Canad. Math. Bull. Vol. **28** (3), 1985

THE CONSTRUCTION OF AUTOMORPHIC FORMS FROM THE DERIVATIVES OF A GIVEN FORM II

BY

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Dedicated to the memory of Robert Arnold Smith

ABSTRACT. Explicit constructions of polynomials of preassigned degree and weight in the derivatives of a given automorphic form are described and studied, supplementing the results of an earlier paper. It turns out that the problem is essentially one concerning symmetric functions rather than automorphic forms.

1. **Introduction**. In an earlier paper [2] under the same title I exhibited a family of 'basic' polynomials ψ_m ($m = 2, 3, \ldots$) in the derivatives of a given automorphic form f, which, together with f itself, sufficed to determine all automorphic forms expressible as polynomials in f and its derivatives. This result is restated in Theorem 4 of §5, but is, in some respects, not as useful as it might be, since a power of f, which could be negative, is involved as well as the basic polynomials, and no indication was given as to how the latter were to be combined in order to cancel out such a negative power and so produce an actual polynomial. This defect came to my notice recently when [3] I needed to express a certain newform as a polynomial of degree 2 in the derivatives of certain theta functions.

The object of the present paper is to show how such polynomials can be constructed in a more explicit way. The somewhat surprising fact emerges that the problem is essentially one concerning symmetric functions of a particular type and has relatively little to do with automorphic forms.

The symmetric function aspect is studied in §§2–4, after which applications to automorphic forms are given.

2. **Statement of the main result**. Throughout, n denotes a sufficiently large positive integer and $\sigma_1, \sigma_2, \ldots, \sigma_n$ are algebraically independent complex numbers, which we regard as the elementary symmetric functions of n complex numbers, which are, therefore, the roots of the algebraic equation

$$\sum_{\lambda=0}^{n} (-1)^\lambda \sigma_{n-\lambda} x^\lambda = 0, \qquad \text{where} \quad \sigma_0 = 1.$$

Received by the editors March 9, 1984 and, in revised form, May 10, 1984.

AMS Subject Classification (1980): Primary 10D05, Secondary 10D12, 10D15.

The actual values of these roots are irrelevant for our purpose, since we shall only be concerned with the properties of the σ_λ and the corresponding power sums $s_1, s_2, \ldots$; these are related to the σ_λ by Newton's formulae:

$$\lambda\sigma_\lambda = \sum_{\mu=1}^{\lambda} \sigma_{\lambda-\mu} s_\mu (-1)^{\mu-1} \qquad (1 \leqslant \lambda \leqslant n). \tag{2.1}$$

A polynomial of positive degree r and weight $m \geqslant 0$ in the σ_λ is a linear combination with complex coefficients of terms of the form

$$\sigma_{\nu_1}\sigma_{\nu_2}\ldots\sigma_{\nu_r}, \tag{2.2}$$

where

$$\nu_1 + \nu_2 + \ldots + \nu_r = m. \tag{2.3}$$

Any such polynomial can be expressed as a polynomial in the power sums s_μ $(1 \leqslant \mu \leqslant n)$ and we are interested in those polynomials that, when so expressed, are independent of the first power sum s_1. They clearly constitute a vector space over $\mathbb{C}$; we call this space $A(r, m)$.

The non-negative integers ν_j $(j = 1, 2, \ldots, r)$ are the components of a vector $\boldsymbol{\nu}$, say, and form a partition of m into r non-negative summands. When these are in descending order,

$$\nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_r \geqslant 0,$$

we say that $\boldsymbol{\nu}$ is in *standard form*. The number of partitions (in standard form) of m into r or fewer positive integral summands is, as usual, denoted by $p_r(m)$; by convention, $p_r(0) = 1$.

We denote by $V(r)$ the set of all vectors $\boldsymbol{\nu}$ containing r non-negative integral components and by $V(r, m)$ the subset for which (2.3) holds. The subset of $V(r, m)$ consisting of vectors in standard form is denoted by $V^*(r, m)$. We shall always assume that $n \geqslant m$.

