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Proposition 1.1.2. The statement is false as written. For a counterexample, fix a prime p, take V to be of dimension p, and take G to be the semidirect product of the trace-zero torus $\mathbb{G}_m^{p-1} = \ker(\times\colon \mathbb{G}_m^p \to \mathbb{G}_m)$ by the group $\mathbb{Z}/p\mathbb{Z}$ acting via the standard permutation representation on the ambient torus $\mathbb{G}_m^p$. There is a unique ascending filtration $H^0, H^1, \ldots$ on the group $(\mathbb{Q}_p/\mathbb{Z}_p)^{p-1} \cong (\mu_{p^\infty})^{p-1} \subseteq \mathbb{G}_m^{p-1}$ characterized by the properties that $H^{n(p-1)} = (p^{-n}\mathbb{Z}_p/\mathbb{Z}_p)^{p-1}$ and the groups $H^{n(p-1)+i}/H^{n/(p-1)}$ for $i = 0, \ldots, p-1$ form a unipotent flag for the action of $\mathbb{Z}/p\mathbb{Z}$; put $G^r = H^n$ for $n = \max\{0, \lfloor -r \rfloor\}$.

For a variant of this counterexample, take G as before, but define G^r as follows: for $r > 0$ take $G^r = H_n$ for $n = \max\{0, \lfloor 1/r \rfloor\}$, and for $r \leqslant 0$ take $G^r = G$.

The error in the proof occurs in the second paragraph, where it is asserted that 'By applying (d) finitely many times ... we can split X as a direct sum of G-stable summands, each of which is G^{r+}-isotypical'. This fails in the given examples because the action of $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{Z}_p^{p-1}$ is unipotent mod p.

To correct the statement, we add two additional conditions. The first of these is:

(g) *For all nonnegative integers g, h and every G-stable subspace W of $(V^\vee)^{\otimes g} \otimes V^{\otimes h}$, if $G^{-\infty+}$ acts trivially on W and G acts on W via a finite abelian group, then G acts trivially on W itself.*

This eliminates the original counterexample: the action of G on the trace-zero subspace of $V^\vee \otimes V$ factors through $\mathbb{Z}/p\mathbb{Z}$ but is not trivial. If F contains all roots of unity (or in the context of Lemma 1.1.4, just a primitive pth root of unity), then the following condition is equivalent:

(g′) *For all nonnegative integers g, h and every one-dimensional G-stable subspace W of $(V^\vee)^{\otimes g} \otimes V^{\otimes h}$, if $G^{-\infty+}$ acts trivially on W and G acts trivially on some power of W, then G acts trivially on W itself.*

The second additional condition is:

(h) *For each $r \in \mathbb{R}$, there exists $s_0 \in (r, +\infty)$ with the following property: for any $s \in (r, s_0]$, if G^t/G^s is finite and abelian for all $t \in (r, s]$, then G^{r+}/G^s is finite.*

This eliminates the variant counterexample.

For later use (see the discussion of Theorem 2.3.17 below), we observe that without condition (g), it is already possible to show that the groups G^r are finite for all $r \in \mathbb{R}$. To wit, let S be

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the set of $r \in \mathbb{R}$ for which G^r is finite. By (a), the set S is up-closed. By (b), the set S does not contain its infimum. If this infimum is finite, we obtain a contradiction by combining Lemma 1.1.1 with (h); it follows that G^r is finite for all $r \in \mathbb{R}$.

We now prove the modified version of Proposition 1.1.2.

Proof of (modified) Proposition 1.1.2. There is no harm in assuming at once that F is algebraically closed. As noted above, the current hypotheses suffice to imply that G^r is finite for all $r \in \mathbb{R}$; it thus remains to derive a contradiction under the assumption that $G^{-\infty+}$ is infinite.

By Lemma 1.1.1, there exists $s_0 \in \mathbb{R}$ such that $G^{-\infty+}/G^{s_0}$ embeds into a product of finitely many copies of $\mathbb{Q}/\mathbb{Z}$. By decreasing s_0 suitably, we may ensure in addition that $G^{-\infty+}/G^{s_0}$ is nontrivial and has no nontrivial finite quotients.

