} \\ &\leqslant C(1+t)^{-\frac{3}{4}} \|(u,\sigma,v)(0)\|_{L^1} + \int_0^t (1+t-\tau)^{-\frac{3}{4}} \|(F_1,F_2,F_3)(\tau)\|_{L^1\cap L^2} \,\mathrm{d}\tau \\ &\leqslant C\delta_0 \left((1+t)^{-3/4} + \int_0^t (1+t-\tau)^{-3/4} (1+\tau)^{-5/4} \mathrm{d}\tau \right) \\ &\leqslant C\delta_0 (1+t)^{-3/4}, \end{aligned} \tag{5.12}$$

for any $0 \leqslant t \leqslant T$. Thus, this together with the fact that $\|(u^h, \sigma^h, v^h)\|_{L^2} \leqslant \|\nabla(u, \sigma, v)\|_{L^2}$, we get

$$\|(u,\sigma,v)(t)\|_{L^2} \lesssim \|(u^l,\sigma^l,v^l)(t)\|_{L^2} + \|\nabla(u,\sigma,v)(t)\|_{L^2} \leqslant C\delta_0(1+t)^{-3/4} \tag{5.13}$$

which implies (2.9).

Step 3: Multiplying $(2.1)_1$ by ϱ, integrating over $\mathbb{R}^3$ and using Cauchy-Schwarz inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\varrho(t)\|_{L^2}^2 = -\langle \operatorname{div} u, \varrho\rangle - \langle u\cdot\nabla\varrho, \varrho\rangle \lesssim \|\nabla u\|_{H^2} \|\varrho\|_{L^2}. \tag{5.14}$$

Next, applying the operator ∇^k to $(2.1)_1$, Multiplying it by $\nabla^k\varrho$, and integrating over $\mathbb{R}^3$, for $1 \leqslant |k| \leqslant 2$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla^k \varrho(t)\|_{L^2}^2 = -\left\langle \nabla^k \operatorname{div} u, \nabla^k \varrho\right\rangle - \left\langle \nabla^k (u\cdot\nabla\varrho), \nabla^k\varrho\right\rangle. \tag{5.15}$$

It is easy to obtain

$$\left|\left\langle \nabla^k \operatorname{div} u, \nabla^k\varrho\right\rangle\right| \lesssim \|\nabla^{k+1} u\|_{L^2} \|\nabla^k\varrho\|_{L^2}, \tag{5.16}$$

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and

$$\left|\left\langle\nabla^k(u\cdot\nabla\rho),\nabla^k\rho\right\rangle\right|\lesssim\left|\left\langle(u\cdot\nabla\nabla^k\rho),\nabla^k\rho\right\rangle\right|+\sum_{|m|=0}^{|k|-1}\left|\left\langle(\nabla^{k-m}u)\cdot\nabla\nabla^m\rho),\nabla^k\rho\right\rangle\right|$$
$$\lesssim\|\nabla u\|_{H^2}\|\nabla\varrho\|_{H^1}.$$

Thus, for $1\leqslant|k|\leqslant 2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k\rho(t)\|_{L^2}^2\lesssim\|\nabla u\|_{H^2}\|\nabla\varrho\|_{H^1}. \tag{5.17}$$

Combining (5.14) with (5.17), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\rho(t)\|_{H^2}\lesssim\|\nabla u\|_{H^2}\lesssim(1+t)^{-5/4}\delta_0.$$

Integrating the above inequality from 0 to t, we obtain (2.7). For (2.10), making use of the above estimates and (2.1), we have

$$\begin{aligned}\|\partial_t(\rho,u,\sigma,v)\|_{L^2}&\leqslant C(\|\nabla u\|_{L^2}+\|\nabla\sigma\|_{L^2}+\|\nabla v\|_{H^1})\\&\leqslant CN_0(1+t)^{-5/4}.\end{aligned}$$

for any $0\leqslant t\leqslant T$. Thus, we get (2.10).

Therefore, we have complete the proof of proposition 2.2.

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