

ON THE EXCEPTIONAL SET OF TRANSCENDENTAL ENTIRE FUNCTIONS IN SEVERAL VARIABLES

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Abstract

We prove that any subset of $\overline{\mathbb{Q}}^m$ (closed under complex conjugation and which contains the origin) is the exceptional set of uncountably many transcendental entire functions over \mathbb{C}^m with rational coefficients. This result solves a several variables version of a question posed by Mahler for transcendental entire functions [*Lectures on Transcendental Numbers*, Lecture Notes in Mathematics, 546 (Springer-Verlag, Berlin, 1976)].

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1. Introduction

An analytic function f over a domain $\Omega \subseteq \mathbb{C}$ is said to be an *algebraic function* over $\mathbb{C}(z)$ if there exists a nonzero polynomial $P \in \mathbb{C}[X, Y]$ for which $P(z, f(z)) = 0$, for all $z \in \Omega$. A function which is not algebraic is called a *transcendental function*.

The study of the arithmetic behaviour of transcendental functions started in 1886 with a letter of Weierstrass to Strauss, proving the existence of such functions taking \mathbb{Q} into itself. Weierstrass also conjectured the existence of a transcendental entire function f for which $f(\overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}$ (as usual, $\overline{\mathbb{Q}}$ denotes the field of all algebraic numbers). Motivated by results of this kind, he defined the *exceptional set* of an analytic function $f : \Omega \rightarrow \mathbb{C}$ as

$$S_f = \{\alpha \in \overline{\mathbb{Q}} \cap \Omega : f(\alpha) \in \overline{\mathbb{Q}}\}.$$

Thus, Weierstrass' conjecture can be rephrased as: *does there exist a transcendental entire function f such that $S_f = \overline{\mathbb{Q}}$?* This conjecture was settled in 1895 by Stäckel [4],

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who proved, in particular, that for any $\Sigma \subseteq \overline{\mathbb{Q}}$, there exists a transcendental entire function f for which $\Sigma \subseteq S_f$.

In his classical book [1], Mahler introduced the problem of studying S_f for various classes of functions. After discussing a number of examples, Mahler posed several problems about the admissible exceptional sets for analytic functions, one of which is as follows. Here $B(0, \rho)$ denotes the closed ball with centre 0 and radius ρ in \mathbb{C} .

PROBLEM 1.1. Let $\rho \in (0, \infty]$ be a real number. Does there exist for any choice of $S \subseteq \overline{\mathbb{Q}} \cap B(0, \rho)$ (closed under complex conjugation and such that $0 \in S$) a transcendental analytic function $f \in \mathbb{Q}[[z]]$ with radius of convergence ρ for which $S_f = S$?

In 2016, Marques and Ramirez [3] proved that the answer to this question is ‘yes’ provided that $\rho = \infty$ (that is, for entire functions). Indeed, they proved the following more general result about the arithmetic behaviour of certain entire functions.

LEMMA 1.2 [3, Theorem 1.3]. *Let A be a countable set and let \mathbb{K} be a dense subset of \mathbb{C} . For each $\alpha \in A$, fix a dense subset $E_\alpha \subseteq \mathbb{C}$. Then there exist uncountably many transcendental entire functions $f \in \mathbb{K}[[z]]$ such that $f(\alpha) \in E_\alpha$ for all $\alpha \in A$.*

This result was improved by Marques and Moreira in [2] giving an affirmative answer to Mahler’s Problem 1.1 for any $\rho \in (0, \infty]$.

In this paper, we consider Mahler’s Problem 1.1 in the context of transcendental entire functions of several variables. Although the previous definitions extend to the context of several variables in a very natural way, we shall include them here for the sake of completeness.

