COLLINEATIONS OF PROJECTIVE MOULTON PLANES

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1. Introduction. In the article "Moulton Planes" **(10)**, I studied F. R. Moulton's construction over any field containing a multiplicative subgroup of index 2. In "Collineations of Affine Moulton Planes" **(11)**, I determined the collineations between two arbitrary *affine* Moulton planes.

The purpose now is to describe the collineations between two *projective* Moulton planes. Since the affine collineations are known from **(11)**, we are concerned with collineations mapping ideal lines onto ordinary lines. Notations and conventions of **(10)** and **(11)** are retained. We treat the collineations from $M_\phi(F)$ onto $M_\psi(K)$, Moulton planes over the respective fields F and K, relative to the respective maps ϕ and ψ. Functions ϕ and ψ are order-preserving on their respective domains; and for arbitrary negatives $n_0 \in F$, $n_0{}' \in K$, the maps

$$x \longrightarrow [\phi(x) - n_0 x] \quad \text{and} \quad x \longrightarrow [\psi(x) - n_0' x]$$

map F onto F and K onto K respectively. Both ϕ and ψ are "normal" in the sense that they fix 0 and 1 **(10**, Lemma 1**)**. Neither ϕ nor ψ is the identity. (Otherwise, one of the Moulton planes would be Desarguesian, and the collineations, if any, classical.)

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2. Collineations mapping Y_∞ onto Y_∞'. In a (non-Desarguesian) Moulton plane, the ideal point of the y-axis is the only point Q on l_∞ (the ideal line) for which the plane is $Q - l_\infty$ transitive. Since $Q - l_\infty$ transitivity for distinct choices of Q on l_∞ implies the Little Desargues' Theorem from l_∞, it would imply in this case the full Desarguesian condition; cf. **(10**, Theorem 4**)**.

Since the ideal point on the y-axis plays a unique role in the *affine* geometry of a Moulton plane, every *affine* collineation from $M_\phi(F)$ onto $M_\psi(K)$ maps $Y_\infty = \{x = 0\} \cap l_\infty$ onto $Y_\infty' = \{x' = 0\} \cap l_\infty'$. The projective situation is quite different! In the first place, there exist planes $M_\phi(F)$ and $M_\psi(K)$ and collineations of $M_\phi(F)$ onto $M_\psi(K)$ sending Y_∞ to Y_∞' but failing to map l_∞ onto l_∞'. (An example is provided by the Moulton plane over the near-field of order 9, which will be treated separately in Theorem A.) In the second place, a projective collineation need not even *map* Y_∞ onto Y_∞'. (Examples

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are provided by the generalized Moulton planes of Pickert (**9**, p. 93); cf., also J. C. D. Spencer (**12**, Theorem 8).) I shall prove (Theorem 3) the surprising result that the Moulton-Pickert planes are the *only* ones (up to isomorphism) from which $M_\phi(F)$ and $M_\psi(K)$ may be chosen if a collineation of $M_\phi(F)$ onto $M_\psi(K)$ is to exist mapping Y_∞ onto a point other than Y_∞'.

The case of order 9 is exceptional. If F has order >9, a collineation from $M_\phi(F)$ onto $M_\psi(K)$ *cannot* map Y_∞ onto Y_∞' unless it maps the pair $(\{x = 0\}, l_\infty\})$ onto $(\{x' = 0\}, l_\infty')$. This property will be established by Lemmas 2 and 3. In reading those Lemmas, it may help to consider the following example. Let ϕ be any sign-preserving automorphism on F, with $x \to [\phi(x) - n_0 x]$ "onto" for every negative $n_0 \in F$. (If F is finite and ϕ "normal" so that $\phi(0) = 0$, $\phi(1) = 1$, as in (**10**, Lemma 1), ϕ is necessarily an automorphism; sign-preservation and the "onto" property being automatic for ϕ (**10**, Corollaries 1 and 2).) Fix $n_0 < 0$ $(n_0 \in F)$; for $x \neq 0$, map (x, y) onto (x', y'), where $x' = \phi\tau(n_0/x)$, $y' = \phi\tau(y/x)$; τ being $\mathfrak{J}$ (the identity) or ϕ^{-1} according as $x > 0$ or $x < 0$; map $(0, c)$ onto the ideal point that corresponds to slope c/n_0; and map the ideal point of slope r onto $(0, \phi(r))$. Substitution in $y' = b + m \circ x'$ shows that a collineation is obtained on $M_\phi(F)$.

LEMMA 1. *Suppose that α is a non-affine collineation from $M_\phi(F)$ onto $M_\psi(K)$ such that $\alpha(Y_\infty) = Y_\infty'$, ϕ (and hence ψ) being non-trivial. Then $\alpha = \beta\gamma$, where β is a collineation of $M_\phi(F)$ onto $M_\psi(K)$ for which $\beta(Y_\infty) = Y_\infty'$, $\beta(\{y = 0\}) = \{y' = 0\}$, $\beta(l_\infty) = \alpha(l_\infty)$, $\beta[(x, y)]$ has the same abscissa as $\alpha[(x, y)]$ for all (ordinary) points $(x, y) \in M_\phi(F)$; γ is a collineation on $M_\psi(K)$ fixing Y_∞' while mapping an ordinary point (x, y) onto $(x, y + a \circ x + k)$, for appropriate constants $a, k \in K$, and the ideal point of slope r onto the ideal point of slope $(r + a)$.*

The plane $M_\phi(F)$ is necessarily (Y_∞, Y_∞)-transitive and $M_\psi(K)$ (Y_∞', Y_∞')-transitive; the left-distributive law, $(a + b) \circ d = a \circ d + b \circ d$, holds in both F and K; functions ϕ and ψ are additive on their respective domains.

If no confusion arises, the symbol $\circ$ will be used to denote either the operation on F relative to ϕ or the operation on K relative to ψ. Where necessary, a distinction will be made: $a \circ_{(\phi)} b$ for the Moulton-product "a times b" on F; $a \circ_{(\psi)} b$ for the corresponding product on K.

Proof. By (**10**, Theorem 3), $M_\psi(K)$ is (Y_∞', l_∞')-transitive. A non-affine collineation from $M_\phi(F)$ onto $M_\psi(K)$ mapping Y_∞ onto Y_∞' but l_∞ onto a line $\neq l_\infty'$ ensures that $M_\psi(K)$ is (Y_∞', l')-transitive for distinct choices of l' through Y_∞'. This implies (Y_∞', Y_∞')-transitivity on $M_\psi(K)$. Since there exists a collineation of $M_\phi(F)$ onto $M_\psi(K)$ under which Y_∞ and Y_∞' correspond, $M_\phi(F)$ is (Y_∞, Y_∞)-transitive. The left-distributive laws for $\circ$ and the additivity of functions ϕ, ψ follow from (**10**, Theorem 5).

Using the left-distributive law, one checks easily that γ (as defined in the Lemma) is a collineation on $M_\psi(K)$, for *any* constants $a, k \in K$. Choose a

and k such that $\{y' = a \circ x' + k\}$ is the α-image in $M_\psi(K)$ of the x-axis in $M_\phi(F)$. Then $\alpha\gamma^{-1}$ is a collineation satisfying the requirements for β. Put $\beta = \alpha\gamma^{-1}$, and the proof is complete.

LEMMA 2. *Let α be a collineation of $M_\phi(F)$ onto $M_\psi(K)$ such that $\alpha(Y_\infty) = Y_\infty'$. If $\alpha(l_\infty) = l_\infty'$, if $\alpha(l_\infty) = \{x' = 0\}$, or if $\alpha(\{x = 0\}) = l_\infty'$, then α maps the set $(\{x = 0\}, l_\infty)$ onto $(\{x' = 0\}, l_\infty')$.*

Proof. (i) If α is affine, then $\alpha(\{x = 0\}) = \{x' = 0\}$ (**11**, Lemma 5).

(ii) Suppose that $\alpha(l_\infty) = \{x' = 0\}$. Let ρ be a collineation on $M_\psi(K)$ which sends (x', y') to (px', py'), for some $p > 0$, $1 \neq p \in K$, l_∞' being pointwise fixed. The conjugate $\alpha\rho\alpha^{-1}$ is affine on $M_\phi(F)$, hence fixes $\{x = 0\}$. This is impossible unless $\alpha(\{x = 0\}) = l_\infty'$.

(iii) Suppose that $\alpha(\{x = 0\}) = l_\infty'$. Apply (ii) to α^{-1}, with $M_\phi(F)$ and $M_\psi(K)$ interchanged. We conclude that $\alpha(l_\infty) = \{x' = 0\}$.

LEMMA 3. *Any collineation of $M_\phi(F)$ onto $M_\psi(K)$ which carries Y_∞ onto Y_∞' maps the set $(\{x = 0\}, l_\infty)$ onto $(\{x' = 0\}, l_\infty')$, provided F has order >9.*

Proof. Suppose a collineation fails to map $(\{x = 0\}, l_\infty)$ onto $(\{x' = 0\}, l_\infty')$. By Lemma 2, it either carries $\{x = 0\}$ onto $\{x' = 0\}$ and l_∞ onto a "finite" line; or it carries neither $\{x = 0\}$ nor l_∞ onto a line of the pair $(\{x' = 0\}, l_\infty')$. If such a collineation, say α^*, exists, then $\alpha^*(l_\infty) = \{x' = l\}$ and $\alpha^*(\{x = 0\}) = \{x' = k\}$ where $l \neq 0$, $k \in K$. Choose $p > 0 \in K$ with $p \neq 1$, $p \neq (k/l)$, and (in case $k \neq 0$) $p \neq (l/k)$. (This is possible since K has more than three distinct positives.) Define $\alpha = \alpha^*\rho(\alpha^*)^{-1}$, where ρ is the collineation on $M_\psi(K)$ which sends (x', y') to (px', py') but fixes l_∞' pointwise. If $\alpha^*(\{x = 0\}) = \{x' = 0\}$, then $\alpha(\{x = 0\}) = \{x = 0\}$, but α displaces l_∞. If $\alpha^*(\{x = 0\}) = \{x' = k \neq 0\}$, then α maps $(\{x = 0\}, l_\infty)$ onto a pair of lines both distinct from $\{x = 0\}$ and l_∞. In the rest of the proof, we confine our attention to such a collineation α, on $M_\phi(F)$, and show that its existence would provide a contradiction. By Lemma 1, we may assume that $\alpha(\{y = 0\}) = \{y = 0\}$.