Now take any vector $\boldsymbol{u} \in \mathbb{C}^r$ and let its components be $u_1, u_2, \ldots, u_r$. We denote the power sums of these r complex numbers by $S_\mu (\mu \geqslant 1)$ and make the assumption that

$$S_1 = u_1 + u_2 + \ldots + u_r = 0. \tag{2.4}$$

The set of all such $\boldsymbol{u} \in \mathbb{C}^r$ is denoted by $U(r)$.

For any $\boldsymbol{\nu} \in V(r, m)$ and $\boldsymbol{u} \in U(r)$ write

$$\sigma(\boldsymbol{\nu}) = \sigma_{\nu_1}\sigma_{\nu_2}\ldots\sigma_{\nu_r}, \qquad \boldsymbol{u}^{\boldsymbol{\nu}} = u_1^{\nu_1}u_2^{\nu_2}\ldots u_r^{\nu_r} \tag{2.5}$$

and put

$$[\boldsymbol{u}^{\boldsymbol{\nu}}] = [u_1^{\nu_1}u_2^{\nu_2}\ldots u_r^{\nu_r}] \tag{2.6}$$

to denote the monomial symmetric function of $u_1, u_2, \ldots, u_r$ containing $\boldsymbol{u}^{\boldsymbol{\nu}}$ as a typical term. This is the sum of $\boldsymbol{u}^{\boldsymbol{\nu}}$ and all the other different terms obtained from it by permuting the suffixes. We write $\pi(\boldsymbol{\nu})$ for the number of terms in $[\boldsymbol{u}^{\boldsymbol{\nu}}]$; it is independent

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of $\boldsymbol{u}$. The symmetric function $[\boldsymbol{u}^{\nu}]$ is, of course, expressible as a polynomial in the S_{μ}. Now define, for $\boldsymbol{u} \in U(r)$,

$$P_{r,m}(\boldsymbol{u}) = \sum_{\nu \in V(r,m)} \sigma(\nu)\boldsymbol{u}^{\nu} = \sum_{\nu \in V^*(r,m)} \sigma(\nu)[\boldsymbol{u}^{\nu}] \tag{2.7}$$

$$= \sum_{\nu \in V(r,m)} \sigma(\nu)[\boldsymbol{u}^{\nu}]/\pi(\nu), \tag{2.8}$$

and note that

$$P_{r,0}(u) = \sigma_0^r = 1, \qquad P_{r,1}(u) = 0. \tag{2.9}$$

We denote by $B(r,m)$ the vector space generated by the polynomials $P_{r,m}(\boldsymbol{u})$ for different $\boldsymbol{u} \in U(r)$.

THEOREM 1. *For arbitrary integers $r \geqslant 1$, $m \geqslant 0$ we have*

$$A(r,m) = B(r,m), \tag{2.10}$$

their common dimension being

$$d_r(m) := p_r(m) - p_r(m-1), \tag{2.11}$$

where $d_r(0) = 1$.

We defer the proof of the theorem to §3, but note here a number of consequences.

COROLLARY 1. (i) $A(1,m) = B(1,m) = 0$ (*the zero space*).
(ii) $d_2(m)$ *is* 1 *for even m and* 0 *for odd m; accordingly, $A(2,m) = 0$ for odd m.*
(iii) $\sigma_0 A(r-1, r+1) = A(r, r+1)$ *for $r > 1$.*

Part (ii) may be deduced from the fact that $p_r(m)$ is the coefficient of x^m in the power series

$$\{(1-x)(1-x^2)\dots(1-x^r)\}^{-1}$$

and from this we also deduce that $d_r(m)$ is the number of partitions of m into positive summands greater than unity and not exceeding r. For $r = 3$ one can easily show that

$$d_3(m) = \frac{1}{12}\left\{2m + 5 + 3(-1)^m + 4\left(\frac{m+1}{3}\right)\right\}, \tag{2.12}$$

where the last term contains a Legendre symbol. Incidentally, $d_3(m)$ is the number of entire modular forms of weight $2m$ for the modular group.

