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TENSOR PRODUCTS OF l^2-VALUED MEASURES

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Duchon (1967, 1969) and Duchon and Kluvánek (1967) considered the problems of the existence of countably additive tensor products for vector measures. Duchon and Kluvánek (1967) showed that a countably additive product with respect to the inductive tensor topology always exists while Kluvánek (1970) presented an example which showed that this was not the case for the projective tensor topology. Kluvánek considered yet another tensor product topology which is stronger than the inductive topology but weaker than the projective tensor topology and showed that a countably additive product for two vector measures always exists for this particular tensor topology, see Kluvánek (1973). He has conjectured that this topology is the strongest tensor topology (given by a cross norm) which always admits products for any two arbitrary vector measures. In this note we use an example of Kluvánek (1974) to show that this conjecture is indeed true when one of the factors in the tensor product is l^2 and the other factor is metrizable. The construction used also clarifies a conjecture made by Swartz (to appear) concerning products of Hilbert space valued measures.

We first describe the tensor product topology used by Kluvánek (1973); this topology is obtained from the dissertation of Jacobs (see Kluvánek (1973)). Let X, Y and Z be locally convex Hausdorff spaces with the topology of X (respectively Y, Z) generated by the family of seminorms $\mathcal{P}$ (respectively $\mathcal{Q}, \mathcal{R}$). (For convenience we assume that all vector spaces are real.) For $p \in \mathcal{P}$ and $q \in \mathcal{Q}$ define a cross norm $r = p \otimes_l q$ on $X \otimes Y$ by

$$(1) \qquad r(u) = \inf \sup p\,(\Sigma \alpha_i q(y_i) x_i),$$

where the supremum is computed over all real α_i with $|\alpha| \leqq 1$ and the infimum is computed over all representations $u = \Sigma x_i \otimes y_i$, $x_i \in X$, $y_i \in Y$. $X \otimes Y$ equipped with the locally convex topology l generated by the family of semi-norms $\{p \otimes_l q : p \in \mathcal{P}, q \in \mathcal{Q}\}$ is denoted by $X \otimes_l Y$; if X and Y are both normed spaces, then $X \otimes_l Y$ is obviously a normed space and we always assume it is normed by $\| \; \| \otimes_l \| \; \|$.

We now give a description of the dual of $X \otimes_l Y$ when Y is a normed space.

A continuous linear operator $B: X \to Y'$ is absolutely summing if there exist a continuous semi-norm p on X and a constant $C_p \geqq 0$ such that

$$\sum_{i=1}^{n} \| Bx_i \| \leqq C_p \sup \left\{ \sum_{i=1}^{n} |\langle x', x_i \rangle| : x' \in U^0 \right\} \tag{2}$$

for every finite sequence x_i in X, where U^0 is the polar in X' of $U = \{x : p(x) \leqq 1\}$ and $\| \ \|$ is the dual norm in Y'. (Such mappings are referred to by Floret and Wloka (1968) as absolutely summing, but this terminology differs from that of Pietsch (1965) §2; when X is metrizable the absolutely summing operators of Pietsch (1965) and Floret and Wloka (1968) coincide. Let $\mathcal{P}(X, Y')$ denote the space of all absolutely summing operators from X into Y'; if X is normed, $\mathcal{P}(X, Y')$ is also normed by taking p above to be the norm of X and defining the norm (called the absolutely summing norm) of B, $\pi(B)$, to be the infimum of all constants C_p satisfying (2) see Pietsch ((1965), 2.2.3) and Floret and Wloka ((1968), Anhang 3.6).

Definition 1. Let Y be a normed space and let b be a continuous bilinear form on $X \times Y$ (thus, b is an element of the dual of the projective tensor product $X \otimes_\pi Y$, Treves ((1967) 43.4). Then b is said to be absolutely summing if the associated linear map $B: X \to Y'$ ($\langle Bx, y \rangle = b(x, y)$, $x \in X$, $y \in Y$) is an absolutely summing operator from X into Y'. The space of all such bilinear forms is denoted by $S(X, Y)$; if X is also normed, we equip $\mathcal{S}(X, Y)$ with the absolutely summing norm, π, from $\mathcal{P}(X, Y')$. (Note any absolutely summing operator $B: X \to Y'$ induces an absolutely summing bilinear form on $X \times Y$, i.e., $\mathcal{P}(X, Y') = \mathcal{S}(X, Y)$.)