Given any positive q, $1 \neq q \in F$, a map β: $(x, y) \to (qx, qy)$, with ideal points fixed, defines a collineation leaving $(0, 0)$ linewise invariant but fixing no other elements. For each such β, we can form a collineation $\gamma = \alpha^{-1}\beta\alpha$, with α as in the preceding paragraph. If $\alpha[(0, 0)] = (s_0, 0)$ and $\alpha(X_\infty) = (u_0, 0)$, the collineation γ (for each choice of β) fixes $(s_0, 0)$ linewise and $\{x = u_0\}$ pointwise, but has no other fixed elements.

Let $\{y' = b' + r' \circ x'\}$ denote the γ-image of a line $\{y = b + r \circ x\}$ for any $b, r \in F$. Since $\{x = u_0\}$ is pointwise fixed,

(i) $b + r \circ u_0 = b' + r' \circ u_0$.

Since γ fixes $(s_0\, 0)$ linewise, moves l_∞ to $\{x' = w_0\}$, and moves $\{x = z_0\}$ to l_∞; we have

(ii) $b' + r' \circ w_0 = r \circ w_0 - r \circ s_0$

and

(iii) $b + r \circ z_0 = r' \circ z_0 - r' \circ s_0$.

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These three relations determine r' and b' as functions of b and r; for $b, r \in F$. From (i) and (iii), using the left-distributive law (Lemma 1),

$$b = b' + (r' - r) \circ u_0 = (r' - r) \circ z_0 - r' \circ s_0.$$

Combining this with (ii), we obtain

$$b' = -(r \circ s_0) - (r' - r) \circ w_0 = -(r' \circ s_0) + (r' - r) \circ z_0 - (r' - r) \circ u_0.$$

For $r = 0$, $\{y = b\} = (u_0\ b) \cup X_\infty$ corresponds in one-to-one fashion to

$$\{y' = r' \circ x' - r' \circ w_0\} = (u_0, b) \cup (w_0, 0).$$

Thus, with $r = 0$,

$$r' \circ s_0 + r' \circ u_0 = r' \circ w_0 + r' \circ z_0,$$

for all $r' \in F$. The product $s_0 u_0$ can be assumed $\geqslant 0$. (Otherwise $s_0 u_0 < 0$, and the condition for $r = 0$ takes the form

$$d_0 \cdot \phi(r') + e_0 r' = f_0 \cdot \phi(r') + g_0 r',$$

with d_0, e_0, f_0, g_0 constants $\in F$; the (unordered) pair $\{d_0, e_0\}$ being $\{s_0, u_0\}$, and $d_0 \neq f_0$. Thus, $\phi(r') = Ar'$ for all $r' \in F$, where A is a constant $\in F$; putting $r' = 1 = \phi(1)$ gives $A = 1$ and $\phi = \mathfrak{J}$, a contradiction.) In order that $s_0 u_0 \geqslant 0$ for every choice of γ, the product $t_0 w_0$ must also be non-negative, where $(t_0, 0) = \gamma[(0, 0)]$.

It is convenient to note that any collineation on $M_\phi(F)$, say δ, which fixes Y_∞ and maps $(\{x = 0\}, l_\infty)$ onto $(\{x = a\}, \{x = d\})$ with $ad > 0$ must satisfy $a = -d$, so that $-1 > 0$. (By Lemma 1, we can take $\delta[(0, 0] = (a, 0)$ and $\delta(X_\infty) = (d, 0)$. Let β_0 denote the collineation fixing ideal points and sending (x, y) to $(dx/a, dy/a)$, in particular, $(a, 0)$ to $(d, 0)$. The map $\delta\beta_0\delta^{-1}$ sends $(0, 0)$ to X_∞; hence X_∞ to $(0, 0)$, by Lemma 2. This is possible only if β_0 sends $(d, 0)$ to $(a, 0)$; $d^2 = a^2$; and (since $a \neq d$) $d = -a$).

The two remaining possibilities for $\gamma = \alpha^{-1}\beta\alpha$ are:

(I) $s_0 u_0 > 0$, in which case $s_0 = -u_0$ (taking $\delta = \alpha$);

(II) $s_0 u_0 = 0$, which occurs only if $u_0 \neq 0$ and $s_0 = 0$.

In Case I, the relation

$$r' \circ s_0 + r' \circ u_0 = r' \circ w_0 + r' \circ z_0, \qquad \text{for all } r' \in F,$$

reduces to $0 = r' \circ w_0 + r' \circ z_0$. Since $\phi \neq \mathfrak{J}$ we have $w_0 z_0 \geqslant 0$. Thus $0 = r'(w_0 + z_0)$ or $0 = \phi(r') \cdot (w_0 + z_0)$. Since r' does not vanish identically, we have $w_0 = -z_0$.

Case $\mathrm{I_a}$. If $w_0 = z_0 = 0$, γ (hence also β) interchanges two distinct points. It follows that β is given by $(x, y) \to (-x, -y)$.

Case $\mathrm{I_b}$. If $w_0 = -z_0 \neq 0$, $w_0 t_0 > 0$. (We have noted that $w_0 t_0 \geqslant 0$. Since γ fixes no values of x other than u_0 and s_0, it follows that $t_0 \neq 0$.) Taking $\gamma = \delta$, $a = t_0$, $d = w_0$ (as above), we conclude that $-t_0 = w_0 = -z_0$, whence

$t_0 = z_0$. Thus $\gamma^2[(0, 0)] = X_\infty$; and $\gamma^2(X_\infty) = (0, 0)$, by Lemma 2. The interchange between $(0, 0)$ and X_∞ under γ^2 implies the interchange of two abscissae under β^2, possible only if β^2 maps each ordinary point (x, y) onto $(-x, -y)$.

In Cases 1_a and 1_b, we have shown that either β or β^2 is given by $(x, y) \to (-x, -y)$. Hence, β sends (x, y) to (ex, ey), for $e > 0 \in F$, with $e^4 = 1$. Since the field F has order >9, F contains more than four distinct positives, and β can be chosen as follows: $\beta[(x, y)] = (qx, qy)$, with $q > 0$ in F but $q^4 \neq 1$. This yields a contradiction.

In Case II, the identity

$$r' \circ u_0 = r' \circ w_0 + r' \circ z_0, \qquad \text{for all } r' \in F,$$

again gives $\phi = \mathfrak{J}$ unless u_0, w_0, z_0 have a common sign. (None of the three is 0, because $s_0 = 0$.)

Since $\gamma[(0, b)] = (0, b')$, the intercept b' depends only on b. A relation between b' and b derives from the fact that γ maps $(0, b) \cup (u_0, b)$ onto $(w_0, 0) \cup (u_0, b)$: any point (x', y') with $x'u_0 > 0$ on the image of $\{y = b\}$ satisfies

$$y' = [b/(u_0 - w_0)]\cdot x' + bw_0/(w_0 - u_0),$$

whence $b' = bw_0/(w_0 - u_0)$. Since

$$\gamma[(z_0, b + r \circ z_0)] = P_\infty(r'),$$

the invariance of lines through $(0, 0)$ gives $r' = r + (b/z_0)$ or $r + (\phi^{-1}(b/z_0))$, the latter using additivity of ϕ, according as u_0 (hence also z_0) > 0 or u_0 (and hence z_0) < 0. Furthermore, invariance of lines through the origin shows that (x, y) and (x', y') $\{= \gamma[(x, y)]\}$ are related by $y' = x'\cdot(y/x)$ if $xx' > 0$; and by $y' = x'\cdot\phi(y/x)$ or $y' = x'\cdot\phi^{-1}(y/x)$ if $xx' < 0$, according as $x > 0$ or $x < 0$. Choose x_0 so that $x_0 u_0 < 0$, letting x_0' be the image of x_0 under γ. Substitute for b', r' in $y' = b' + r' \circ x'$, putting $y = b + r \circ x_0$, $x' = x_0'$ with y' determined (as above) by x_0, x_0'. The result is $\phi(Ab) = Bb$ or $\phi^{-1}(Ab) = Bb$ (all $b \in F$) according as $x_0' < 0$ or $x_0' > 0$, the constants being

$$A = 1/z_0 \quad \text{and} \quad B = (1/x_0) - w_0/[x_0'\cdot(w_0 - u_0)] \qquad \text{if } x_0 x_0' > 0$$

$$A = 1/x_0 \quad \text{and} \quad B = (1/z_0) + w_0/[x_0'\cdot(w_0 - u_0)] \qquad \text{if } x_0 x_0' < 0.$$

Putting $Ab = u$, and using $\phi(1) = 1$, we get $\phi = \mathfrak{J}$, a contradiction.

This completes the proof of Lemma 3.

3. The Moulton plane of order 9. In **(11)**, all affine collineations of non-Desarguesian Moulton planes were shown to be sign-preserving or sign-reversing on x, *except for planes of order* 9. The extension from affine to projective collineations is likewise exceptional on $M_\phi(F_9)$; and again the reason is a shortage of elements from which to make a certain choice. Since there is only one (non-Desarguesian) Moulton plane of order 9 (except for notational changes), we consider the group of collineations on one such plane.