Part (iii) follows since $\sigma_0 A(r-1, r+1) \subseteq A(r, r+1)$ and $d_{r-1}(r+1) = d_r(r+1)$.

3. **Proof of Theorem 1**. We show first that $A(r, m)$ has dimension $d_r(m)$, and may clearly assume that $m \geqslant 1$.

Any member F of $A(r,m)$ can be written as

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$$(3.1) \qquad F = \sum_{\nu \in V(r,m)} a(\nu)\sigma(\nu) \qquad (a(\nu) \in \mathbb{C}),$$

where $a(\nu)$ takes the same value for all permutations of the components $\nu_1, \nu_2, \ldots, \nu_r$. Our object is to determine conditions that the $a(\nu)$ must satisfy. We regard F as being expressed as a function of the power sums s_μ $(1 \leqslant \mu \leqslant n)$ and we therefore require that $\partial F/\partial s_1 = 0$.

To determine the consequences of this condition we require the following

LEMMA 1. *If the σ_k are expressed as functions of the s_μ $(1 \leqslant \mu \leqslant n)$, then*

$$(3.2) \qquad \frac{\partial \sigma_k}{\partial s_1} = \sigma_{k-1} \qquad (1 \leqslant k \leqslant n).$$

Similarly, if the s_k are expressed in terms of the σ_μ then

$$(3.3) \qquad \frac{\partial s_k}{\partial \sigma_1} = k w_{k-1}$$

where w_{k-1} is the complete (Wronskian) symmetric function of order $k - 1$.

These results are of some interest and are, no doubt, well known; we shall only require (3.2).

Clearly both relations hold for $k = 1$. The proofs in the general case follow easily by induction on using Newton's formulae (2.1).

If we differentiate (3.1) partially with respect to s_1 and apply (3.2) we obtain

$$(3.4) \qquad 0 = \sum_{\nu \in V(r,m)} a(\nu)\sigma(\nu) \left\{\frac{\sigma_{\nu_1 - 1}}{\sigma_{\nu_1}} + \ldots + \frac{\sigma_{\nu_r - 1}}{\sigma_{\nu_r}}\right\},$$

where we make the convention that $\sigma_{-1} = 0$. It follows that

$$0 = \sum_{z \in V(r,m-1)} \sigma(z)\{a(z_1 + 1, z_2, \ldots, z_r) + \ldots + a(z_1, \ldots, z_{r-1}, z_r + 1)\}$$

$$= \sum_{z \in V^*(r,m-1)} \sigma(z)\{a(z_1 + 1, z_2, \ldots, z_r) + \ldots + a(z_1, \ldots, z_{r-1}, z_r + 1)\}\pi(z)$$

from which we deduce that

$$(3.5) \qquad a(z_1 + 1, z_2, \ldots, z_r) + \ldots + a(z_1, \ldots, z_{r-1}, z_r + 1) = 0$$

for all $z \in V^*(r, m - 1)$.

We now show that these $p_r(m - 1)$ equations are linearly independent. For this purpose we arrange the $p_r(m)$ variables $a(\nu)$, where $\nu \in V^*(r, m)$, in lexicographical order, so that ν precedes ν' (where $\nu' \neq \nu$) when, for some $k \geqslant 0$,

$$\nu_\lambda = \nu'_\lambda (\lambda \leqslant k), \qquad \nu_{k+1} > \nu'_{k+1}.$$

We then have a $p_r(m - 1) \times p_r(m)$ matrix V of coefficients of the variables $a(\nu)$ in which the columns correspond to different $\nu \in V^*(r, m)$ and are taken in the lexi-

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cographical order just described. The arguments of the r unknown variables in (3.5) determine r, not necessarily different, vectors in $V^*(r, m)$ and each of the equations determines a different leading vector (according to the lexicographical order). The corresponding columns in V form a square matrix V_0 with the property that each row has its first nonzero entry in a different column. Accordingly, V_0 is nonsingular, and so the $p_r(m-1)$ equations (3.5) are linearly independent. This proves that the functions F form a vector space of dimension $d_r(m)$.