Since the projective tensor topology is stronger than the topology l, see Kluvánek (1970), the dual of $X \otimes_l Y$ is a subspace of $B(X, Y)$, the continuous bilinear forms on $X \times Y$, and since l is stronger than the inductive topology, Kluvánek (1970), the dual of $X \otimes_l Y$ contains the integral forms on $X \times Y$. We now describe the dual of $X \otimes_l Y$ as a subspace of $B(X, Y)$.

Theorem 2. *Let $(Y, \| \ \|)$ be a normed space. A bilinear form b on $X \times Y$ is continuous on $X \otimes_l Y$ iff b is absolutely summing. If X is normed, the dual of $X \otimes_l Y$ with the dual norm is isometrically isomorphic to $\mathcal{S}(X, Y)$ equipped with the absolutely summing norm.*

Proof. Let $b \in B(X, Y)$ and let $B: X \to Y'$ be the associated linear operator.

Suppose b is absolutely summing and let $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. Then there exist $p \in \mathcal{P}$ and $C_p \geqq 0$ satisfying (2).

Then

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$$|b(u)| \leqq \sum_{i=1}^{n} \|B(\|y_i\|x_i)\| \leqq C_p \sup\left\{\sum_{i=1}^{n} |\langle x', |\,|y_i|\,|x_i\rangle| : x' \in U^0\right\}$$
$$= C_p \sup\left\{ \Big|\sum_{i=1}^{n} \alpha_i \|y_i\| \langle x', x_i\rangle\Big| : x' \in U^0, |\alpha_i| \leqq 1\right\}$$
$$= C_p \sup\left\{p\Big(\sum_{i=1}^{n} \alpha_i \|y_i\| x_i\Big) : |\alpha_i| \leqq 1\right\}.$$

Thus $|b(u)| \leqq C_p p \otimes_l \|\ \|(u)$ and b is continuous on $X \otimes_l Y$; moreover, if X is normed, $\|b\| \leqq \pi(B)$.

Suppose b is continuous on $X \otimes_l Y$. Then there exist $p \in \mathscr{P}$ and a constant k such that $|b(u)| \leqq kp \otimes_l \|\ \|(u)$ for $u \in X \otimes Y$. Set $U = \{x : p(x) \leqq 1\}$, let $\{x_i : 1 \leqq i \leqq n\} \subseteq X$ and let $\varepsilon > 0$. Pick $y_i \in Y$ such that $\|y_i\| = 1$ and $\langle Bx_i, y_i\rangle + \varepsilon/n > \|Bx_i\|$. If $u = \Sigma_{i=1}^{n} x_i \otimes y_i$, then

$$\sum_{i=1}^{n} \|Bx_i\| - \varepsilon \leqq \sum_{i=1}^{n} \langle Bx_i, y_i\rangle = b(u) \leqq k \sup\left\{p\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) : |\alpha_i| \leqq 1\right\}$$
$$= k \sup\left\{\Big|\sum_{i=1}^{n} \alpha_i \langle x', x_i\rangle\Big| : x' \in U^0, |\alpha_i| \leqq 1\right\} = k \sup\left\{\sum_{i=1}^{n} |\langle x', x_i\rangle| : x' \in U^0\right\}$$

so that $\Sigma_{i=1}^{n} \|Bx_i\| \leqq k \sup\{\Sigma_{i=1}^{n} |\langle x', x_i\rangle| : x' \in U^0\}$. Hence B is absolutely summing and if X is normed, $\pi(B) \leqq \|b\|$.

It follows from Theorem 2 that when $X = Y = l^2$ the ε, l and π-topologies are all distinct; for the dual of the π-topology is the space of all bounded linear maps on l^2, the dual of the l-topology is the space of absolutely summing (or Hilbert-Schmidt) operators, and the dual of the ε-topology is the space of integral (nuclear) operators.

We now consider the product of vector measures. Let $\mathscr{M}, \mathscr{N}$ be σ-algebras of subsets of the sets S, T. Let $v : X \times Y \to Z$ be a separately continuous bilinear map. If $\mu : \mathscr{M} \to X$ and $\nu : \mathscr{N} \to Y$ are countably additive set functions (vector measures), their product $\lambda = \mu \times \nu$ with respect to v is defined on measurable rectangles $A \times B$, $A \in \mathscr{M}$, $B \in \mathscr{N}$, by $\lambda(A \times B) = v(\mu(A), \nu(B))$. If $\mathscr{A}$ is the algebra generated by the measurable rectangles, then λ has an obvious finitely additive extension (still denoted by $\mu \times \nu = \lambda$) from $\mathscr{A}$ to Z, Swartz (to appear) and Duchon and Kluvánek (1967). If Σ is the σ-algebra generated by $\mathscr{A}$, we say that μ admits a v-product with respect to ν (and conversely) if λ is countably additive on $\mathscr{A}$ and has a (necessarily unique) countably additive extension from Σ to Z; if every X-valued measure (on an arbitrary σ-algebra $\mathscr{M}$) admits products with respect to ν, we say that X admits v-products with respect to ν.