THEOREM A. *Let* ϕ *denote the automorphism* $x + jy \to x - jy$ $(x, y \in$ *the prime subfield,* $F_3)$ *on* F_9, *the field of order* 9. As in **(11)**, F_9 consists of elements $x + jy$, where $j^2 = -1$. *The group* C *of collineations on* $M_\phi(F_9)$ *consists of all the affine collineations, together with the products* $\omega\sigma_q$, *for all* $\omega \in A$ (the affine group on $M_\phi(F_9)$), *and corresponding to each* $q \in F_9$ *one* σ_q *that maps* l_∞ *onto* $\{x' = q\}$. *The general affine collineation* ω *is a succession:* $(x, y) \to (x', y')$ *followed by* $(x', y') \to (x'', y'')$. *The functions* x' *and* y' *are given by* $y' = lk\cdot\alpha(y)$, $x' = l\cdot\alpha(x)$ *or* $[(lk)\alpha(x)]/\phi(k)$, *according as* $\alpha(x) \geqslant 0$ *or* $\alpha(x) <; 0$ l, k *being arbitrary non-zero elements of* F_9, *and* α *an arbitrary* F_3*-linear transformation of* F_9 *onto itself. The map* $(x', y') \to (x'', y'')$ *involves* $x'' = x'$, $y'' = y' + d \circ x' + c$, *with* d *and* c *arbitrary in* F_9. *On* l_∞, ω *sends* $P_\infty(r)$ *to* $P_\infty[k\cdot\alpha(r) + d]$ *or to* $P_\infty\{\phi[k\alpha(r)] + d\}$ *according as* $l > 0$ *or* $l < 0$.

If $q = 0$, σ_q *may be defined as a map interchanging* $(0, y)$ *with* $P_\infty[\phi(y)]$ *and otherwise sending* (x, y) *to* (x', y'), *with* $x' = 1/[\phi(x)]$ *or* $1/x$, $y' = \phi(y/x)$ *or* y/x, *each according as* $x > 0$ *or* $x < 0$.

If $q \neq 0$, σ_q *may be defined to map* $P_\infty(r)$ *onto* $(q, -rq)$, $(-j, y)$ *onto* $P_\infty(jy)$ *or* $P_\infty[\phi(jy)]$ *according as* $q > 0$ *or* $q < 0$, *otherwise sending* (x, y) *to* (x', y'). *Here*

$$x' = q(x - j)/(x + j), \quad y' = -qy/(x + j) \qquad \text{if } x \geqslant 0 \text{ (and } \neq -j),$$

$$x' = -q(x - j)/[\phi(x) - j],$$

$$y' = -qyj + q[(jx + 1)\phi(y)]/[\phi(x) - j] \qquad \text{if } x < 0.$$

The Group C *comprises* 10 *right cosets of* A, *one for each admissible image of* l_∞. *Point* Y_∞ *is fixed by all collineations* $\in C$.

Proof. The affine group A was determined in **(11)**, where $M_\phi(F_9)$ was treated separately. Given $\omega_1 \in A$ and a collineation γ_1 on $M_\phi(F_9)$ that displaces l_∞, the product $\omega_1\gamma_1$ is a collineation sending l_∞ to $\gamma_1(l_\infty)$. Conversely, if γ_1, γ_2 are collineations of $M_\phi(F_9)$ for which $\gamma_1(l_\infty) = \gamma_2(l_\infty)$, then $\gamma_1\gamma_2^{-1} \in A$. Hence, the collineations mapping l_∞ onto a given line form a single right coset of A, completely determined by any one of its members. It will, therefore, be sufficient to verify the collineation-property for σ_q, as defined above, *provided we show that* Y_∞ *is fixed by every map* $\in C$.

If $q = 0$, we substitute for x' and y' in $\{y' = b' + r' \circ x'\}$, with $x' \neq 0$, and (using the automorphism property of ϕ) obtain $\{y = \phi(r') + [\phi(b')] \circ x\}$, for $x \neq 0$. This is consistent with the interchange $(0, y) \leftrightarrow P_\infty[\phi(y)]$.

If $q \neq 0$, the collineation property for σ_q is implied by the property for σ_1: In fact, for each $q \neq 0$, σ_q is the product of σ_1 followed by $(x, y) \to (qx, qy)$; l_∞ being pointwise fixed in the latter if $q > 0$, but $P_\infty(r)$ being interchanged with $P_\infty(\phi(r))$ in the latter if $q < 0$.

Let us verify that σ_1 is a collineation. The formulae relating x' to x show that $x \to x'$ fixes the negatives setwise, also the set of non-negatives and ∞. For $x \geqslant 0$ $(x \neq -j)$, substitution of

$$x' = (x - j)/(x + j), \qquad y' = -y/(x + j)$$

in $\{y' = b' + r'x'\}$ gives

$$\{y = -(b' + r')x + j(r' - b')\}.$$

For $x < 0$, we substitute

$$x' = (x - j)/[-\phi(x) + j],$$

and

$$y' = -yj + [(jx + 1)\phi(y)]/[\phi(x) - j]$$

in $\{y' = b' + \phi(r')\cdot x'\}$. Replacing y by $-[\phi(b' + r')]\cdot x + j(r' - b')$, and $\phi(y)$ by $-(b' + r')\cdot\phi(x) - j[\phi(r' - b')]$, we obtain an equation that holds identically for all negative x (easily checked if we use the fact that $\phi(x) = -1/x$, for all $x < 0$):

$$[j\phi(x) + 1]\cdot[-\phi(b' + r')\cdot x + j(r' - b')] - (jx + 1)\cdot\{-(b' + r')\cdot\phi(x) - j[\phi(r' - b')]\} = b'\cdot[-\phi(x) + j] + \phi(r')\cdot(x - j).$$

The definition of σ_1 on l_∞ and $\{x = -j)$ is consistent with the calculation just given; and σ_1 is one-to-one on the points of $M_\phi(F_9)$. Hence, σ_1 is a collineation—and so is σ_q, for all non-zero $q \in F_9$.

Unless Y_∞ is fixed by all collineations of $M_\phi(F_9)$, there is a collineation γ for which $\gamma(Y_\infty) \neq Y_\infty$. Since $M_\phi(F_9)$ is (Y_∞, Y_∞)-transitive (**10**, Corollary 3), the plane is also $(\gamma(Y_\infty), \gamma(Y_\infty))$-transitive, thus Desarguesian—a contradiction.

This completes the proof.

Remark. The order of C is 311,040.—This agrees with the known order of C over the near-field of order 9, also with the order over the almost-field whose plane is dual to $M_\phi(F_9)$; cf. (**5**, Appendix II and **1**, p. 139).

Proof. The general affine collineation ω can be chosen in 31,104 ways (eight choices for each non-zero constant l, k; six choices for $\alpha(j)$ and hence for α itself; eighty-one choices for $(x', y') \to (x', y' + d \circ x' + c)$). The ten cosets of A in C contain $(31{,}104)\cdot(10) = 311{,}040$ collineations.

4. Moulton planes of order > 9. If the order of F exceeds 9, (**11**, Theorem **1**) gives a necessary and sufficient condition for the existence of an affine collineation from $M_\phi(F)$ onto $M_\psi(K)$ and its general form. Hence, the existence and general form of all collineations from $M_\phi(F)$ onto $M_\psi(K)$ can be settled by treating the non-affine collineations. Theorem 1, below, gives a necessary and sufficient condition for the existence and general form of those non-affine collineations which map Y_∞ onto Y_∞'. Theorems 2 and 3 determine all other non-affine collineations and characterize the Moulton planes from which such non-affine collineations may arise.

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THEOREM 1. (*Part* 1_a). *Let ϕ be non-trivial on a field F of order >9. A non-affine collineation mapping Y_∞ onto Y_∞' exists from $M_\phi(F)$ onto $M_\psi(K)$ if and only if ϕ is additive and there is a sign-preserving isomorphism α(from F onto K), together with non-zero constants $b_1 \in F$, $s_0 \in K$, such that*

$$(\dagger)\quad [\lambda\alpha(b/b_1)] \circ_{(\psi)} \{s_0\alpha[\tau_u(b_1 u)/b_1]\} = s_0\alpha[\tau_u(bu)/b_1], \qquad \textit{for all } b, u \in F;$$

τ_u being $\mathfrak{J}$ (on F) or ϕ^{-1} according as $u > 0$ or $u < 0$, and λ being $\mathfrak{J}$ (on K) or ψ^{-1} as $s_0 > 0$ or $s_0 < 0$.

(*Part* 1_b). *The most general non-affine collineation, from $M_\phi(F)$ onto $M_\psi(K)$, which maps Y_∞ onto Y_∞' has the form $\gamma = \nu\delta$, with δ on $M_\psi(K)$ given by $(x, y) \to (x, y + a \circ x + k)$ for some $a, k \in K$ and the ideal point of slope r moving to that of slope $(r + a)$; ν, from $M_\phi(F)$ to $M_\psi(K)$, given for constants $b_1 \in F$, $s_0 \in K$, by*

$$(x, y) \to (x', y') = (\mu(x), s_0\alpha[\tau_x(y/x)/b_1]),$$

if $x \neq 0$, where $\mu(x) = s_0\alpha\{[\tau_x(b_1/x)]/b_1\}$, and α satisfies the identity ($\dagger$) in b and $x \neq 0$. The ideal point $P_\infty(r)$, for $r \in F$, is mapped by ν onto $(0, b') = (0, s_0\alpha(r/b_1))$. The point $(0, b)$ is mapped by ν onto $P_\infty(r')$, where $r' = \lambda\alpha(b/b_1)$.