We next verify that G centralizes $G^{-\infty+}$. For each $s \in \mathbb{R}$, the conjugation action of G on the finite group G^s must be trivial on the identity connected component of G, so this action factors through the component group $\pi_0(G)$ of G. For each prime p, let H_p be the union of the p-Sylow subgroups of G^s/G^{s_0} for $s \leqslant s_0$; it then suffices to check that G centralizes H_p provided that the latter is nonzero. Since H_p has no nontrivial finite quotients, it is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ for some positive integer n, and the action of $\pi_0(G)$ corresponds to a homomorphism $\rho\colon \pi_0(G) \to \mathrm{GL}_n(\mathbb{Z}_p)$, which we wish to show is trivial. By (d) plus Tannaka–Krein duality, the image of $\pi_0(G)$ in $\mathrm{GL}_n(\mathbb{F}_p)$ is unipotent; consequently, the image of ρ is a p-group. Suppose by way of contradiction that this image is nontrivial; then the kernel of ρ is contained in a normal subgroup G' of G such that $G/G' \cong \mathbb{Z}/p\mathbb{Z}$. By duality again, for some g, h, there exists a G-stable subspace W of $(V^\vee)^{\otimes g} \otimes V^{\otimes h}$ on which G' acts trivially but G acts (completely reducibly and) nontrivially. However, since $G^{-\infty+} \subseteq \ker(\rho) \subseteq G'$, this constitutes a violation of (g); this yields a contradiction and shows that indeed G centralizes $G^{-\infty+}$.

By duality yet again, we can choose g, h so that $(V^\vee)^{\otimes g} \otimes V^{\otimes h}$ contains a G-stable subspace X on which G^{s_0} acts trivially but G^s acts nontrivially for some $s < s_0$. By Maschke's theorem, X admits a $G^{-\infty+}$-isotypical decomposition, which is respected by G since G centralizes $G^{-\infty+}$; since $G^{-\infty+}$ is infinite, this decomposition admits a summand Y for which the image of the action of $G^{-\infty+}$ on Y is isomorphic to an infinite subgroup of $\mathbb{Q}/\mathbb{Z}$. Put $W = \wedge^{\dim(Y)} Y$; this space occurs as a G-invariant subspace of $(V^\vee)^{\otimes g} \otimes V^{\otimes h}$ for some g, h. However, the image of $G^{-\infty+}$ in $\mathrm{GL}(W)$ is again isomorphic to an infinite subgroup of $\mathbb{Q}/\mathbb{Z}$, which is a contradiction. This yields that $G^{-\infty+}$ is finite, as claimed. $\square$

Remark 1.1.3. The fiber functor ω should be assumed to be neutral. More seriously, additional conditions are needed in order to imply the added hypotheses (g) and (h) of Proposition 1.1.2.

(iv) For each $V \in \mathcal{C}$ with $r(V) = -\infty$, if $G(V)$ is finite abelian, then it is trivial. (This implies (g).)

(v) For each $r \in \mathbb{R}$, there exists $s_0 \in (r, +\infty)$ with the following property: for any $s \in (r, s_0]$ and any $V \in \mathcal{C}$ with $r(V) < s$, if $G^t(V)$ is finite and abelian for all $t \in (r, s]$, then $G^{r+}(V)$ is finite. (This implies (h).)

See the discussions of Theorems 2.3.17 and 3.8.16 below.

Corollary 2.2.7. In the statement of (b), $\log \omega - \log \rho + s$ should be $\log \omega - \log \rho + c$. More seriously, the application of [Ked10, Theorem 6.7.4] is not sufficient; see the erratum for [Ked10, Lemma 6.8.1] for an alternate approach that gives the value $\delta = s(\omega/s)^{a(n_1,n_2,m)}$ for some $a(n_1, n_2, m)$.

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Lemma 2.3.9. Given the inclusion $V_2' \to \bigoplus_{m=0}^{p-1}(V_1' \otimes X_m)$, it is asserted that 'since V_2' is indecomposable, we have $V_2' \subseteq V_1' \otimes X_m$ for some m'. We clarify this point here.

For each irreducible constituent Z of V_2', there must be an index m for which Z also occurs as an irreducible constituent of $V_1' \otimes X_m$. In fact, this index must be unique because V_1' is refined. (For $m' \neq m$, the intrinsic subsidiary radii of $(V_1' \otimes X_m)^\vee \otimes (V_1' \otimes X_{m'}) \cong ((V_1')^\vee \otimes V_1) \otimes X_{m'-m}$ are all equal to ω^p, so this differential module cannot contain a trivial constituent.)

