Canad. Math. Bull. Vol. **37** (3), 1994 pp. 306–309

NORMAL-PRESERVING LINEAR MAPPINGS

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ABSTRACT. Let H be a Hilbert space, $\dim H \geq 3$, and $\mathcal{B}(H)$ the algebra of all bounded linear operators on H. We characterize bijective linear mappings on $\mathcal{B}(H)$ that preserve normal operators.

The problem of characterizing linear mappings on matrix and operator algebras that leave invariant certain functions, subsets or relations has attracted the attention of many mathematicians in the last few decades [6]. The present work is motivated by the following two results of this kind. Kunicki and Hill [5] proved that if M_n is the algebra of all $n \times n$ complex matrices with $n \geq 3$, and if ϕ is a normal-preserving linear mapping on M_n, then either range of ϕ consists of normal operators, or there exists a unitary matrix U, a scalar c, and a linear functional f such that ϕ has one of the following forms:

(i) $\phi(A) = cU^*AU + f(A)I$ for all $A \in M_n$,

(ii) $\phi(A) = cU^*A^{\mathrm{tr}}U + f(A)I$ for all $A \in M_n$.

In the infinite-dimensional case we have the following result of Choi, Jafarian and Radjavi [3]: Let H be an infinite-dimensional Hilbert space and $\mathcal{B}(H)$ the algebra of all linear bounded operators on H. If ϕ is a bijective adjoint-preserving and normal-preserving linear mapping on $\mathcal{B}(H)$ then ϕ has one of the forms (i) or (ii), where $U \in \mathcal{B}(H)$ is a unitary operator, f is a linear functional on $\mathcal{B}(H)$, and A^{tr} denotes the transpose of A relative to any basis of H, fixed in advance.

The mapping $\phi: \mathcal{B}(H) \longrightarrow \mathcal{B}(H \oplus H)$ given by $\phi(A) = A \oplus A$ shows that in the infinite-dimensional case we have a normal-preserving linear mapping which is not of the form (i) or (ii). Moreover, its range does not consist entirely of normal operators, and therefore, the result of Kunicki and Hill can not be extended to the infinite-dimensional case.

It is the aim of this note to show that the assumption that ϕ is adjoint-preserving is superfluous in the result of Choi, Jafarian, and Radjavi, thus showing that the description of all bijective linear normal-preservers in the infinite-dimensional case is the same as in the finite-dimensional case.

Our approach is different from the one used in the finite-dimensional case [5]. It is based on the following result which was proved in [1].

THEOREM 1. *Let $\mathcal{A}$ and $\mathcal{A}'$ be centrally closed prime algebras over a field F, such that the characteristic of F is different from* 2 *and* 3. *Let $\phi: \mathcal{A} \longrightarrow \mathcal{A}'$ be a bijective linear mapping satisfying $[\phi(x^2), \phi(x)] = 0$ for all $x \in \mathcal{A}$. Here, $[u, v]$ denotes the commutator*

Supported by a grant from the Ministry of Science of Slovenia.
Received by the editors December 30, 1990.
AMS subject classification: 47B15, 47D30.

$uv - vu$. If neither $\mathcal{A}$ nor $\mathcal{A}'$ satisfies S_4, the standard polynomial identity of degree 4, then

$$\phi(x) = c\varphi(x) + p(x)$$

for all $x \in \mathcal{A}$, where $c \in F$, $c \neq 0$, φ is an isomorphism or an antiisomorphism of $\mathcal{A}$ onto $\mathcal{A}'$, and p is a linear mapping from $\mathcal{A}$ into the center of $\mathcal{A}'$.

Let H be a complex Hilbert space with $\dim H \geq 3$. It is easy to see that $\mathcal{B}(H)$ is a prime algebra, that is, $A\mathcal{B}(H)B = 0$, where $A, B \in \mathcal{B}(H)$, implies $A = 0$ or $B = 0$. Moreover, $\mathcal{B}(H)$ is centrally closed over the field of complex numbers [7]. By standard PI theory [4], a prime ring R satisfies S_4 if and only if R is commutative or R embeds in $M_2(K)$ for some field K. Thus, if $\dim H \geq 3$, then the algebra $\mathcal{A} = \mathcal{A}' = \mathcal{B}(H)$ satisfies all the assumptions of Theorem 1. Obviously, a linear mapping p from $\mathcal{B}(H)$ into the center of $\mathcal{B}(H)$ is of the form $p(A) = f(A)I$ for some linear functional f defined on $\mathcal{B}(H)$. Here, I denotes the identity operator on H. It is well-known [2] that every automorphism φ of $\mathcal{B}(H)$ is inner, that is, $\varphi(A) = V^{-1}AV, A \in \mathcal{B}(H)$, for some invertible operator $V \in \mathcal{B}(H)$. It follows that every antiautomorphism φ of $\mathcal{B}(H)$ is of the form $\varphi(A) = V^{-1}A^{\mathrm{tr}}V$, where $V \in \mathcal{B}(H)$ is invertible and A^{tr} denotes the transpose of V relative to a fixed but arbitrary orthonormal basis.

Now we are ready to prove our result.