An analytic function f over a domain $\Omega \subseteq \mathbb{C}^m$ (we also say that f is *entire* if $\Omega = \mathbb{C}^m$) is said to be *algebraic* over $\mathbb{C}(z_1, \dots, z_m)$ if it is a solution of a polynomial functional equation

$$P(z_1, \dots, z_m, f(z_1, \dots, z_m)) = 0 \quad \text{for all } (z_1, \dots, z_m) \in \Omega,$$

for some nonzero polynomial $P \in \mathbb{C}[z_1, \dots, z_m, z_{m+1}]$. A function which is not algebraic is called a transcendental function. (We remark that an entire function in several variables is algebraic if and only if it is a polynomial function just as in the case of one variable.) Let \mathbb{K} be a subset of \mathbb{C} and let f be an analytic function on the polydisc $\Delta(0, \rho) := B(0, \rho_1) \times \dots \times B(0, \rho_m) \subseteq \mathbb{C}^m$ for some $\rho = (\rho_1, \dots, \rho_m) \in (0, \infty]^m$. We say that $f \in \mathbb{K}[[z_1, \dots, z_m]]$ if

$$f(z_1, \dots, z_m) = \sum_{(k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m} c_{k_1, \dots, k_m} z_1^{k_1} \cdots z_m^{k_m},$$

with $c_{k_1, \dots, k_m} \in \mathbb{K}$ for all $(k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m$ and for all $(z_1, \dots, z_m) \in \Delta(0, \rho)$.

The exceptional set S_f of an analytic function $f : \Omega \subseteq \mathbb{C}^m \rightarrow \mathbb{C}$ is defined as

$$S_f := \{(\alpha_1, \dots, \alpha_m) \in \Omega \cap \overline{\mathbb{Q}}^m : f(\alpha_1, \dots, \alpha_m) \in \overline{\mathbb{Q}}\}.$$

For example, let $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ and $g : \mathbb{C}^2 \rightarrow \mathbb{C}$ be the transcendental entire functions given by

$$f(w, z) = e^{w+z} \quad \text{and} \quad g(w, z) = e^{wz}.$$

By the Hermite–Lindemann theorem,

$$S_f = \{(\alpha, -\alpha) : \alpha \in \overline{\mathbb{Q}}\} \quad \text{and} \quad S_g = (\overline{\mathbb{Q}} \times \{0\}) \cup (\{0\} \times \overline{\mathbb{Q}}).$$

In general, if $P_1(X, Y), \dots, P_n(X, Y) \in \overline{\mathbb{Q}}[X, Y]$, then the function

$$f(w, z) = \exp\left(\prod_{k=1}^n P_k(w, z)\right)$$

has the exceptional set given by

$$S_f = \bigcup_{k=1}^n \{(\alpha, \beta) \in \overline{\mathbb{Q}}^2 : P_k(\alpha, \beta) = 0\}.$$

We refer the reader to [1, 5] (and references therein) for more about this subject.

In the main result of this paper, we shall prove that every subset S of $\overline{\mathbb{Q}}^m$ (under some mild conditions) is the exceptional set of uncountably many transcendental entire functions of several variables with rational coefficients.

THEOREM 1.3. *Let m be a positive integer. Then, every subset S of $\overline{\mathbb{Q}}^m$, closed under complex conjugation and such that $(0, \dots, 0) \in S$, is the exceptional set of uncountably many transcendental entire functions $f \in \mathbb{Q}[[z_1, \dots, z_m]]$.*

To prove this theorem, we shall provide a more general result about the arithmetic behaviour of a transcendental entire function of several variables.

THEOREM 1.4. *Let X be a countable subset of \mathbb{C}^m and let \mathbb{K} be a dense subset of \mathbb{C} . For each $u \in X$, fix a dense subset $E_u \subseteq \mathbb{C}$ and suppose that if $(0, \dots, 0) \in X$, then $E_{(0, \dots, 0)} \cap \mathbb{K} \neq \emptyset$. Then there exist uncountably many transcendental entire functions $f \in \mathbb{K}[[z_1, \dots, z_m]]$ such that $f(u) \in E_u$ for all $u \in X$.*

Theorem 1.4 is a several variables extension of the one-variable result due to Marques and Ramirez [3, Theorem 1.3].