For $m \in \{0, \ldots, p-1\}$, let Y_m be the kernel of $V_2' \to \bigoplus_{m' \neq m}(V_1' \otimes X_{m'})$. The map $\bigoplus_{m=0}^{p-1} Y_m \to V_2'$ is injective: if Z were an irreducible submodule of the kernel, then Z would project nontrivially to Y_m for at least one value of m, but the previous paragraph would then imply that this value is unique and this would contradict the manifest injectivity of $Y_m \to V_2'$. The map $\bigoplus_{m=0}^{p-1} Y_m \to V_2'$ is also surjective: if Z were a constituent of the cokernel, then by the previous paragraph Z would occur in $V_1' \otimes X_m$ for some m, but then $V_2'/Y_m \to \bigoplus_{m' \neq m}(V_1' \otimes X_{m'})$ would be an injective map for which Z occurs in the source but not the target.

We thus conclude that $V_2' = \bigoplus_{m=0}^{p-1} Y_m$. Since V_2' is indecomposable, there can only be one index m for which $Y_m \neq 0$. For that m, the map $V_2' \to V_1' \otimes X_m$ is injective.

Lemma 2.3.11. In the last paragraph of the proof, the V_i' need not be indecomposable, so Lemma 2.3.9 does not apply directly. However, we may decompose V_i' as a direct sum $\bigoplus_j V_{ij}'$ of indecomposable summands and then apply Lemma 2.3.9 to produce $m_{ij} \in \{0, \ldots, p-1\}$ such that $IR((V_0')^\vee \otimes V_{ij}' \otimes W_{m_{ij}}) > \omega^p$. For $j' \neq j$, we also have $IR((V_{ij}')^\vee \otimes V_{ij'}' \otimes W_{m_{ij'} - m_{ij}}) > \omega^p$; since V_i' is refined, this implies that $m_{ij'} = m_{ij}$. That is, the m_{ij} are all equal to a common value m_i, and we may take $V' = \bigoplus_{i=0}^r V_i' \otimes W_{m_i}$, as originally claimed.

Theorem 2.3.17. In light of the correction to Proposition 1.1.2, we no longer can apply it directly to yield this result. As explained in the discussion of Proposition 1.1.2 above, to prove that the groups $G^r(V)$ are finite, it is not necessary to check condition (iv) of Remark 1.1.3; since conditions (i)–(iii) are already confirmed in the original text, we need only supplement them by verifying condition (v). (Note that in light of Remark 2.3.18, the filtration of G must be truncated at some index $s' \in (s, 1)$ in order for condition (iii) to be satisfied.)

Before doing so, however, we should point out that the fiber functor ω as defined is not neutral, so Remark 1.1.3 does not apply directly (see above). Instead, one should extend scalars from E to a larger differential field over which every differential module over E becomes trivial (e.g., a Picard–Vessiot extension).

To establish condition (v) of Remark 1.1.3, we use the following lemma.

Lemma. *With notation as in Theorem 2.3.17, define $G^{r-} = \bigcup_{s<r} G^s$. For any $r < \omega$, if $G^s(V)$ is a finite abelian p-group for all $s < r$, then $G^{r-}(V)$ is finite.*

Proof. We will use throughout the following observation: for $X \in [V]$ and $s \in (0,1)$, the multiplicity of s among the intrinsic subsidiary radii of X is equal to the dimension of the space $\omega(X)^{G(V)^{s-}}/\omega(X)^{G(V)^s}$. In particular, for $s < r$ this quantity depends only on the isomorphism class of $\omega(X)$ as a representation of $G^{r-}(V)$.

Let H be the dual group of $G^{r-}(V)$; it is a finitely generated $\mathbb{Z}_p$-module. Choose an irreducible constituent V_i of V and let $\chi_1, \ldots, \chi_n \in H$ be the distinct characters occurring in $\omega(V_i)$; these are all $G(V)$-conjugate, and so occur to the same multiplicity d. For each nonnegative integer k, let W_k be the differential submodule of $(\wedge^d V_i)^{\otimes p^k}$ with the property that $\omega(W_k)$ is the union of the one-dimensional subspaces of $\omega((\wedge^d V)^{\otimes p^k})$ on which $G^{r-}(V)$ acts via the characters $\chi_1^{\otimes p^k d}$, $\ldots, \chi_n^{\otimes p^k d}$. By the previous paragraph, it will suffice to check that there exists a constant $c > 1$

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independent of k such that $IR(W_{k+1}) \geqslant \max\{r, cIR(W_k)\}$; there is nothing to check unless $IR(W_k) < r$.