Proof of 1_a. By Lemma 3, every collineation of $M_\phi(F)$ onto $M_\psi(K)$ which sends Y_∞ to Y_∞' must map $(\{x = 0\}, l_\infty)$ onto $(\{x' = 0\}, l_\infty')$. Let γ be a non-affine collineation, from $M_\phi(F)$ onto $M_\psi(K)$, mapping Y_∞ onto Y_∞'. It follows that γ maps $\{x = 0\}$ and l_∞ onto l_∞' and $\{x' = 0\}$ respectively. By Lemma 1, functions ϕ, ψ are additive; and $\gamma = \nu\delta$, where δ, on $M_\psi(K)$, moves the ideal point of slope r to that of slope $(r + a)$ and sends (x, y) to $(x, y + a \circ x + k)$, line $\{y' = a \circ x' + k\}$ being the γ-image [in $M_\psi(K)$] of $\{y = 0\}$ in $M_\phi(F)$; and where ν is a collineation of $M_\phi(F)$ onto $M_\psi(K)$ sending the x-axis of $M_\phi(F)$ to the x'-axis of $M_\psi(K)$. Conversely, let δ be defined on $M_\psi(K)$ by $(x, y) \to (x, y + a \circ x + k)$ (with the ideal point of slope r moving to that of slope $(r + a)$); let ν be a non-affine collineation sending Y_∞ to Y_∞' and $\{y = 0\}$ in $M_\phi(F)$ to $\{y' = 0\}$ in $M_\psi(K)$. It follows that $\gamma = \nu\delta$ is also non-affine and sends Y_∞ to Y_∞'. Hence a non-affine collineation mapping Y_∞ onto Y_∞' exists if and only if there is one mapping Y_∞, X_∞, and $O = (0, 0)$ of $M_\phi(F)$ onto the respective points Y_∞', $O' = (0, 0)$, and X_∞' of $M_\psi(K)$.

Let ν: $(x, y) \to (x', y')$ determine such a collineation. Since $Y_\infty \to Y_\infty'$, x' depends only on x. Since y' (for $x \neq 0, \infty$) depends only on the slope of $(x, y) \cup (0, 0)$, ν is given, for $x \neq 0, \infty$, by $x' = \mu(x)$, $y' = \sigma\tau_x(y/x)$; with $\tau_x = \mathfrak{J}$ (the identity on F) or ϕ^{-1} according as $x > 0$ or $x < 0$, and σ a single-valued function from F onto K.

Let us verify that σ is additive (cf. Figure 1). Given $v, w \in F$, denote by V, W, Z, P, U_∞, the respective points $(1, v)$, $(1, w)$, $(1, z) = (1, v + w)$, $(0, w)$, $\{y = v \circ x\} \cap l_\infty$, of $M_\phi(F)$; and by $V', W', Z', P_\infty', U'$, their respective images under ν in $M_\psi(K)$. Because $z = v + w$, lines PZ and OV are "parallel." (That

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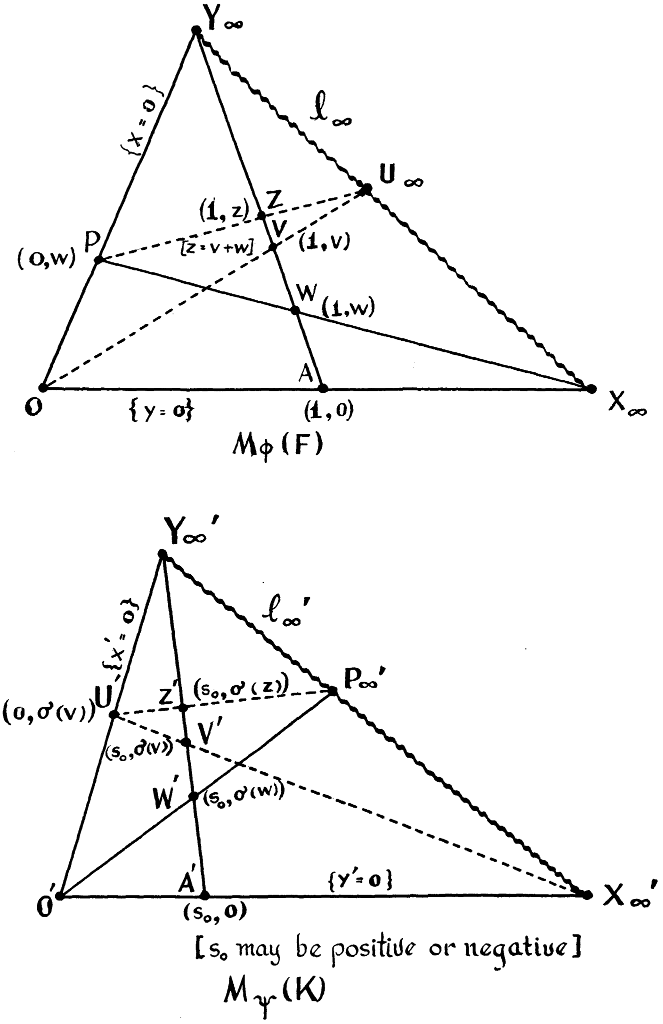

FIGURE 1

is, P, Z, U_∞ are collinear.) Denoting $\mu(1)$ by s_0, the points V', W', Z' are given by $(s_0, \sigma(v))$, $(s_0, \sigma(w))$, $(s_0, \sigma(z))$ respectively. Using $\nu[(0, 0)] = X_\infty'$, $\nu(X_\infty) = O'$, the collinearity of X_∞', V', U' follows, also that of P_∞', W', O'; we conclude the collinearity of U', Z', P_∞' (whence $O'W'$ is parallel to $U'Z'$), and $\sigma(z) = \sigma(v) + \sigma(w)$. Since $z = v + w$, $\sigma(v + w) = \sigma(v) + \sigma(w)$, and σ is additive.

Calling $\{y' = b' + r' \circ_{(\psi)} x'\}$ the image under ν of a "non-vertical" line $\{y = b + r \circ_{(\phi)} x\}$, we see that b' is a function of r alone; and r' a function only of b. For any $r \in F$, ν carries $\{y = r \circ_{(\phi)} x\}$ onto $\{y' = b'\}$. Hence, putting

$y' = \sigma\tau_x(y/x)$, $x = 1$, $y = r$, and $\tau_x = \mathfrak{J}$ gives $\sigma(r) = b'$. For any $b \in F$, ν carries $\{y = b\}$ onto $\{y' = r' \circ_{(\psi)} x'\}$. Hence, putting $y' = \sigma\tau_x(y/x)$, $x = 1$, $y = b$, and $\tau_x = \mathfrak{J}$ in $y' = r' \circ_{(\psi)} x'$ gives $r' \circ_{(\psi)} \mu(1) = r' \circ_{(\psi)} s_0 = \sigma(b)$. It follows that $r' = \lambda[(\sigma(b))/s_0]$, with $\lambda = \mathfrak{J}$ (the identity on K) or ψ^{-1} according as $s_0 > 0$ or $s_0 < 0$. Thus necessary relations of b' to r and of b to r' are known in terms of σ. Since ν maps $\{y = b\}$ onto $\{y' = r' \circ x'\}$, we conclude that $r' \circ_{(\psi)} \mu(x) = \sigma\tau_x(b/x)$, for all $x \in F$. From this equality, $\mu(x)$ is obtained explicitly by putting $r' = 1$ and letting b_1 be the value of b corresponding to $r' = 1$: $x' = \mu(x) = \sigma\tau_x(b_1/x)$, for $x \neq 0, \infty$. Note that $s_0 = \mu(1) = \sigma(b_1)$. Substitution for y', r', and x' in $y' = r' \circ x'$ shows the necessity of the condition

$$[\lambda\{(\sigma(b))/s_0\}] \circ_{(\psi)} [\sigma\tau_x(b_1/x)] = \sigma\tau_x(b/x),$$

for all b and all non-zero $x \in F$. Putting $1/x = u$, we get

$$(*) \quad [\lambda\{(\sigma(b))/s_0\}] \circ_{(\psi)} [\sigma\tau_u(b_1 u)] = \sigma\tau_u(bu), \qquad \text{for all } b \text{ and for all } u \in F$$

(even for $u = 0$, since $\sigma(0) = 0$).

Assume, conversely, that ϕ is additive; and let σ denote a one-to-one additive function from F onto K, with $b_1 \neq 0$ a fixed element of F and $s_0 = \sigma(b_1) \in K$, such that (*) holds, τ_u and λ being defined as above. We shall construct a one-to-one map ν of $M_\phi(F)$ onto $M_\psi(K)$ (carrying O, X_∞, Y_∞ onto the respective points X_∞', O', Y_∞'], and check that ν is a collineation.

For $x \neq 0, \infty$, define $x' = \mu(x) = \sigma\tau_x(b_1/x)$, $y' = \sigma\tau_x(y/x)$, and let ν map (x, y) onto (x', y'). (Note that $\mu(1) = \sigma(b_1) = s_0$.) Given $b \in F$, define $\nu[(0, b)] = P_\infty(r')$, the ideal point of slope r' in $M_\psi(K)$, where $r' = \lambda[(\sigma(b))/s_0]$. Given $r \in F$, let $P_\infty(r)$ be the ideal point of slope r in $M_\phi(F)$ and define $\nu[P_\infty(r)] = (0, \sigma(r))$. We note first that $y' = r' \circ x'$ if and only if $y = b$, r' having the required value $\lambda[(\sigma(b))/s_0]$: in fact, $y' = r' \circ x'$ amounts to

$$\sigma\tau_x(y/x) = \{\lambda[(\sigma(b))/s_0]\} \circ_{(\psi)} \{\sigma\tau_x(b_1/x)\},$$

for all non-zero $x \in F$, whence the identity (*) gives $y = b$ for all $x \neq 0$; $\nu[(0, b)] = P_\infty(r')$ and $\nu(X_\infty) = \nu[P_\infty(0)] = (0, 0)$, by definition of ν. Finally, we substitute $y' = \sigma\tau_x(y/x)$, $b' = \sigma(r)$, and $r' \circ x' = \sigma\tau_x(b/x)$ in $y' = b' + r' \circ x'$, for $x'[\neq 0, \infty] \in K$. Using the additivity of σ and ϕ^{-1} we obtain:

$$\sigma(y/x) = \sigma[r + (b/x)], \qquad \text{if } x > 0 \text{ in } F;$$

$$\sigma\phi^{-1}(y/x) = \sigma\phi^{-1}[\phi(r) + (b/x)], \qquad \text{if } x < 0 \text{ in } F.$$

Thus, for all non-zero $x \in F$, $y = b + r \circ_{(\phi)} x$. Since $\nu[(0, b)] = P_\infty(r')$ for all $b \in F$, and $\nu[P_\infty(r)] = (0, b') = (0, \sigma(r))$ for all $r \in F$, ν is a collineation.