2. Proofs

2.1. Proof that Theorem 1.4 implies Theorem 1.3. In the statement of Theorem 1.4, choose $X = \overline{\mathbb{Q}}^m$ and $\mathbb{K} = \mathbb{Q}^* + i\mathbb{Q}$. Write $S = \{u_1, u_2, \dots\}$ and $\overline{\mathbb{Q}}^m/S = \{v_1, v_2, \dots\}$ (one of them may be finite) and define

$$E_u := \begin{cases} \overline{\mathbb{Q}} & \text{if } u \in S, \\ \mathbb{K} \cdot \pi^n & \text{if } u = v_n. \end{cases}$$

By Theorem 1.4, there exist uncountably many transcendental entire functions

$$f(z_1, \dots, z_m) = \sum_{k_1 \geq 0, \dots, k_m \geq 0} c_{k_1, \dots, k_m} z_1^{k_1} \cdots z_m^{k_m}$$

in $\mathbb{K}[[z_1, \dots, z_m]]$ such that $f(u) \in E_u$ for all $u \in \overline{\mathbb{Q}}^m$. Define $\psi(z_1, \dots, z_m)$ as

$$\psi(z_1, \dots, z_m) := \frac{f(z_1, \dots, z_m) + \overline{f(\overline{z_1}, \dots, \overline{z_m})}}{2}.$$

By the properties of the conjugation of power series,

$$\psi(z_1, \dots, z_m) = \sum_{(k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m} \operatorname{Re}(c_{k_1, \dots, k_m}) z_1^{k_1} \cdots z_m^{k_m}$$

is a transcendental entire function in $\mathbb{Q}[[z_1, \dots, z_m]]$ since $\operatorname{Re}(c_{k_1, \dots, k_m})$ is rational and nonzero for all $(k_1, \dots, k_m) \in \mathbb{Z}_{\geq 0}^m$ by construction. (Here, as usual, $\operatorname{Re}(z)$ denotes the real part of the complex number z .)

Therefore, it suffices to prove that $S_\psi = S$. In fact, since S is closed under complex conjugation, if $u \in S$, then $\overline{u} \in S$ and thus $f(u)$ and $\overline{f(\overline{u})}$ are algebraic numbers and so is $\psi(u)$. (Observe also that $f(0, \dots, 0) = c_{0, \dots, 0} \in \mathbb{Q}$.) In the case in which $u = v_n$, for some n , we can distinguish two cases. When $v_n \in \mathbb{R}^m$, then $\psi(u) = \operatorname{Re}(f(v_n))$ is transcendental, since $f(v_n) \in \mathbb{K} \cdot \pi^n$. For $v_n \notin \mathbb{R}^m$, we have $\overline{v_n} = v_l$ for some $l \neq n$. Thus, there exist nonzero algebraic numbers γ_1, γ_2 such that

$$\psi(v_n) = \frac{\gamma_1 \pi^n + \gamma_2 \pi^l}{2},$$

which is transcendental, since $\overline{\mathbb{Q}}$ is algebraically closed and π is transcendental. In conclusion, $\psi \in \mathbb{Q}[[z_1, \dots, z_m]]$ is a transcendental entire function whose exceptional set is S .

2.2. Proof of Theorem 1.4. Let us proceed by induction on m . The case $m = 1$ is covered by Lemma 1.2. Suppose that the theorem holds for all positive integers $k \in [1, m - 1]$. That is, if \mathbb{K} is a dense subset of \mathbb{C} , X is a countable subset of \mathbb{C}^k and E_u is a dense subset in \mathbb{C} for each $u \in X$, then there exist uncountably many transcendental entire functions $f \in \mathbb{K}[[z_1, \dots, z_k]]$ such that $f(u) \in E_u$ for all $u \in X$, for any integer $k \in [1, m - 1]$.