Let T be the unique maximal torus inside the group $\mathrm{GL}(\omega(W_k))$ containing the image of $G^{r-}(V)$. The Weyl group of $\mathrm{GL}(\omega(W_k))$ is the symmetric group S_n. The highest-weight vectors are the orbits of S_n on the character group of T; these orbits may be identified with partitions $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0)$ of nonnegative integers into at most n parts. Order these partitions lexicographically, and write $|\lambda|$ for $\lambda_1 + \cdots + \lambda_n$. The irreducible representation of $\mathrm{GL}(\omega(W_k))$ with highest weight λ may be identified with $\omega(S^\lambda W_k)$, where S^λ is the *Schur functor* associated to λ; see [FH91, ch. 15] for a concrete discussion. We may decompose $\omega(S^\lambda W_k)$ into eigenspaces $X_{\lambda,\mu}$ indexed by partitions μ with $|\mu| = |\lambda|$, with the dimension $d_{\lambda,\mu}$ of $X_{\lambda,\mu}$ being given by the Weyl character formula; the only fact we need here is that $d_{\lambda,\mu} = 1$ if $\mu = \lambda$ and $d_{\lambda,\mu} = 0$ if $\mu > \lambda$. This implies the existence of an isomorphism

$$W_{k+1} \cong \prod_{|\lambda|=p} (S^\lambda W_k)^{\otimes e_\lambda}$$

of virtual representations of $G^{r-}(V)$ for some integers e_λ with $e_{(p,0,\dots,0)} = 1$.

Apply Corollary 2.1.6 to find a cyclic vector $\mathbf{v}$ for W_k and then write $D^n(\mathbf{v}) = a_0\mathbf{v} + \cdots + a_{n-1}D^{n-1}(\mathbf{v})$ with $a_0, \dots, a_{n-1} \in E$ as in Proposition 2.2.6. Let $\alpha_1, \dots, \alpha_n$ be the multiset of roots of the polynomial $Q(T) = T^n - a_{n-1}T^{n-1} - \cdots - a_0$ in some algebraic closure of E; we then have $-\log|\alpha_i| = \log IR(W_k) - \log\omega + \log\rho$ for $i = 1, \dots, n$. Using the quantitative form of Corollary 2.2.7 described above, we see that there exists $c > 1$ depending on n, p, r (but not k) such that for any partition λ with $|\lambda| = p$, the following two multisets coincide in their values not exceeding $\log\max\{cIR(W_k), r\} - \log\omega + \log\rho$. (In principle, the choice of c depends also on $IR(W_k)$, but since $\log IR(W_k) < r < \omega$ we may make the choice uniform in this parameter.)

- The multiset consisting of $\log s - \log\omega + \log\rho$ for s ranging over the intrinsic subsidiary radii of $S^\lambda W_k$.
- The multiset consisting of $-\log|\pi_1\alpha_1 + \cdots + \pi_n\alpha_n|$ taken with multiplicity $d_{\lambda,\mu}$, for each partition μ and each distinct permutation $(\pi_1, \dots, \pi_n)$ of $(\mu_1, \dots, \mu_n)$.

This formally implies an analogous assertion with $S^\lambda W_k$ replaced by $\prod_\lambda (S^\lambda W_k)^{\otimes e_\lambda}$, or equivalently by W_{k+1}, and $d_{\lambda,\mu}$ replaced by $\sum_\lambda e_\lambda d_{\lambda,\mu}$, or equivalently by 1 if $\mu = (p, 0, \dots, 0)$ and 0 otherwise. This implies at once that $IR(W_{k+1}) \geqslant \max\{cIR(W_k), r\}$, as desired. □

Using the lemma, we may apply Proposition 1.1.2 to see that $G^{r-}(V)$ is finite for $r < \omega^p$. It follows that for $r < \omega$, the set $(0, r]$ contains only finitely many distinct values which can occur as $IR(W)$ as W runs over $[V]$. Using Proposition 2.3.5 and Lemma 2.3.6, we inductively deduce the same statement for $r \leqslant \omega^{p^{-h}}$ for $h = 0, 1, \dots$. We thus obtain condition (v) of Remark 1.1.3 for general r, and applying Proposition 1.1.2 again finishes the proof.