We have now proved a necessary and sufficient condition for the existence of the collineation ν in terms of σ, $b_1 \in F$, and $s_0 = \sigma(b_1)$. It remains to show that this condition is equivalent to the one stated in the Theorem.

Assume that σ is a one-to-one additive function from F onto K, with $b_1 \neq 0$ a fixed element of F and $s_0 = \sigma(b_1) \in K$, such that (*) holds, τ_u being $\mathfrak{J}$ (on F)

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or ϕ^{-1} according as $u > 0$ or $u < 0$, and λ being $\mathfrak{J}$ (on K) or ψ^{-1} as $s_0 > 0$ or $s_0 < 0$. Define $\alpha(t) = (1/s_0)\cdot\sigma(b_1 t)$, for all $t \in F$. The identity (*), with $[\sigma(b)]/s_0$ replaced by $\alpha(b/b_1)$, $\sigma\tau_u(b_1 u)$ by $s_0\alpha[\tau_u(b_1 u)/b_1]$, and $\sigma\tau_u(bu)$ by $s_0\alpha[\tau_u(bu)/b_1]$, becomes the formula (†). Clearly, $\alpha(0) = 0$ and $\alpha(1) = 1$.

The proof that α is a sign-preserving isomorphism is nearly the same as that for α in (**11**, Theorem 1). It will be sketched here—detailing only the steps which involve changes.

To conclude that α is an isomorphism, it will be enough to know that α is multiplicative. Let S be the set of all positive x for which $\alpha(x) > 0$. If $s \in S$, then $\tau_s = \mathfrak{J}$, and the basic identity (*) becomes

$$\alpha(b/b_1)\cdot\alpha(s) = \alpha[(b/b_1)\cdot s] \qquad \text{for all } b \in F,$$

regardless of the sign of s_0. Thus $\alpha(xs) = \alpha(x)\cdot\alpha(s)$ for any $s \in S$ and any $x \in F$. As in (**11**), S forms a multiplicative subgroup of P, and we may assume the existence of $q_0 > 0$ in F with $\alpha(q_0) < 0$. As before, $P - S$ forms a single coset $(q_0 S)$: in fact, given any positive q_1 for which $\alpha(q_1) < 0$, the basic identity gives $\alpha(q_0^{-1} q_1) = \psi\alpha(q_0^{-1})\cdot\alpha(q_1)$ or $\psi^{-1}\alpha(q_0^{-1})\cdot\alpha(q_1)$ as $s_0 > 0$ or $s_0 < 0$; in either case $q_0^{-1} q_1 \in S$ since $\alpha(q_0^{-1})$ and $\alpha(q_1)$ are both negative. The rest of the proof that α is multiplicative and the proof that α preserves signs proceed exactly as in the quoted theorem.

Assume, finally, that α is *known* to be a sign-preserving isomorphism of F onto K, with $b_1 \in F$ and $s_0 \in K$ non-zero constants, such that (†) holds, λ and τ_u being as above. Define $\sigma(x) = s_0\cdot\alpha(x/b_1)$ for all $x \in F$. Clearly, σ is one-to-one from F onto K, and $\sigma(b_1) = s_0$. Replacing α by the corresponding expression in σ transforms the assumed condition to

$$[\lambda\{(\sigma(b))/s_0\}] \circ_{(\psi)} [\sigma\tau_u(b_1 u)] = \sigma\tau_u(bu),$$

for all b, $u \in F$. This condition is necessary and sufficient for the existence of ν, hence also for the required collineation $\gamma = \nu\delta$.

Proof of 1_b. The explicit determination of ν in $\gamma = \nu\delta$ is obtained at once if we rewrite the known formulae in terms of α rather than of σ.

COROLLARY 1. *The group C of collineations fixing Y_∞ in a plane $M_\phi(F)$ is either (i) the affine group A or (ii) a group consisting of the affine collineations and one coset of non-affine collineations which interchange $\{x = 0\}$ with l_∞.*

The group C is larger than A if and only if ϕ is additive and there is a sign-preserving automorphism α on F, together with non-zero constants b_1, $s_0 \in F$ such that

$$[\lambda\alpha(b/b_1)] \circ_{(\phi)} \{s_0\alpha[\tau_u(b_1 u)/b_1]\} = s_0\alpha[\tau_u(bu)/b_1],$$

for all $b, u \in F$, τ_u being $\mathfrak{J}$ (on F) or ϕ^{-1} according as $u > 0$ or $u < 0$, and λ being $\mathfrak{J}$ (on F) or ϕ^{-1} as $s_0 > 0$ or $s_0 < 0$.

Proof. The group structure of C follows at once from Lemma 3.

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The condition for the existence of non-affine collineations that fix Y_∞ is obtained by specializing $\phi = \psi$ and $F = K$ in Theorem 1.

Remark. The explicit collineations fixing Y_∞ are easily found by specializing $M_\phi(F) = M_\psi(K)$ in **(11**, Theorem 1**)** and in the above theorem.

COROLLARY 2. *If there is a non-affine collineation from $M_\phi(F)$ onto $M_\psi(K)$ sending Y_∞ to Y_∞', then $M_\phi(F)$ is (Y_∞, Y_∞)-transitive and $M_\psi(K)$ is (Y_∞', Y_∞')-transitive.*

Proof. Since ϕ and ψ are additive, the corollary follows from **(10,** Theorem 5**)**.

EXAMPLES. (i) *The collineation σ_0 on $M_\phi(F_9)$* (Theorem A of this paper) *involved*

$$(x, y) \rightarrow (x', y') = (\phi\tau(q_0/x), \phi\tau(y/x)),$$

$x \neq 0, \infty$; $\tau = \mathfrak{J}$ or ϕ^{-1} as $x > 0$ or $x < 0$; with an appropriate interchange of l_∞ and $\{x = 0\}$.

(ii) *On an arbitrary Moulton plane $M_\phi(F)$, the formulae*

$$x' = \phi\tau(q_0/x), \qquad y' = \phi\tau(y/x), \qquad \textit{for } x \neq 0$$

and q_0 an arbitrary negative (positive) constant, determine a collineation if and only if ϕ is an automorphism (an automorphism of order 2).

Proof. Substitution for x', y' in $y' = b' + r' \circ x'$ gives:
for $q_0 > 0$: $\quad y = b'x + q_0 \cdot \phi(r')$ or

$$y = x \cdot [\phi^{-1}\{b' + r' \cdot \phi(q_0/x)\}] \text{ as } x < 0 \text{ or } x > 0;$$

and for $q_0 < 0$: $\quad y = b'x + q_0 r'$ or

$$y = x \cdot [\phi^{-1}\{b' + \phi(r') \cdot \phi(q_0/x)\} \text{ as } x < 0 \text{ or } x > 0.$$

Thus, a collineation is determined if and only if:

$$\phi\{\phi^{-1}(b') + [\phi(r') \cdot q_0/x]\} = b' + r' \cdot \phi(q_0/x) \qquad \text{if } q_0 > 0,$$
$$\phi\{\phi^{-1}(b') + [r' q_0/x]\} = b' + \phi(r') \cdot \phi(q_0/x) \qquad \text{if } q_0 < 0,$$

for all b', r', and $x > 0 \in F$.

If $q_0 > 0$, the values $b' = 0$, $x = q_0$ give the necessity of $\phi = \phi^{-1}$. In either case, $b' = 0$ gives $\phi(uv) = \phi(u) \cdot \phi(v)$ for all $u \in F$ and v such that $vq_0 \geqslant 0$. The additivity of ϕ follows at once.