We now show that $P_{r,m}(\boldsymbol{u})$, as defined by (2.7, 8), belongs to $A(r, m)$, where, as previously, we assume that $m \geqslant 1$. We therefore have to show that, for each $z \in V(r, m-1)$,

$$u_1^{z_1+1} u_2^{z_2} \ldots u_r^{z_r} + \ldots + u_1^{z_1} u_2^{z_2} \ldots u_r^{z_r+1} = 0,$$

which is clearly true, by (2.4). It follows that

$$B(r, m) \subseteq A(r, m). \tag{3.6}$$

It remains to show that, if $F \in A(r, m)$, then F can be expressed as a linear combination of the polynomials $P_{r,m}(\boldsymbol{u})$ for different $\boldsymbol{u} \in U(r)$.

By Corollary 1(i), this is trivially true for $r = 1$ and we therefore dispose next of the case $r = 2$ and may assume that m is even, say $m = 2N$. Since $d_2(2N) = 1$, it is enough to find a single nonzero polynomial in $B(2, 2N)$. For this purpose we take $\boldsymbol{u} = (-1, 1)$, put $v_1 = \lambda$, $v_2 = 2N - \lambda$ and so we have

$$\begin{aligned} P_{2,m}(\boldsymbol{u}) &= \sum_{\lambda=0}^{2N} (-1)^\lambda \sigma_\lambda \sigma_{2N-\lambda} \\ &= 2 \sum_{\lambda=0}^{N-1} (-1)^\lambda \sigma_\lambda \sigma_{2N-\lambda} + (-1)^N \sigma_N^2, \end{aligned} \tag{3.7}$$

which clearly does not vanish identically.

From now on we assume that $r \geqslant 3$ and that F is given by (3.1), where the $a(\boldsymbol{v})$ satisfy (3.5). In order to express F as a linear combination of polynomials in $B(r, m)$ it is only necessary to choose these polynomials so that the coefficient of $\sigma(\boldsymbol{v})$ is $a(\boldsymbol{v})$ for vectors $\boldsymbol{v}$ corresponding to columns in the matrix V that are not columns of V_0; for the remaining coefficients will take the correct values in view of the $p_r(m-1)$ equations (3.5). The corresponding set of $d_r(m)$ standard vectors we denote by $V_1^*(r, m)$ and note that for them we have

$$v_1 = v_2. \tag{3.8}$$

In what follows we shall be concerned with not identically vanishing polynomials $f \in \mathbb{C}[x_1, \ldots, x_k]$, where $k \geqslant 1$. Such a polynomial is a sum of terms of the form $A x_1^{v_1} x_2^{v_2} \ldots x_k^{v_k}$, where $A \neq 0$, and the leading term is that term that takes precedence by lexicographical ordering the sets of exponents, as described previously. The leading terms of two polynomials are said to be *essentially distinct* if their ratio is not independent of $x_1, x_2, \ldots, x_k$.

LEMMA 2. *Let* $f_\lambda \in \mathbb{C}[x_1, \ldots, x_k]$ $(\lambda = 1, 2, \ldots, q)$ *be q polynomials with essentially different leading terms. Then it is possible to find q vectors* $\boldsymbol{x}^{(\mu)} = (x_1^{(\mu)}, \ldots, x_k^{(\mu)})$ $(1 \leqslant \mu \leqslant q)$ *such that*

$$\det f_\lambda(\boldsymbol{x}^{(\mu)}) \neq 0. \tag{3.9}$$

PROOF. This is certainly true when $q = 1$. We assume that the result is true for some $q \geqslant 1$ and prove its truth for $q + 1$. We may assume that the $q + 1$ polynomials are ordered so that f_{q+1} has the highest leading term, and we take $\boldsymbol{x}^{(\mu)}$ $(\mu = 1, 2, \ldots, q)$ so that the $q \times q$ determinant with entries $f_\lambda(\boldsymbol{x}^{(\mu)})$ $(1 \leqslant \lambda \leqslant q)$ does not vanish. Then, if no vector $\boldsymbol{x}^{(q+1)}$ can be found to make the corresponding $(q + 1) \times (q + 1)$ determinant vanish, it follows that there exist constants $c_\lambda (1 \leqslant \lambda \leqslant q + 1)$ such that $c_{q+1} \neq 0$ and

$$c_1 f_1(\boldsymbol{x}) + \ldots + c_{q+1} f_{q+1}(\boldsymbol{x}) = 0$$

for all $\boldsymbol{x} \in \mathbb{C}^k$. By considering the leading terms on the left-hand side we obtain a contradiction. The lemma follows.