Now, let X be a countable subset of \mathbb{C}^m and E_u a fixed dense subset of \mathbb{C} for all $u \in X$. Without loss of generality, we can assume that $(0, \dots, 0) \in X$. In this case, by hypothesis, $\mathbb{K} \cap E_{(0, \dots, 0)} \neq \emptyset$. To apply the induction hypothesis, we consider the partition of X given by

$$X = \bigcup_{S \in \mathcal{P}_m} X_S,$$

where \mathcal{P}_m denotes the powerset of $[1, m] = \{1, \dots, m\}$ and X_S denotes the set of all $z = (z_1, \dots, z_m)$ in $X \subseteq \mathbb{C}^m$ such that $z_i \neq 0$ if and only if $i \in S$. In particular, $X_\emptyset = \{(0, \dots, 0)\}$ and $X_{[1, m]} = X \cap (\mathbb{C} \setminus \{0\})^m$.

Given $S = \{i_1, \dots, i_k\}$ in $\mathcal{Q}_m = \mathcal{P}_m \setminus \{\emptyset, [1, m]\}$ and $z = (z_1, \dots, z_m)$ in \mathbb{C}^m , we denote by z_S the element $(z_{i_1}, \dots, z_{i_k}) \in \mathbb{C}^k$. To simplify the exposition, we will assume that $i_1 < \dots < i_k$ for all $S \in \mathcal{Q}_m$. Our goal is to show that there exist uncountably many ways to construct a transcendental entire function $f \in \mathbb{K}[[z_1, \dots, z_m]]$ given by

$$f(z_1, \dots, z_m) = a_0 + \left(\sum_{S \in \mathcal{Q}_m} \left(\prod_{i \in S} z_i \right) f_S(z_S) \right) + f^*(z_1, \dots, z_m),$$

where $a_0 \in E_{(0, \dots, 0)} \cap \mathbb{K}$ and, for each $S = \{i_1, \dots, i_k\} \in \mathcal{Q}_m$, the function $f_S : \mathbb{C}^k \rightarrow \mathbb{C}$ is a transcendental entire function such that

$$f_S(u_S) \in \frac{1}{\alpha_{i_1} \cdots \alpha_{i_k}} \cdot (E_u - \Theta_{S,u})$$

for all $u = (\alpha_1, \dots, \alpha_m) \in X_S$ with

$$\Theta_{S,u} = a_0 + \sum_{T \in \mathcal{Q}_m, T \neq S} \left(\prod_{i \in T} \alpha_i \right) f_T(u_T) \in \mathbb{C}.$$

By the induction hypothesis, f_S exists for all $S \in \mathcal{Q}_m$ (noting that if E_u is a dense subset of \mathbb{C} , then $(\alpha_{i_1} \cdots \alpha_{i_k})^{-1} \cdot (E_u - \Theta_{S,u})$ is also a dense set). Moreover, we want the function $f^*(z_1, \dots, z_m) \in \mathbb{K}[[z_1, \dots, z_m]]$ to satisfy the condition

$$f^*(u) \in \left(E_u - a_0 - \sum_{S \in \mathcal{Q}_m} \left(\prod_{i \in S} \alpha_i \right) f_S(u_S) \right) \tag{2.1}$$

for all $u = (\alpha_1, \dots, \alpha_m) \in X_{[1,m]}$, and $f^*(z_1, \dots, z_m) = 0$ whenever $z_i = 0$ for some i with $1 \leq i \leq m$. Under these conditions, it is easy to see that if $S \in \mathcal{Q}_m$ and $u \in X_S$, then $f^*(u) = 0$ and $f(u) \in E_u$.

To construct the function $f^* : \mathbb{C}^m \rightarrow \mathbb{C}$, let us consider an enumeration $\{u_1, u_2, \dots\}$ of $X_{[1,m]}$, where we write $u_j = (\alpha_1^{(j)}, \dots, \alpha_m^{(j)})$. We construct a function $f^* \in \mathbb{K}[[z_1, \dots, z_m]]$ given by

$$f^*(z_1, \dots, z_m) = \sum_{n=m}^{\infty} P_n(z_1, \dots, z_m) = \sum_{i_1 \geq 1, \dots, i_m \geq 1} c_{i_1, \dots, i_m} z_1^{i_1} \cdots z_m^{i_m},$$

where P_n is a homogeneous polynomial of degree n and the coefficients $c_{i_1, \dots, i_m} \in \mathbb{K}$ will be chosen so that f^* will satisfy the desired conditions.