Let $\mathscr{N}$ denote the Borel sets of the unit interval $[0,1]$. From Kluvánek's example (1974) it follows that there exists a vector measure $\nu : \mathscr{N} \to l^2$ such that the range of ν contains the unit ball of l^2 (if m is the measure of Kluvánek (1974), the measure $m - m$ contains a ball centered at the origin so multiplication by a suitable constant will give ν). The measure ν is fixed throughout the remainder of the paper.

THEOREM 3. *Let $B: X \to l^2$ be linear, continuous and let v be the associated bilinear form on $X \times l^2$ given by $v(x, y) = (y, Bx)$, where $(,)$ denotes the inner product on l^2. If $\mu: \mathcal{M} \to X$ is a vector measure which admits v-products with respect to ν, then for any disjoint sequence $\{E_i\} \subseteq \mathcal{M}$ $\Sigma \| B\mu(E_i)\| < \infty$.*

PROOF. Pick $z_m = \nu(H_m)$, $H_m \in \mathcal{N}$, in the range of ν such that $\| z_m \| = 1$ and $(z_m, Bx_m) = \| Bx_m \|$, where $x_m = \mu(E_m)$. Then $E_m \times H_m \in \mathcal{A}$, the algebra generated by the measurable rectangles, and $\{E_m \times H_m\}$ is disjoint so $\mu \times \nu$ being countably additive on Σ, the σ-algebra generated by the measurable rectangles, implies

$$\Sigma \,|\, \lambda(E_m \times H_m)\,| = \Sigma\,(z_m, Bx_m) = \Sigma \,\| Bx_m \| < \infty.$$

COROLLARY 4. *Let B and v be as in Theorem 3. If X admits v-products with respect to ν, then B is absolutely summing (or $v \in \mathcal{S}(X, l^2)$).*

PROOF. If Σx_m is an unconditionally convergent series in X, then this series induces a vector measure μ on the power set $\mathcal{P}(\mathbf{N})$ of the positive integers $\mathbf{N}$ via $\mu(E) = \Sigma_{n \in E} x_n$. Taking $E_m = \{m\}$ in Theorem 3 gives $\Sigma \| Bx_m \| < \infty$ or B is absolutely summing.

We now apply this corollary to the tensor product of vector measures. If $v: X \times Y \to X \hat{\otimes}_\varepsilon Y$ ([17]) §43) or $v: X \times Y \to X \hat{\otimes}_l Y$ is the natural tensor product map, then X admits v-products with respect to any Y-valued vector measure, Duchon and Kluvánek (1967) and Kluvánek (1973), but if $v: X \times Y \to X \hat{\otimes}_\pi Y$, this is no longer the case, see Kluvánek (1970) and Bagby and Swartz. Kluvánek conjectured that the topology l is the strongest tensor topology on $X \otimes Y$ which admits products in the sense above; we show that this is the case when X is metrizable and $Y = l^2$. Let τ be a tensor topology on $X \otimes l^2$ which is stronger than the inductive topology (ε-topology) and weaker than the projective topology (π-topology) and let $X \hat{\otimes}_\tau l^2$ denote the completion of the tensor product with respect to τ.

THEOREM 5. *Let X be metrizable and suppose τ is such that X admits v-products with respect to ν, where v is the natural tensor map from $X \times l^2$ into $X \hat{\otimes}_\tau l^2$. Then l is stronger than τ.*

PROOF. By Corollary 4 the dual of $X \otimes_\tau l^2$ is a subspace of $\mathcal{S}(X, l^2)$. That is, the injection $j: X \otimes_l l^2 \to X \otimes_\tau l^2$ is weakly continuous. But $X \otimes_l l^2$ is obviously metrizable so j is actually continuous with respect to the topologies l and τ Treves ((1967), Lemma 37.6), i.e. l is stronger than τ.

Theorem 3 also contains several other implications for products of Hilbert space valued measures which we now give.