We now take the vector $\boldsymbol{u} \in U(r)$ to be of the particular form

$$\boldsymbol{u} = (1, x_1, x_2, \ldots, x_{r-2}, -1 - x_1 - x_2 - \ldots . x_{r-2}),$$

where the x_λ $(1 \leqslant \lambda \leqslant r - 2)$ are complex numbers. Then, for $\boldsymbol{\nu} \in V_1^*(r, m)$, $[\boldsymbol{u}^{\boldsymbol{\nu}}]$ is a polynomial with integral coefficients in the $r - 2$ variables $x_1, x_2, \ldots, x_{r-2}$ and has leading term

$$(-1)^{\nu_1} x_1^{\nu_1+\nu_2} x_2^{\nu_3} \ldots x_{r-2}^{\nu_{r-1}}.$$

Because of (3.8), these leading terms are all essentially different. Therefore, by Lemma 2, we can find q vectors $\boldsymbol{x}^{(\mu)} \in \mathbb{C}^{r-2}$ $(1 \leqslant \mu \leqslant q = d_r(m))$ such that (3.9) holds, where f_λ $(1 \leqslant \lambda \leqslant q)$ runs through the q polynomials $[\boldsymbol{u}^{\boldsymbol{\nu}}]$.

Let $\boldsymbol{u}_\mu$ and $[\boldsymbol{u}_\mu^{\boldsymbol{\nu}}]$ denote respectively the values of the vector $\boldsymbol{u}$ and polynomial $[\boldsymbol{u}^{\boldsymbol{\nu}}]$ at $\boldsymbol{x}^{(\mu)}$. It follows that it is possible to solve the $d_r(m)$ linear equations

$$\sum_{\mu=1}^{q} [\boldsymbol{u}_\mu^{\boldsymbol{\nu}}] z_\mu = a(\boldsymbol{\nu}) \pi(\boldsymbol{\nu}) \qquad (\boldsymbol{\nu} \in V_1^*(r, m)),$$

for the q unknowns z_μ $(1 \leqslant \mu \leqslant q)$. Hence

$$F = \sum_{\mu=1}^{q} z_\mu P_{r,m}(\boldsymbol{u}_\mu)$$

and this completes the proof of the theorem.

4. The functions $P_{r,m}(\boldsymbol{u})$.

THEOREM 2. *For* $r \geqslant 2$, $m \geqslant 2$ *and* $\boldsymbol{u} \in U(r)$,

$$(-1)^{m-1} m P_{r,m}(\boldsymbol{u}) = \sum_{\mu=0}^{m-2} (-1)^{\mu} s_{m-\mu} S_{m-\mu} P_{r,\mu}(\boldsymbol{u}). \tag{4.1}$$

PROOF. We work in terms of formal power series and write

$$g(x) = \sum_{\lambda=0}^{n} (-1)^{\lambda} \sigma_{\lambda} x^{\lambda} \tag{4.2}$$

from which we deduce, by the usual methods of symmetric function theory, that

$$\frac{g'(x)}{g(x)} = -\sum_{\lambda=1}^{\infty} s_{\lambda} x^{\lambda-1}. \tag{4.3}$$

Accordingly, by (2.7) and (4.2),

$$F_r(\boldsymbol{u}, x) := \sum_{m=0}^{\infty} (-1)^m P_{r,m}(\boldsymbol{u}) x^m = \prod_{j=1}^{r} g(u_j x)$$

and, on differentiating, we obtain, by (4.2),

$$\sum_{m=1}^{\infty} (-1)^m m P_{r,m}(\boldsymbol{u}) x^{m-1} = F_r(\boldsymbol{u}, x) \sum_{j=1}^{r} u_j g'(u_j x)/g(u_j x)$$

$$= -F_r(\boldsymbol{u}, x) \sum_{\lambda=1}^{\infty} s_{\lambda} S_{\lambda} x^{\lambda-1}.$$

We deduce (4.1) by taking the coefficient of x^{m-1} and noting that $S_1 = 0$.