The first condition is

$$|c_{i_1, \dots, i_m}| < s_{i_1 + \dots + i_m} := \frac{1}{\binom{i_1 + \dots + i_m - 1}{m-1} (i_1 + \dots + i_m)!},$$

where $c_{i_1, \dots, i_m} \neq 0$ for infinitely many m -tuples of integers $i_1 \geq 1, \dots, i_m \geq 1$. These conditions will be used to guarantee that f^* is an entire function. Let $L(P)$ denote the length of the polynomial $P(z_1, \dots, z_m) \in \mathbb{C}[[z_1, \dots, z_m]]$ given by the sum of the absolute values of its coefficients. Since

$$|P_n(z_1, \dots, z_m)| \leq L(P_n) \max\{1, |z_1|, \dots, |z_m|\}^n,$$

for all $n \geq m$ and (z_1, \dots, z_m) belonging to the open ball $B(0, R)$,

$$|P_n(z_1, \dots, z_m)| < \frac{\binom{n-1}{m-1}}{\binom{n-1}{m-1}n!} \max\{1, R\}^n = \frac{\max\{1, R\}^n}{n!},$$

since $P_n(z_1, \dots, z_m)$ has at most $\binom{n-1}{m-1}$ monomials of degree n . Hence, the series $\sum_{n \geq m} P_n(z_1, \dots, z_m)$ converges uniformly in any of these balls. Thus, f^* is a transcendental entire function such that $f^*(0, z_2, \dots, z_m) = f^*(z_1, 0, z_3, \dots, z_m) = f^*(z_1, z_2, \dots, 0) = 0$.

To obtain the coefficients $c_{i_1, \dots, i_m} \in \mathbb{K}$ such that f^* satisfies the condition (2.1), we consider a hyperplane $\pi(n, j)$ for positive integers n and j with $1 \leq j \leq n$, given by

$$\pi(n, j) : \mu_{n,1}^{(j)} z_1 + \dots + \mu_{n,m}^{(j)} z_m - \lambda_n^{(j)} = 0,$$

and such that if u_j, u_{n+1} and the origin are noncollinear, then $\pi(n, j)$ is a hyperplane containing u_j and parallel to the line passing through the origin and the point u_{n+1} , and, if u_j, u_{n+1} and the origin are collinear, then $\pi(n, j)$ is a hyperplane containing u_j and perpendicular to the line passing through the origin and the point u_{n+1} . Note that in both cases, $\lambda_n^{(j)} \neq 0$ and u_{n+1} does not belong to any hyperplane $\pi(n, j)$ with $1 \leq j \leq n$.

Now, we define the polynomials $A_0(z_1, \dots, z_m) := z_1 \cdots z_m$ and

$$A_n(z_1, \dots, z_m) := \prod_{j=1}^n (\mu_{n,1}^{(j)} z_1 + \dots + \mu_{n,m}^{(j)} z_m - \lambda_n^{(j)})$$

for all $n \geq 1$. By the definition of $\pi(n, j)$, we have $A_n(u_j) = 0$ for $1 \leq j \leq n$. Since u_{n+1} and the origin do not belong to $\pi(n, j)$, we also have $A_n(0, \dots, 0) \neq 0$ and $A_n(u_{n+1}) \neq 0$ for all $n \geq 1$. Thus, we can define the function

$$f_{1,0}^*(z_1, \dots, z_m) := \delta_{1,0} A_0(z_1, \dots, z_m) = \delta_{1,0} z_1 \cdots z_m$$

such that $\Theta_1 + f_{1,0}^*(u_1) \in E_{u_1}$ and $0 < |\delta_{1,0}| < s_m/m$, where

$$\Theta_j := a_0 + \sum_{S \in \mathcal{Q}_m} \left(\prod_{i \in S} \alpha_i^{(j)} \right) f_S(u_{j,S}),$$

and $u_{j,S} = (\alpha_{i_1}^{(j)}, \dots, \alpha_{i_k}^{(j)})$ for $S = \{i_1, \dots, i_k\}$, for all integers $j \geq 1$.