THEOREM 2. *Let H be a Hilbert space such that $\dim H \geq 3$, and let $\phi\colon \mathcal{B}(H) \longrightarrow \mathcal{B}(H)$ be a bijective linear mapping. Assume that $\phi(N)$ is a normal operator whenever $N \in \mathcal{B}(H)$ is normal. Then there exist a unitary operator $U \in \mathcal{B}(H)$, a linear functional f on $\mathcal{B}(H)$, and a scalar c such that ϕ has one of the forms*

(i) $\phi(A) = cU^*AU + f(A)I$ for all $A \in \mathcal{B}(H)$,

(ii) $\phi(A) = cU^*A^{\mathrm{tr}}U + f(A)I$ for all $A \in \mathcal{B}(H)$. Here, A^{tr} denotes the transpose of A relative to a fixed basis.

PROOF. Pick an arbitrary Hermitian operator $S \in \mathcal{B}(H)$. Then $S^2 + \lambda S$ is normal for every complex number λ. Consequently, $\phi(S^2) + \lambda\phi(S)$ is normal which further implies $[\phi(S^2), \phi(S)^*] = 0$. By the assumption, $\phi(S)$ is normal. This yields together with Fuglede's theorem [8, Corollary 1.18] that

$$[\phi(S^2), \phi(S)] = 0 \tag{1}$$

for all Hermitian operators $S \in \mathcal{B}(H)$.

Let S and T be arbitrary Hermitian operators from $\mathcal{B}(H)$. Replacing S in (1) by $S + T$ we get

$$\big([\phi(ST+TS), \phi(S)] + [\phi(S^2), \phi(T)]\big) + \big([\phi(T^2), \phi(S)] + [\phi(ST+TS), \phi(T)]\big) = 0.$$

Putting $-T$ instead of T in the above equation we obtain

$$-\big([\phi(ST+TS), \phi(S)] + [\phi(S^2), \phi(T)]\big) + \big([\phi(T^2), \phi(S)] + [\phi(ST+TS), \phi(T)]\big) = 0.$$

Comparing these two relations we see that

$$[\phi(ST+TS),\phi(S)]+[\phi(S^2),\phi(T)]=0 \tag{2}$$

for all Hermitian $S, T \in \mathcal{B}(H)$.

Let us decompose an arbitrary operator A from $\mathcal{B}(H)$ as $A = S+iT$ where S and T are Hermitian operators. We have

$$[\phi(A^2),\phi(A)] = \\ i\big([\phi(S^2),\phi(T)]+[\phi(ST+TS),\phi(S)]\big)-\big([\phi(ST+TS),\phi(T)]+[\phi(T^2),\phi(S)]\big).$$

Applying (2) we get that

$$[\phi(A^2),\phi(A)]=0$$

is valid for every operator $A \in \mathcal{B}(H)$.

Using Theorem 1 we see that there exist an invertible operator $V \in \mathcal{B}(H)$, a nonzero complex number c, and a linear functional f on $\mathcal{B}(H)$ such that ϕ is either of the form

$$\phi(A)=cV^{-1}AV+f(A)I, \quad A\in\mathcal{B}(H), \tag{3}$$

or

$$\phi(A)=cV^{-1}A^{\mathrm{tr}}V+f(A)I, \quad A\in\mathcal{B}(H), \tag{4}$$

where the transposition is taken in any basis fixed in advance.

In order to complete the proof we have to show that V is a scalar multiple of a unitary operator. First, we shall fix some notation. For any $x, y \in H$ we shall denote the scalar product of these two vectors by y^*x, while xy^* will denote the rank one operator defined by $(xy^*)z = (y^*z)x$ for $z \in H$. Note that every operator of rank one can be written in this form. It is easy to see that a nonzero operator xy^* is normal if and only if y is a scalar multiple of x.

Let us first consider the case that ϕ is of the form (3). Then for every nonzero vector x the operator $V^{-1}xx^*V = (V^{-1}x)(V^*x)^*$ must be normal, or equivalently, for every nonzero vector x there exists a complex number λ_x such that $V^*x = \lambda_x V^{-1}x$. If x and y are linearly independent vectors from H then

$$\lambda_x V^{-1}x+\lambda_y V^{-1}y = V^*(x+y) = \lambda_{x+y}V^{-1}x+\lambda_{x+y}V^{-1}y$$

implies that $\lambda_x = \lambda_{x+y} = \lambda_y$. So, λ_x does not depend on the choice of x, and consequently, there exists a positive real number λ such that $V^*V = VV^* = \lambda I$. It follows that $\lambda^{-1/2}V$ is a unitary operator.

In the case that ϕ is of the form (4) we consider the mapping $\varphi(A) = \phi(A^{\mathrm{tr}})$ which preserves normality. Using the same approach as above we prove that also in this case V is a scalar multiple of a unitary operator.

REMARKS. Note that whatever discontinuity the mapping ϕ may have is inherited by the linear functional f, and the "essential" part of ϕ, that is, $cU^*(\cdot)U$ or $cU^*(\cdot)^{\mathrm{tr}}U$,

https://doi.org/10.4153/CMB-1994-046-1 Published online by Cambridge University Press

is automatically continuous. It is worth observing that besides some computations we needed only two nontrivial statements for characterizing normal-preserving linear mappings on $\mathcal{B}(H)$: Theorem 1 and Fuglede's theorem, which holds true in every C^*-algebra. Thus, the same method can be applied in order to characterize normal-preserving linear mappings on a much larger class of C^*-algebras.

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https://doi.org/10.4153/CMB-1994-046-1 Published online by Cambridge University Press