Denote by $D(m)$ $(m \geqslant 2)$ the set of partitions π of m into summands $v_1, v_2, \ldots, v_k$ (k arbitrary), where

$$v_1 \geqslant v_2 \geqslant \ldots \geqslant v_k \geqslant 2$$

and write

$$s_{\pi} = s_{v_1} s_{v_2} \ldots s_{v_k}, \quad S_{\pi} = S_{v_1} S_{v_2} \ldots S_{v_k}, \tag{4.4}$$

$$\epsilon_{\pi} = (-1)^{m-k} \quad , \quad \varphi_{\pi} = \epsilon_{\pi} s_{\pi} S_{\pi}. \tag{4.5}$$

Observe that φ_{π} depends on r only through the factor S_{π}.

THEOREM 3. *Let $r \geqslant 2$, $m \geqslant 2$ and $\boldsymbol{u} \in U(r)$. Then*

$$P_{r,m}(\boldsymbol{u}) = \sum_{\pi \in D(m)} b_{\pi} \varphi_{\pi} \tag{4.6}$$

where the coefficients b_{π} are positive rational numbers independent of r and are defined inductively as follows:

$$b_0 = 1, \qquad b_{\pi} = \frac{1}{m} \sum b_{\rho},$$

the summation being extended over all partitions ρ of non-negative integers that become π after the addition of a single positive summand; the trivial partition of zero is denoted by $\rho = 0$.

This follows in a straightforward manner by induction. In the examples that follow we write

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$$\varphi_\lambda = (-1)^{\lambda-1} s_\lambda S_\lambda. \tag{4.7}$$

$$2P_{r,2} = \varphi_2, \quad 3P_{r,3} = \varphi_3, \quad 4P_{r,4} = \varphi_4 + \frac{1}{2}\varphi_2^2,$$

$$5P_{r,5} = \varphi_5 + \frac{5}{6}\varphi_3\varphi_2, \quad 6P_{r,6} = \varphi_6 + \frac{3}{4}\varphi_4\varphi_2 + \frac{1}{3}\varphi_3^2 + \frac{1}{8}\varphi_2^3,$$

$$7P_{r,7} = \varphi_7 + \frac{7}{10}\varphi_5\varphi_2 + \frac{7}{12}\varphi_4\varphi_3 + \frac{7}{24}\varphi_3\varphi_2^2,$$

$$8P_{r,8} = \varphi_8 + \frac{2}{3}\varphi_6\varphi_2 + \frac{8}{15}\varphi_5\varphi_3 + \frac{1}{4}\varphi_4^2 + \frac{1}{4}\varphi_4\varphi_2^2 + \frac{2}{9}\varphi_3^2\varphi_2 + \frac{1}{48}\varphi_2^4.$$

Note, in particular, that when $r = m = 8$, $d_8(8) = 7$ and that we need to take more than one vector $\boldsymbol{u}$ to represent some polynomials in $A(8,8)$. For, if we take

$$f = as_4^2 + bs_4s_2^2 + cs_2^4,$$

it is not possible to find $\boldsymbol{u}$ to make

$$a = \frac{1}{4}S_4^2, \quad b = \frac{1}{4}S_4S_2^2, \quad c = \frac{1}{48}S_2^4,$$

unless $b^2 = 12\,ac$.

COROLLARY 2. *Let ω_r be a primitive rth root of unity, where $r \geqslant 2$ and let $\boldsymbol{u}_r$ be the vector with components ω_r^λ $(0 \leqslant \lambda < r)$. Then*

$$P_{r,r}(\boldsymbol{u}_r) = (-1)^{r-1}s_r.$$

This follows at once from Theorem 3 by considering the values taken by S_λ.