Since \mathbb{K} is a dense subset of \mathbb{C} , we can choose $\delta_{1,1}$ such that the coefficient $c_{1,1, \dots, 1}$ of $z_1 \cdots z_m$ in the function

$$f_{1,1}^*(z_1, \dots, z_m) := f_{1,0}^*(z_1, \dots, z_m) + \delta_{1,1} z_1 \cdots z_m A_1^{(1)}(z_1, \dots, z_m)$$

belongs to \mathbb{K} with $|c_{1,1, \dots, 1}| < s_m$. Therefore, we take

$$f_1^*(z_1, \dots, z_m) := f_{1,1}^*(z_1, \dots, z_m),$$

where $P_1(z_1, \dots, z_m) = c_{1,1, \dots, 1} z_1 \cdots z_m$.

Recursively, we can construct a function $f_{n,0}^*(z_1, \dots, z_m)$ given by

$$f_{n,0}^*(z_1, \dots, z_m) := f_{n-1}^*(z_1, \dots, z_m) + \delta_{n,0} z_1^n z_2 \cdots z_m A_{n-1}(z_1, \dots, z_m)$$

where we take $\delta_{n,0} \neq 0$ in the ball $B(0, s_{n+m-1}/(n+m-1))$ such that

$$\Theta_n + f_{n,0}^*(u_n) \in E_{u_n}.$$

This is possible since E_{u_n} is a dense subset of \mathbb{C} and all coordinates of u_n are nonzero.

Since \mathbb{K} is a dense subset of \mathbb{C} , if we consider the ordering of the monomials of degree $n+m-1$ given by the lexicographical order of the exponents, then we can choose $\delta_{n,l}$ such that the coefficient c_{j_1, \dots, j_m} of the l th monomial $z_1^{j_1} \cdots z_m^{j_m}$ in

$$f_{n,l}^*(z_1, \dots, z_m) := f_{n,l-1}^*(z_1, \dots, z_m) + \delta_{n,l} z_1^{j_1} \cdots z_m^{j_m} A_n(z_1, \dots, z_m)$$

belongs to \mathbb{K} with $|c_{j_1, \dots, j_m}| < s_{n+m-1}$. Thus, we define

$$f_n^*(z_1, \dots, z_m) := f_{n,L}^*(z_1, \dots, z_m),$$

where $L = \binom{n+m-2}{m-1}$ is the number of distinct monomials of degree $n+m-1$. Then $f_n^*(z_1, \dots, z_m)$ is a polynomial function such that $c_{j_1, \dots, j_m} \in \mathbb{K}$ for every m -tuple (j_1, \dots, j_m) such that $j_1 + \dots + j_m \leq n+m-1$.

Finally, this construction implies that the functions f_n^* converge to a transcendental entire function $f^* \in \mathbb{K}[[z_1, \dots, z_m]]$ as $n \rightarrow \infty$ such that

$$f^*(u_j) = f_n^*(u_j) = f_j^*(u_j)$$

for all $n \geq j \geq 1$. Let $f : \mathbb{C}^m \rightarrow \mathbb{C}$ be the entire function given by

$$f(z_1, \dots, z_m) = a_0 + \left(\sum_{S \in Q_m} \left(\prod_{i \in S} z_i \right) f_S(z_S) \right) + f^*(z_1, \dots, z_m).$$

Then $f(u) \in E_u$ for all $u \in X \subset \mathbb{C}^m$. Since f is an entire function that is not a polynomial, it follows that f is transcendental. Note that there are uncountably many ways to choose the constants $\delta_{n,j}$. This completes the proof.

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