Definition 3.4.2. The notation $\langle x \rangle$ is not defined; it is not the usage of Definition 3.1.4. Rather, for $x \in \mathbb{Q}_p$, one writes $\langle x \rangle$ for the smallest positive rational number a such that one of $x - a, x + a$ is in $\mathbb{Z}_p$.

Corollary 3.4.6. In the statement and the proof, 'p-adic non-Liouville number' should be 'p-adic Liouville number'.

Corollary 3.4.7. The proof given is incorrect: given the correction to Corollary 3.4.6, we cannot rule out the possibility that $a - b$ is a p-adic Liouville number. See [Ked10, Lemma 13.4.3] for a correct argument.

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Remark 3.4.14. We provide a more detailed derivation of part (a) of this statement in the case $0 \notin I$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a basis of $M_{1,J}$, and choose $\mathbf{e}'_1, \ldots, \mathbf{e}'_{n'} \in M_J$ lifting a basis of $M_{2,J}$. By [Ked10, Lemma 13.5.4], there exists $k > 0$ such that for each $\zeta \in K^{\mathrm{alg}}$ with $\zeta^{p^m} = 1$, the matrix of action of ζ^* on $\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{e}'_1, \ldots, \mathbf{e}'_{n'}$ has operator norm at most p^{km}. Of course this remains true after enlarging k, which we will do several times hereafter.

By hypothesis, after enlarging k, for each m there exist elements $\mathbf{v}_{m,A_1,j} \in M_{1,J}$ such that $\zeta^*(\mathbf{v}_{m,A_1,j}) = \zeta^{a_{1,j}}\mathbf{v}_{m,A_1,j}$ for all $\zeta \in K^{\mathrm{alg}}$ with $\zeta^{p^m} = 1$, and the matrix S_{m,A_1} defined by $\mathbf{v}_{m,A_1,j} = \sum_i (S_{m,A_1})_{ij}\mathbf{e}_i$ is invertible and satisfies $|S_{m,A_1}|_J, |S^{-1}_{m,A_1}|_J \leqslant p^{km}$. After enlarging k again, there also exist elements $\mathbf{v}'_{m,A_2,j} \in M_J$ in the span of $\mathbf{e}'_1, \ldots, \mathbf{e}'_{n'}$ such that $\zeta^*(\mathbf{v}'_{m,A_2,j}) \equiv \zeta^{a_{2,j}}\mathbf{v}_{m,A_2,j} \pmod{M_{1,J}}$ for all $\zeta \in K^{\mathrm{alg}}$ with $\zeta^{p^m} = 1$, and the matrix S_{m,A_2} defined by $\mathbf{v}'_{m,A_2,j} = \sum_i (S_{m,A_2})_{ij}\mathbf{e}'_i$ is invertible and satisfies $|S_{m,A_2}|_J, |S^{-1}_{m,A_2}|_J \leqslant p^{km}$. By replacing $\mathbf{v}'_{m,A_2,j}$ with

$$\mathbf{v}''_{m,A_2,j} = \frac{1}{p^m} \sum_{\zeta^{p^m}=1} \zeta^{-a_{2,j}}\zeta^*(\mathbf{v}'_{m,A_2,j}),$$

we ensure that $\zeta^*(\mathbf{v}''_{m,A_2,j}) = \zeta^{a_{2,j}}\mathbf{v}''_{m,A_2,j}$. The change-of-basis matrix $S_{m,A}$ from $\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{e}'_1, \ldots, \mathbf{e}'_{n'}$ to $\mathbf{v}_{m,A_1,1}, \ldots, \mathbf{v}_{m,A_1,n}, \mathbf{v}''_{m,A_2,1}, \ldots, \mathbf{v}''_{m,A_2,n'}$ is an upper-triangular block matrix with diagonal blocks S_{m,A_1}, S_{m,A_2}; by comparing to the original basis, we see that the off-diagonal block of $S_{m,A}$ is bounded in norm by p^{4km}. We then have $|S_{m,A}|_J, |S^{-1}_{m,A}|_J \leqslant p^{6km}$.