To show that ϕ is an automorphism, we shall check that $\phi(u_1 u_2) = \phi(u_1)\phi(u_2)$ for $u_1 q_0 < 0$ and $u_2 q_0 < 0$. Choose $n_0 < 0$ such that $n_0 + 1 \geqslant 0$. (Unless this is possible, $1 + n < 0$ for all $n < 0$; while $1 + (1/n) = (n + 1)/n > 0$ for all $n < 0$; a contradiction because $1/n$ ranges over the negative as n does.) Using

$$\phi\{n_0 \cdot (n_0 + 1)\} = \phi(n_0) \cdot \phi(n_0 + 1)$$

and the additivity of ϕ, we have

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$$\phi(n_0^2) + \phi(n_0) = \phi(n_0^2 + n_0) = \phi\{n_0 \cdot (n_0 + 1)\} = \phi(n_0) \cdot \phi(n_0 + 1)$$
$$= \phi(n_0) \cdot [\phi(n_0) + 1] = [\phi(n_0)]^2 + \phi(n_0).$$

Subtraction of $\phi(n_0)$ from the extremes proves that $\phi(n_0^2) = [\phi(n_0)]^2$. With $u_1 = n_0 v_1$ and $u_2 = n_0 v_2$ (so that $v_1 q_0 > 0$ and $v_2 q_0 > 0$), the required conclusion is obtained:

$$\phi(u_1 u_2) = \phi(n_0^2 v_1 v_2) = \phi(n_0^2) \cdot \phi(v_1) \cdot \phi(v_2) = [\phi(n_0)]^2 \cdot \phi(v_1) \cdot \phi(v_2)$$
$$= [\phi(n_0) \cdot \phi(v_1)] \cdot [\phi(n_0) \cdot \phi(v_2)] = \phi(n_0 v_1) \cdot \phi(n_0 v_2) = \phi(u_1) \cdot \phi(u_2).$$

Remark. The remaining case for a collineation from $M_\phi(F)$ to $M_\psi(K)$ is that in which Y_∞ does *not* map onto Y_∞'. Such a situation can arise. In fact, J. C. D. Spencer (**12**, Theorem 8) has proved the existence of collineations on $M_\phi(F)$ displacing Y_∞ if $M_\phi(F)$ is the generalized Moulton plane of Pickert (**9**, p. 93), i.e., if F is an ordered field and if $\phi(m) = m$ for $m \geqslant 0$ in F, $\phi(m) = q_0 m$ for $m < 0$ in F, q_0 $(\neq 1)$ being a positive constant in F.

The theorems in this section describe the most general planes $M_\phi(F)$ and $M_\psi(K)$ that admit a collineation from $M_\phi(F)$ onto $M_\psi(K)$ failing to map Y_∞ onto Y_∞'. The totality of collineations from $M_\phi(F)$ onto $M_\psi(K)$ will be determined for this case. Perhaps the most striking new result is an extended "converse" of Spencer's theorem—viz., there cannot exist a collineation from $M_\phi(F)$ onto $M_\psi(K)$ that fails to map Y_∞ onto Y_∞' unless F and K are ordered (under the given pseudo-orders) and $M_\phi(F)$, $M_\psi(K)$ are isomorphic to a generalized Pickert-Moulton plane.

THEOREM 2. *Let F be an ordered field with a, h arbitrary elements and p, q arbitrary positives in F. For $x \in F$, define*

$$\phi(x) = \begin{cases} p(x-h) + a & \text{if} \quad x - h \geqslant 0, \\ pq(x-h) + a & \quad x - h < 0. \end{cases}$$

Then ϕ preserves the ordering of F; $x \to \phi(x) - n_0 x$ defines a map of F onto itself for each $n_0 < 0 \in F$; and ϕ determines a Moulton plane $M_\phi(F)$, which is Desarguesian if and only if $q = 1$. No generality is sacrificed by assuming that $\phi(0) = 0$ and $\phi(1) = 1$.

Proof. The function ϕ is one-to-one from F onto itself, since ϕ is the resultant of the maps $x \to t = x - h$, $t \to w = pt$ or $w = pqt$ according as $t \geqslant 0$ or $t < 0$, $w \to w + a$. (The "one-to-one onto" property for $t \to w$ uses the fact that p and pq are both positive.) The function $x \to \phi(x) - n_0 x$, for arbitrary $n_0 < 0 \in F$, is likewise the resultant of $x \to t = x - h$, $t \to w = (p - n_0)t$ or $w = (pq - n_0)t$ according as $t \geqslant 0$ or $t < 0$, $w \to w + a - n_0 h$. (The "one-to-one onto" property for $t \to w$ uses the fact that $p - n_0 > 0$ and $pq - n_0 > 0$.)

By (**10**, Theorem 1), a Moulton plane $M_\phi(F)$ will be determined if ϕ preserves the ordering of F. From the definition of ϕ, the latter will preserve the ordering if $[\phi(u) - \phi(v)]$ has the sign of $(u - v)$ for each choice of u and v satisfying $u - h \geqslant 0$, $v - h < 0$. We have, for such u, v,

$$\phi(u) - \phi(v) = p(u - h) - pq(v - h),$$

which is positive since $p(u - h)$ is non-negative and $-pq(v - h)$ is positive; $u - v = (u - h) - (v - h)$ is positive since $u - h$ is non-negative and $-(v - h)$ is positive.

The plane $M_\phi(F)$ is Desarguesian if and only if $q = 1$ (**10**, Theorem 4, interpreted for a function ϕ which may not be normalized; cf. **10**, Lemma 1).

By (**10**, Lemma 1), $M_\phi(F)$ is isomorphic to a plane $M_{\phi'}(F)$ with $\phi'(0) = 0$, $\phi'(1) = 1$. In fact, the transformation used to normalize ϕ in that lemma changes the given ϕ to a function ϕ' of the same form. Thus, no generality is lost if we assume that $\phi(0) = 0$, $\phi(1) = 1$.

Remark. The type of plane constructed in Theorem 2 may appear to generalize the Pickert-Moulton planes, reducing to the latter if $a = h = 0$. The next Theorem shows, however, that *all* planes of the type given in Theorem 2 are isomorphic to Pickert-Moulton planes.

The concept of Lenz-Barlotti *substructure* (**3**; **6**; **9**, pp. 70 and 93; **12**, pp. 253–255) will be useful in what follows. The substructure $S(\pi)$ of a projective plane π includes a point P of π if and only if there exists a line q through P for which π is (P, q)-transitive and includes a line l of π if and only if there exists a point Q for which π is (Q, l)-transitive. It is easily shown (**9**) that for every choice, in the plane, of a line l through a point P, the plane is (P, l)-transitive if and only if P and l both belong to $S(\pi)$. It is also easy to show that $S(\pi)$ contains all the lines of a pencil if it contains two distinct lines thereof; and, dually, $S(\pi)$ contains all points of a range if it contains two distinct points of it.

Any collineation of a projective plane π onto a plane π' maps $S(\pi)$ onto $S(\pi')$, substructure being intrinsic.

THEOREM 3. *Let $M_\phi(F)$ and $M_\psi(K)$ be (non-Desarguesian) Moulton planes; assume the existence of a collineation, from $M_\phi(F)$ onto $M_\psi(K)$, which fails to send Y_∞ onto Y_∞'.*

(*Part* 3_a). *Every non-affine collineation γ, of $M_\phi(F)$ onto $M_\psi(K)$, is given by $\gamma = \tau\alpha\beta$, where τ is a translation of $M_\phi(F)$, β is an affine collineation from $M_\phi(F)$ onto $M_\psi(K)$* (determined by (**11**, Theorem 1)), *and α is a collineation on $M_\phi(F)$ fixing* $(0, 0)$ *linewise and $\{y = h \circ x\}$ pointwise, for some $h \in F$; α is defined as follows:*

$$x' = 1/\{[\lambda((y + 1)/x)] - [(\lambda(h)) \circ x]/x\},$$
$$y' = [\lambda(y/x)]/\{[\lambda((y + 1)/x)] - [(\lambda(h)) \circ x]/x\};$$

provided $x \neq 0$ and $y + 1 \neq h \circ x$; with $\lambda = \mathfrak{J}$ (the identity on F), ϕ, or ϕ^{-1}, according as $x'x > 0$, $x > 0$ and $x' < 0$, or *$x < 0$ and $x' > 0$;*

$$\alpha[(0, y)] = (0, y/(y+1)), \qquad \textit{unless } y = -1;$$

$$\alpha[(x, -1 + h \circ x)] = \begin{cases} P_\infty[h - (1/x)] & \textit{if } x > 0, \\ P_\infty\{\phi^{-1}[\phi(h) - (1/x)]\} & \textit{if } x < 0; \end{cases}$$

$$\alpha[(0, -1)] = Y_\infty; \qquad \alpha(Y_\infty) = (0, 1); \qquad \alpha[P_\infty(h)] = P_\infty(h)$$

and

$$\alpha[P_\infty(r)] = \begin{cases} (1/(r-h), r/(r-h)) & \textit{if } r - h > 0, \\ (1/[\phi(r) - \phi(h)], \phi(r)/[\phi(r) - \phi(h)]) & \textit{if } r - h < 0. \end{cases}$$

Equivalently, the general non-affine collineation of $M_\phi(F)$ onto $M_\psi(K)$ can be written $\beta'\alpha'\tau'$; with β' an affine collineation from $M_\phi(F)$ onto $M_\psi(K)$, τ' a translation of $M_\psi(K)$, and α' the collineation on $M_\psi(K)$ analogous to α.

(Part 3_b). The fields F and K are ordered (under their given pseudo-orders). Both $M_\phi(F)$ and $M_\psi(K)$ are planes of the type described in Theorem 2: for each $x \in F$,

$$\phi(x) = \begin{cases} p(x-h) + a & \textit{if } x - h \geqslant 0, \\ pq(x-h) + a & \textit{if } x - h < 0; \end{cases}$$

$p > 0$, $1 \neq q > 0$, a, h being constants in F. For each $x' \in K$,

$$\psi(x') = \begin{cases} p'(x'-h') + a' & \textit{if } x' - h' \geqslant 0, \\ p'q'(x'-h') + a' & \textit{if } x' - h' < 0; \end{cases}$$

$p' > 0$, $1 \neq q' > 0$, a', h' being constants $\in K$.

The planes $M_\phi(F)$ and $M_\psi(K)$ are isomorphic to a Pickert-Moulton plane $M_\eta(F)$, with $\eta(x) = x$ or qx, for all $x \in F$, according as $x \geqslant 0$ or $x < 0$ ($1 \neq q > 0$ being the same as in the formula, of the preceding paragraph, for ϕ).

There also exists a Pickert-Moulton plane over K, isomorphic to $M_\phi(F)$ and $M_\psi(K)$.

(Part 3_c). *The substructure $S[M_\phi(F)]$ consists of the points on the range $\{x = 0\}$ and the lines of an ideal pencil $[P_\infty(h)]$.*

The substructure $S[M_\psi(K)]$ consists of the points on $\{x' = 0\}$ and the lines of $[P_\infty(h')]$.