REMARK 6. Let v be the inner product on l^2. If $\mu: \mathcal{M} \to l^2$ is a vector measure which admits v-products with respect to ν, then it follows from Theorem 3 (with B the identity operator on l^2) and [11] Theorem 2.2 that μ

must have bounded variation. (Recall a vector measure of bounded variation admits products with respect to any bilinear map and any measure, Duchon (1967) or Swartz (to appear). This fact may help to explain the examples presented in Dudley and Pakula (1972) and Rao (1972).

REMARK 7. Let $B: l^2 \to l^2$ be continuous, linear and let b be the associated bilinear form on l^2 induced by B. According to Swartz (to appear) (Conjecture 12) it was conjectured that l^2 admits b-products iff B is nuclear. The results of Swartz (to appear), Theorem 6, and Corollary 4 show that this conjecture is false and that l^2 admits b-products iff B is absolutely summing (Theorem 8 below). Actually, we can also obtain from Corollary 4 some results pertaining to the ideas discussed in Swartz (to appear). Recall a vector measure $\mu: \mathcal{M} \to l^2$ is dominated with respect to b iff $E_m \downarrow \phi$, $E_m \in \mathcal{M}$, implies $\tilde{\mu}(E_m) \to 0$, where $\tilde{\mu}$ denotes the b-semi-variation of μ,

$$\tilde{\mu}(A) = \sup\{|\Sigma b(\mu(A_i), x_i)| : \|x_i\| \leqq 1,\ \{A_i\} \text{ is a partition of } A\}$$

Swartz (to appear). We then have

THEOREM 8. *The following are equivalent:*
(i) *l^2 admits b-products*
(ii) *l^2 admits b-products with respect to ν*
(iii) *B is absolutely summing (or equivalently Hilbert-Schmidt)*
(iv) *every vector measure $\mu: \mathcal{M} \to l^2$ is dominated with respect to b.*

PROOF. That (i) implies (ii) is clear, (ii) implies (iii) by Corollary 4, and (iv) implies (i) by Swartz (to appear), Theorem 6. Thus we only need to check (iii) implies (iv). If $A \in \mathcal{M}$, $\{A_i: 1 \leqq i \leqq n\}$ is a partition of A and $\|x_i\| \leqq 1$, $1 \leqq i \leqq n$, then

$$\begin{aligned} \left| \sum_{i=1}^{n} b(\mu(A_i), x_i) \right| &\leqq \sum_{i=1}^{n} \|B(\mu(A_i))\| \\ &\leqq \pi(B) \sup\left\{ \sum_{i=1}^{n} |(x', \mu(A_i)| : \|x'\| \leqq 1 \right\} \\ &\leqq \pi(B) \sup\{\operatorname{var}(x'\mu)(A) : \|x'\| \leqq 1\}, \end{aligned} \tag{3}$$

where $\pi(B)$ is the absolutely summing norm of B and $\operatorname{var}(x'\mu)$ is the variation of the scalar measure $x'\mu: A \to (x', \mu(A))$. From (3) and Dunford and Schwartz ((1958), IV.10.5), it follows that μ is dominated with respect to b.

Theorem 9 indicates that Theorem 6 of Swartz (to appear) may be the best general result available for products of vector measures.

In concluding it may also be of interest to note that the general criteria for the existence of countably additive products of vector measures given by Theorem 6 of Swartz (to appear) yields the result of Kluvánek (1973). For let

$\mu : \mathcal{M} \to X$ be a vector measure and $p \otimes_l q = r$ a continuous semi-norm on $X \otimes_l Y$. If $E \in \mathcal{M}$ and $\tilde{\mu}_{q,r}$ is the semi-variation of μ with respect to q, r and the tensor product map (equation (1) of Swartz (to appear))

$$\tilde{\mu}_{q,r}(E) = \sup\{r(\Sigma\mu(E_i)\otimes y_i): \{E_i\} \text{ partition of } E,\ q(y_i) \leqq 1\}$$

$$\leqq \sup\{p(\Sigma\alpha_i q(y_i)\mu(E_i)): \{E_i\} \text{ partition of } E, q(y_i) \leqq 1, |\alpha_i| \leqq 1\}$$
$$\leqq \tilde{\mu}_p(E).$$

where $\tilde{\mu}_p$ is the scalar semi-variation of μ with respect to p Dunford and Schwartz ((1958), IV.10.3). But there is a positive measure $\lambda(=\lambda_p)$ such that $\tilde{\mu}_p(E) \to 0$ if $\lambda(E) \to 0$ so that μ is dominated (with respect to l; Def. 2 of Swartz (to appear)). Hence, Theorem 6 of Swartz (to appear) gives the theorem of Kluvánek (1970).

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