5. **Applications to automorphic forms**. In [2] I was concerned with the determination of all polynomials in the derivatives of an automorphic form f of arbitrary real weight k and multiplier system (MS) v that could be expressed as polynomials in f and its derivatives. Almost no assumptions were made regarding the group Γ to which f belonged other than that it was infinite; but to fix ideas it may be assumed that Γ is a discrete group acting on the upper half-plane. As in [2] write

$$h_\mu = \frac{f^{(\mu)}}{\Gamma(k+\mu).\,\mu!} \; (\mu = 0, 1, 2, \ldots), \tag{5.1}$$

and, for simplicity, restrict attention to *admissible* values of k, i.e. real numbers other than the integers $k = 0, -1, -2, \ldots$

If we now put

$$\sigma_\mu = h_\mu / h_0 \; (\mu \geqslant 0), \quad \psi_\mu = (-1)^{\mu-1} s_\mu h_0^\mu \; (\mu \geqslant 2), \tag{5.2}$$

then ψ_m is a polynomial in the h_μ $(\mu \geqslant 0)$ of degree and weight m and can be expressed explicitly as an $m \times m$ determinant; see equation (6) of [2]. The following theorem was proved in [2].

THEOREM 4. *If $P_{r,m}$ is a polynomial of degree r and weight m in the automorphic form f and its derivatives, then $P_{r,m}$ is expressible as $h_0^{r-m}\, Q_m$, where Q_m is a polynomial of weight m in the functions ψ_μ ($\mu \geqslant 2$). Conversely, given any such polynomial Q_m, an integer $r \leqslant m$ can be found such that $h_0^{r-m}\, Q_m$ is a polynomial of degree r and weight m in f and its derivatives and is an automorphic form.*

It may be noted that, in the theorem, the automorphic *form* $h_0^{r-m}\, Q_m$ has weight (i.e. negative dimension) $rk + 2m$ and MS v^r. For the reasons mentioned in §1, Theorem 4 is not as useful as might be desired.

We now observe that, by (5.2), $h_0^{r-m}\, Q_m = h_0^r\, Q_m^*$, where Q_m^* is a polynomial in the s_μ ($\mu \geqslant 2$) of weight m. Moreover, because $h_0^{r-m}\, Q_m$ is a polynomial in the h_μ ($\mu \geqslant 0$), Q_m^*, when expressed as a polynomial in the σ_μ cannot contain any term having more than r factors σ_μ with $\mu > 0$. This means that $h_0^{r-m}\, Q_m$ is associated with a unique member F of $A(r, m)$; conversely, each $F \in A(r, m)$ gives rise to a polynomial of the type $P_{r,m}$ mentioned in Theorem 4.

Theorem 3 now fills the gap in Theorem 4, in that it shows that every F in $A(r, m)$ can be expressed as a linear combination of the polynomials $P_{r,m}(\boldsymbol{u})$ for different $\boldsymbol{u} \in U(r)$, where $P_{r,m}(\boldsymbol{u})$ is expressed as in (4.6) in terms of the s_μ ($\mu \geqslant 2$) and the coefficients b_π can be determined inductively.

Note that for particular choices of automorphic forms some polynomials in $A(r, m)$ may vanish identically. For example, it was shown on p. 115 of [2] that

$$35\psi_4 + 11\psi_2^2$$

vanishes identically for f equal to the modular discriminant Δ; here $r = m = 4$. This differential equation for Δ is *homogeneous* in the sense that the terms are of constant degree and weight. Inhomogeneous differential equations for automorphic forms also exist; see [2] and, for a fuller discussion, [4] and [5].

In conclusion we remark that the restriction to admissible $k = 0$ can be removed for $k \neq 0$ by considering the function $1/f$ in place of f, or by using Resnikoff's functions $D^\mu f$ in place of the h_μ.

6. **Extension to several automorphic forms**.