The substructure $S[M_\eta(F)]$ consists of the points on $\{x^* = 0\}$ and the lines of $[X_\infty^*]$.

Proof of 3_a. Since there is a collineation γ of $M_\phi(F)$ onto $M_\psi(K)$ for which $\gamma(Y_\infty) \neq Y_\infty'$, the plane $M_\phi(F)$ supports a collineation (for example, $\gamma\sigma\gamma^{-1}$ with σ a non-trivial translation on $M_\psi(K)$) displacing Y_∞.

By (**10**, Theorem 3), $S[M_\phi(F)]$ includes Y_∞ and l_∞; but $S[M_\phi(F)]$ contains no further ideal points (to prevent $M_\phi(F)$ from being a translation plane—hence Desarguesian (**10**, Theorem 4)). It follows that a collineation displacing Y_∞ must map Y_∞ onto an ordinary point. The only ordinary points which may belong to $S[M_\phi(F)]$ are those on $\{x = 0\}$. (Otherwise an elation $(x, y) \to (px, py)$ $(1 \neq p > 0)$ would provide distinct points of the substructure on a line not through Y_∞, and hence an ideal point $\neq Y_\infty$ on $S[M_\phi(F)]$.) Using an $M_\phi(F)$-collineation that displaces Y_∞, and applying translations, we see that $S[M_\phi(F)]$ does include all points of $\{x = 0\}$, and at least the lines of one ideal pencil.

Construction of α. The origin and $\{y = h \circ x\}$, for some $h \in F$, are collineation-images of Y_∞ and l_∞ respectively. Hence, there exists a collineation α fixing $(0, 0)$ linewise and $\{y = h \circ x\}$ pointwise. Using $(0, 0) - \{y = h \circ x\}$ transitivity on the points of the y-axis, we can suppose that $\alpha[(0, -1)] = Y_\infty$, and $\alpha[\{y = -1 + h \circ x\}] = l_\infty$. From the linewise invariance of $(0, 0)$, we determine α on $\{y = -1 + h \circ x\}$.

Assuming $(x, y) \notin \{y = -1 + h \circ x\}$, let α be given by $(x, y) \to (x', y')$. For $x \neq 0$, x' and y' may be calculated from the fact that

$$(x, y) = \{(x, y) \cup (0, 0)\} \cap \{(x, y) \cup (0, -1)\}$$

maps onto $(x', y') = \{(x, y) \cup (0, 0)\} \cap \{x' = z\}$, where z is the abscissa of $\{y = h \circ x\} \cap \{(x, y) \cup (0, -1)\}$. (This is the construction used by Spencer (**12**, p. 254) in her proof of Theorem 8.) Simultaneous solution of $\{t = h \circ s\}$ and $t = [\lambda((y + 1)/x)] \cdot s - 1$, with λ as in the Theorem, gives the formulae for x', y'. Since each of the functions $\mathfrak{J}$, ϕ, ϕ^{-1} is order-preserving, $\{[\lambda((y + 1)/x)] - [(\lambda(h)) \circ x]/x\}$ (when non-zero) has the same sign as $x/(y + 1 - h \circ x)$. Thus, given (x, y) restricted as above, the sign of $(y + 1 - h \circ x)$ determines the sign of x', and the appropriate formulae for x' and y'.

To obtain α on $\{x = 0\}$, let $\{y' = b' + r' \circ x'\}$ be the α-image of $\{y = b + r \circ x\}$ with $b \neq -1$, $r \neq h$. The sign of $(b + 1 + r \circ x - h \circ x)$ is positive for at least two distinct non-zero values $x = x_1, x_2$; and the corresponding points (x_i, y_i) $[i = 1, 2]$ on $\{y = b + r \circ x\}$ are related to their respective α-images (x_i', y_i') on $\{y' = b' + r' \circ x'\}$ by

$$x' = x/(y + 1 - h \circ x), y' = y/(y + 1 - h \circ x).$$

Substituting these for x', y' in $y' = b' + r' \circ x'$ gives

$$y = [b'/(1 - b')] + [r' \circ x - b'(h \circ x)]/(1 - b').$$

Thus $\alpha[(0, b'/(1 - b'))] = (0, b')$; which amounts to $\alpha[(0, y)] = (0, y/(y + 1))$ unless $y = -1$ or $y' = 1$. It follows that $\alpha(l_\infty) = \{y' = 1 + h \circ x'\}$; $\alpha(Y_\infty) = (0, 1)$; and the linewise invariance of $(0, 0)$ gives the expressions for $\alpha[P_\infty(r)]$.

The general collineation from $M_\phi(F)$ *onto* $M_\psi(K)$. If τ, α, β and τ', α', β' are as given in the Theorem (Part 3_a), it is immediate that $\tau\alpha\beta$ is a collineation $M_\phi(F)$ onto $M_\psi(K)$ and that $\beta'\alpha'\tau'$ is too.

Conversely, let γ be any non-affine collineation from $M_\phi(F)$ onto $M_\psi(K)$. Then $\gamma(Y_\infty) \neq Y_\infty'$. Otherwise, by Theorem 1, γ would map $\{x = 0\}$ onto l_∞', a contradiction since $S[M_\phi(F)]$ contains all points of $\{x = 0\}$, while

$S[M_\psi(K)]$ contains only one ideal point, Y_∞'. The order of F exceeds 9 since α displaces Y_∞.

If τ is a translation on $M_\phi(F)$ for which $\tau[\gamma^{-1}(Y_\infty')] = (0, -1)$, then $\tau\alpha$ maps $\gamma^{-1}(Y_\infty')$ onto Y_∞, so that $(\tau\alpha)^{-1}\cdot\gamma$ sends Y_∞ to Y_∞'. By Theorem 1, $(\tau\alpha)^{-1}\cdot\gamma$ maps $(\{x = 0\}, l_\infty)$ onto $(\{x' = 0\}, l_\infty')$, in fact, l_∞ onto l_∞' since the range $\{x = 0\}$ lies in the substructure, while l_∞' does not. Putting $(\tau\alpha)^{-1}\cdot\gamma = \beta$ gives $\gamma = \tau\alpha\beta$, as required. Similarly, γ can be written as $\beta'\alpha'\tau'$.

Proof of 3_b. Letting $(P_\infty(h))$ denote a fixed pencil in $S[M_\phi(F)]$, we define

$$\eta(x) = [\phi(h + x) - \phi(h)]/[\phi(h + 1) - \phi(h)], \qquad \text{for all } x \in F.$$

Clearly, η is one-to-one from F onto F, $\eta(0) = 0$, and $\eta(1) = 1$. Since

$$[\eta(u) - \eta(v)] = [\phi(h + u) - \phi(h + v)]/[\phi(h + 1) - \phi(h)],$$

which has the sign of $(h + u) - (h + v) = u - v$, the function η is order-preserving. The map $x \to \eta(x) - n_0 x$, for fixed $n_0 < 0$, and all $x \in F$, is "onto"; being the resultant of

$$x \to t = \phi(h + x) - (\phi(h + 1) - \phi(h))\cdot n_0\cdot (h + x),$$

itself "onto" since $(\phi(h + 1) - \phi(h))\cdot n_0 < 0$ by **(10**, Theorem 1**)**, followed by

$$t \to \{[t - \phi(h)]/[\phi(h + 1) - \phi(h)]\} + n_0 h.$$

Thus, η determines a Moulton plane $M_\eta(F)$, again by **(10**, Theorem 1**)**.

For any point (x, y) in $M_\phi(F)$, define $\beta[(x, y)] = (x^*, y^*)$, where $y^* = y - h \circ_{(\phi)} x$; and $x^* = x$ if $x \geqslant 0$, $x^* = [\phi(1 + h) - \phi(h)]\cdot x$ if $x < 0$. For any $r \in F$, define $\beta[P_\infty(r)] = P_\infty(r^*)$, where $r^* = r - h$. The map β is one-to-one from $M_\phi(F)$ onto $M_\eta(F)$. To establish the isomorphism between $M_\phi(F)$ and $M_\eta(F)$, we shall prove that β is a collineation. Clearly, the lines through Y_∞, in $M_\phi(F)$, are mapped onto the lines through Y_∞^* fo $M_\eta(F)$. Let $\{y^* = b^* + r^* \circ_{(\eta)} x^*\}$ denote any other line of $M_\eta(F)$. Substitution for x^* and y^* gives the β pre-image of this line:

$$y = b + (h + r^*)x \quad \text{if } x \geqslant 0, \qquad y = b + [\phi(h + r^*)]x \quad \text{if } x < 0.$$

Since $\beta[P_\infty(r)] = P_\infty(r - h)$, β is a collineation. Since $\beta[P_\infty(h)] = P_\infty(0) = X_\infty^*$ in $M_\eta(F)$, and since β carries the y-axis onto the y^*-axis, $S[M_\eta(F)]$ includes the range $\{x^* = 0\}$ and the pencil (X_∞^*). The construction for α, applied to $M_\eta(F)$, gives a collineation α^* on $M_\eta(F)$. The formulae for α^* are essentially those for α, with $h = 0$ and ϕ replaced by η.