The ideas used to construct the polynomials $P_{r,m}(\boldsymbol{u})$ can be extended to several automorphic forms as follows. Let

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C}), \quad Tz = \frac{az + b}{cz + d} \quad (z \in \mathbb{C}) \tag{6.1}$$

and let k be any admissible real number. Put

$$f_T(z) = (cz + d)^{-k} f(Tz), \quad S = cz + d, \quad \lambda = c/S, \tag{6.2}$$

where the functions f and f_T are holomorphic on some open domain $D \subseteq \mathbb{C}$. We write $f_T = f|T(k)$ and put

$$h_q(z, f, k) = \frac{f^{(q)}(z)}{\Gamma(k + q).\, q!} \quad (z \in D, \ q \geqslant 0). \tag{6.3}$$

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By differentiating the equation

$$f(Tz) = (cz + d)^k f(z)$$

successively, we obtain, as in §2 of [2],

$$h_q(Tz, f, k) = S^{k+2q} \sum_{\nu=0}^{q} \frac{\lambda^{q-\nu}}{(q-\nu)!} h_\nu(z, f_T, k). \tag{6.4}$$

If we now introduce the formal power series

$$h(x; z, f, k) = \sum_{q=0}^{\infty} h_q(z, f, k)x^q, \tag{6.5}$$

we find that

$$h(x; Tz, f, k) = S^k e^{\lambda x S^2} h(xS^2; z, f_T, k). \tag{6.6}$$

Now take r functions f_λ and r admissible real numbers k_λ ($\lambda = 1, 2, \ldots, r$) satisfying the conditions stated above, any $\boldsymbol{u} \in U(r)$, and write

$$K = k_1 + k_2 + \ldots + k_r. \tag{6.7}$$

We then deduce from (6.6) that

$$\prod_{\lambda=1}^{r} h(u_\lambda x; Tz, f_\lambda, k_\lambda) = S^K \prod_{\lambda=1}^{r} h(S^2 u_\lambda x; z, f_\lambda | T(k_\lambda), k_\lambda). \tag{6.8}$$

Define

$$F_m(z) = \sum_{\nu \in V(r,m)} h_{\nu_1}(z, f_1, k_1) \ldots h_{\nu_r}(z, f_r, k_r)\boldsymbol{u}^\nu, \tag{6.9}$$

Then $F_m(Tz)$ is the coefficient of x^m on the left-hand side of (6.8), and so we have

$$F_m(Tz) = S^{K+2m} \sum_{\nu \in V(r,m)} \left\{ \prod_{\lambda=1}^{r} h_{\nu_\lambda}(z, f_\lambda | T(k_\lambda), k_\lambda) \right\} \boldsymbol{u}^\nu \tag{6.10}$$

for all $z \in D$. In particular, if

$$f_\lambda | T(k_\lambda) = w_\lambda f_\lambda \quad (1 \leqslant \lambda \leqslant r),$$

we have

$$F_m(Tz) = w_1 w_2 \ldots w_r (cz + d)^{K+2m} F_m(z), \tag{6.11}$$

from which we deduce

THEOREM 5. *Let f_λ be automorphic forms of admissible weight k_λ and MS w_λ $(1 \leqslant \lambda \leqslant r)$, where $r \geqslant 2$, for a discrete infinite group Γ; we write this as $f_\lambda \in \{\Gamma, k_\lambda, w_\lambda\}$. Then, if for every integer $m \geqslant 0$ F_m is defined by (6.9), we have*

$$F_m \in \{\Gamma, K + 2m, w_1 \ldots w_r\}.$$

We immediately deduce

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COROLLARY 3. *Let* $f_\lambda \in \{\Gamma, k_\lambda, w_\lambda\}$ *for* $\lambda = 1, 2$. *Then if*

$$G_m(z) = \sum_{\lambda=0}^{m} (-1)^\lambda h_\lambda(z, f_1, k_1) h_{m-\lambda}(z, f_2, k_2) \quad (m \geqslant 0),$$

$G_m \in \{\Gamma, k_1 + k_2 + 2m, w_1 w_2\}$.

The function G_m is a generalization of the polynomial $P_{2,m}$ of (3.7), where m is even and $f_1 = f_2$. Corollary 3 has already been proved by H. Cohen [1] (Theorem 7.1 and Corollary 7.2).

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