To determine η, let $\{y^* = b^* + r^* \circ x^*\}$ be the image of $\{y = b + r \circ x\}$ under α^*, with $b \neq -1$, $b^* \neq 1$. Since the ideal point on slope r moves to $(1/r, 1)$ if $r > 0$ and to $(1/\eta(r), 1)$ if $r < 0$, r > 0 implies

$$1 = b^* + (r^*/r) = (b/(b + 1)) + (r^*/r),$$

which, solved for r, gives $r = (b + 1)r^*$; while $r < 0$ implies

$$1 = b^* + (\eta(r^*)/\eta(r)) = (b/(b + 1)) + (\eta(r^*)/\eta(r)),$$

whence $\eta(r) = (b+1)\cdot\eta(r^*)$. The pre-image of the ideal point on slope r^* is $(-1/r^*, -1)$ or $(-1/\eta(r^*), -1)$ according as $-r^* > 0$ or $-r^* < 0$; substitution of this pre-image in $y = b + r \circ x$ gives $r = (b+1)r^*$ or $\eta(r) = (b+1)\cdot\eta(r^*)$ according as $-r^* = (-r/(b+1)) > 0$ or $-r/(b+1) < 0$. Let n be an *arbitrary* negative element of F, and choose b so that $-(1+b) = n$. For such b, either $r \geqslant 0$ and $-r/(b+1) \leqslant 0$, or $r < 0$ and $-r/(b+1) > 0$. Hence the relations between r and r^* give $r = (b+1)\cdot r^*$ and $\eta(r) = (b+1)\cdot\eta(r^*)$ regardless of whether $r \geqslant 0$ or $r < 0$. It follows that

$$\eta(-nr^*) = \eta[(b+1)\cdot r^*] = (b+1)\cdot\eta(r^*) = -n\cdot\eta(r^*),$$

for arbitrary negative n, for appropriate b, and for all $r^* \in F$. With $r^* = 1$, we get $\eta(-n) = -n$ for every $n < 0$. Moreover, $-1 < 0$ to avoid a contradiction: If -1 were positive, $n < 0$ would imply $-n < 0$ and $\eta(x) = x$ for all $x < 0$. Since $-np < 0$ for $n < 0$ and $p > 0$, $-np = \eta(-np) = -n\cdot\eta(p)$, the latter obtained by putting $r^* = p$ in $\eta(-nr^*) = -n\cdot\eta(r^*)$; hence, $\eta = \mathfrak{J}$ on the positives.

From $\eta(-n) = -n$, for $n < 0$, and from $-1 < 0$, it follows that η fixes the positives elementwise. The value $r^* = -1$ in $\eta(-nr^*) = -n\cdot\eta(r^*)$ gives $\eta(n) = q\cdot n$ for all $n < 0$, where $q = -\eta(-1) \neq 1$.

To prove that F is ordered, let x_0 and y_0 be arbitrary positive elements of F. Using the fact that α^* maps $\{x = x_0\}$ onto $\{y^* = (-1/x_0)\circ x^* + 1\}$, recalculate the abscissa of $(x_0{}^*, y_0{}^*) = \alpha^*[(x_0, y_0)]$ as the intersection of

$$\{y^* = (-1/x_0)\circ x^* + 1\},$$

with $\{y^* = (y_0/x_0)\circ x^*\}$: $x_0{}^* = x_0/(y_0+1)$ or $x_0{}^* = x_0/(y_0+q)$ according as $x_0{}^* > 0$ or $x_0{}^* < 0$, i.e. as (y_0+1) is positive or negative. From the formulae for α^*, $x_0 > 0$ implies $x_0{}^* = x_0/(y_0+1)$ or $x_0{}^* = 1/\eta[(y_0+1)/x_0] = x_0/[q(y_0+1)]$ as $y_0 + 1 > 0$ or < 0. From $q \neq 1$, it follows that $y_0 + 1 > 0$. Since y_0 is any positive, $p_1 + p_2 = p_1\cdot[1 + (p_2/p_1)] > 0$ for arbitrary positives p_1 and p_2. This shows that F is ordered under its given pseudo-order.

The definition of η, with x replaced by $u - h$, now gives:

$$\phi(u) = \begin{cases} p(u-h) + a & \text{if } u - h \geqslant 0, \\ pq(u-h) + a & \phantom{\text{if }} u - h < 0. \end{cases}$$

Here $a = \phi(h)$ and $p = [\phi(h+1) - \phi(h)]$. The uniqueness of h will be proved in 3_c.

That $M_\psi(K)$ is isomorphic to $M_\eta(F)$ follows from the fact that $M_\psi(K)$ is isomorphic to $M_\phi(F)$. That ψ has the form given in Theorem 2 and that K is ordered follow at once, since an isomorphic Pickert-Moulton plane could be constructed, starting from the latter plane instead of from $M_\phi(F)$.

Proof of 3_c. We have shown that $S[M_\phi(F)]$ contains exactly the points of the range $\{x = 0\}$, and at least the lines of an ideal pencil $[P_\infty(h)]$, for some $h \in F$. According to the isomorphisms γ of $M_\phi(F)$ onto $M_\psi(K)$ and β of $M_\phi(F)$ onto $M_\eta(F)$, $S[M_\psi(K)]$ consists of the points on $\{x' = 0\}$ and at least

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the lines of $[P_\infty(h')]$, while $S[M_\eta(F)]$ consists of the points on $\{x = 0\}$ and at least the lines of $[X_\infty^*]$. To complete the proof, it will be enough to show that $S[M_\eta(F)]$ includes no lines except those of the pencil $[X_\infty^*]$.

Suppose a line not through X_∞^* belonged to $S[M_\eta(F)]$. Then all lines of $M_\eta(F)$ would belong to the substructure; in particular $M_\eta(F)$ would be (Y_∞^*, Y_∞^*)-transitive, and η additive (**11**, Theorem 5). That is impossible: in fact, for $n < 0$ and $u > 0 \in F$, $\phi(n) + \phi(u) = qn + u$ while $\phi(n + u)$ is equal to $n + u$ or $q(n + u)$, $q \neq 1$.

This completes the proof of Theorem 3.

Remark. That the form of α^* given in Theorem 3 is *sufficient* to define a collineation had already been proved by Spencer (**12**, pp. 254–255) for the case of Pickert-Moulton planes.

COROLLARY 3. *Assume that a (non-Desarguesian) Moulton plane $M_\phi(F)$ is isomorphic to $M_\psi(K)$ under a collineation γ for which $\gamma(Y_\infty) \neq Y_\infty'$. Then F is ordered (under its given pseudo-order);*

$$\phi(x) = \begin{cases} p(x-h)+a & \\ pq(x-h)+a & \end{cases} \quad \textit{if} \quad \begin{array}{l} x - h \geqslant 0, \\ x - h < 0, \end{array}$$

and $M_\phi(F)$ supports a collineation α given, for $y \neq -1 + h \circ x$, by $(x, y) \to (x', y')$, with

$$x' = \begin{cases} \dfrac{x}{y+1-h\circ x} \\[2ex] \dfrac{x}{pq(y+1-hx)}, \\[2ex] \dfrac{px}{y+1-ax} \end{cases} \quad y' = \begin{cases} \dfrac{y}{y+1-h\circ x} \\[2ex] \dfrac{pq(y-hx)+ax}{pq(y+1-hx)}, \\[2ex] \dfrac{y-ax+phx}{y+1-ax} \end{cases} \quad \textit{if} \quad \begin{array}{l} y+1-h\circ x > 0, \\[2ex] x \geqslant 0 \textit{ and} \\ y+1-hx < 0, \\[2ex] x \leqslant 0 \textit{ and} \\ y+1-ax < 0; \end{array}$$

$$\alpha[(x, -1 + h\circ x)] = \begin{cases} P_\infty[h - (1/x)] \\ P_\infty[h - (1/(px)] \end{cases} \quad \textit{if} \quad \begin{array}{l} x > 0, \\ x < 0; \end{array}$$

$$\alpha[P_\infty(r)] = \begin{cases} \left(\dfrac{1}{r-h}, \dfrac{r}{r-h}\right) \\[2ex] \left(\dfrac{1}{pq(r-h)}, \dfrac{pq(r-h)+a}{pq(r-h)}\right) \end{cases} \quad \textit{if} \quad \begin{array}{l} r - h > 0, \\[2ex] r - h < 0; \end{array}$$

$$\alpha[P_\infty(h)] = P_\infty(h); \qquad \alpha[(0, -1)] = Y_\infty; \qquad \textit{and}\ \alpha(Y_\infty) = (0, 1).$$

Proof. This is essentially Theorem 3_a, restated in terms of the formulae for ϕ. The expression for α on $\{x = 0\}$ has been incorporated into the formulae for x' and y'. The case $(y + 1 - h \circ x) > 0$ is immediate, since $\lambda = \mathfrak{J}$. Let $(y + 1 - h \circ x) < 0$. Then

$$\begin{array}{l} [(y+1)/x] - h < 0 \\ [(y+1)/x] - a > 0 \end{array} \quad \text{if} \quad \begin{array}{l} x > 0, \\ x < 0. \end{array}$$

The alternative $x > 0$ implies that

$$\phi[(y+1)/x] = pq\cdot\{[(y+1)/x] - h\} + a;$$

also $(y/x) - h < 0$, since F is ordered and since

$$[(y+1)/x] - h = [(y/x) - h] + (1/x) < 0 \qquad \text{with } 1/x > 0,$$

so that

$$\phi(y/x) = pq\{(y/x) - h\} + a.$$

If $x < 0$, then $[(y+1)/x] - a = [(y+1)/x] - \phi(h) > 0$. Hence $(y/x) - \phi(h) > 0$, since

$$[(y+1)/x] - \phi(h) = [(y/x) - \phi(h)] + (1/x) > 0 \qquad \text{with } (1/x) < 0.$$

Here we substitute

$$\phi^{-1}(u) = \begin{cases} h + [(u-a)/p] & \text{if } u - \phi(h) \geqslant 0, \\ h + [(u-a)/pq] & \phantom{\text{if }} u - \phi(h) < 0. \end{cases}$$

The equations for α on $\{y = -1 + h \circ x\}$ and on l_∞ are easily obtained.

The collineation problem for Moulton planes is now completely solved.

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