THEOREM 3.4.20. The given argument alone only implies that the $S_{m_0,A}U$ is the change-of-basis matrix to a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of $M_{[\alpha',\beta']}$ which is fixed by parallel transport by all p-power roots of unity. To deduce that the matrix of action N of D on this basis has entries in K, note that for any $\lambda \in K^\times$, the operations λ^* and D commute; consequently, for ζ a p-power root of unity,

$$\sum_i N_{ij}\mathbf{e}_i = D(\mathbf{e}_i) = (D \circ \zeta^*)(\mathbf{e}_i) = (\zeta^* \circ D)(\mathbf{e}_i) = \sum_i \zeta^*(N_{ij})\mathbf{e}_i.$$

This implies that each N_{ij} belongs to K. To see that N is nilpotent, let λ be any eigenvalue; by [Ked10, Example 9.5.2], we must have $\lambda \in \mathbb{Z}_p$. We can then find a nonzero eigenvector $\mathbf{v}$ of N in the K-span of $\mathbf{e}_1, \ldots, \mathbf{e}_n$, which is fixed by ζ^*; however, by computing ζ^* in terms of N we obtain $\zeta^*(\mathbf{v}) = \zeta^\lambda \mathbf{v}$. This forces $\lambda = 0$, so N is nilpotent as claimed. (Compare the errata for [Ked10, Theorem 13.6.1].)

THEOREM 3.4.22. The given argument implies that the Π_m converge to a projector Π stable under parallel transport by all p-power roots of unity. However, it is not immediately apparent either that the limiting projector is horizontal, or that the summands have the correct exponents.

To remedy this, it is convenient to work in a somewhat larger category. As in the given argument, we assume that $0 \notin I$. Let μ_{p^∞} be the group of p-power roots of unity in an algebraic closure of K. Let $\mathcal{C}$ be the category in which an object is a finite projective R_I-module M together with a semilinear action of μ_{p^∞} on $M \otimes_K K(\mu_{p^\infty})$ satisfying the following conditions.

- The action of $\zeta \in \mu_{p^\infty}$ is semilinear with respect to the substitution $t \mapsto \zeta t$ on R_I.
- The action of μ_{p^∞} is equivariant with respect to the action of $\mathrm{Gal}(K(\mu_{p^\infty})/K)$ on both μ_{p^∞} and $M \otimes_K K(\mu_{p^\infty})$. In particular, for each positive integer n, the action of μ_{p^n} is induced by an action on $M \otimes_K K(\mu_{p^n})$.
- For each closed subinterval J of I, for some (and hence any) basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of M_J, there exists $k > 0$ such that for each positive integer m, for each $\zeta \in \mu^{p^m}$, the matrix $E(\zeta)$ defined by $\zeta^*(\mathbf{e}_j) = \sum_i E(\zeta)_{ij}\mathbf{e}_i$ satisfies $|E(\zeta)| \leqslant p^{km}$.

We use the following facts about $\mathcal{C}$.

- Every μ_{p^∞}-stable submodule of an object of $\mathcal{C}$ is saturated. This follows from the fact that every finitely generated μ_{p^∞}-stable ideal of $R_I \otimes_K K(\mu_{p^\infty})$ is either zero or trivial (the support of such an ideal cannot be both finite and nonempty).
- The condition on $|E(\zeta)|$ is preserved under passage to μ_{p^∞}-stable subquotients. Using the previous point, it follows that $\mathcal{C}$ is an abelian category. Beware that the condition on $|E(\zeta)|$ is not preserved under formation of extensions; however, for an extension which does belong to $\mathcal{C}$, the discussion of Remark 3.4.14 given above carries over.
- The category $\mathcal{C}$ contains the differential modules over R_I satisfying the Robba condition as a *nonfull* subcategory. This is a consequence of [Ked10, Lemma 13.5.4].
- Neither the final part of the proof of [Ked10, Theorem 13.5.5] (starting from 'We next choose $A\ldots$'), nor any of the proof of [Ked10, Theorem 13.5.6], makes any direct reference to differentiation, only to parallel transport. Consequently, both parts of Theorem 3.4.16 remain true for objects of $\mathcal{C}$.
- The given argument for Theorem 3.4.22 applies without change to an object M of $\mathcal{C}$. That is, if M has an exponent admitting a nontrivial Liouville partition A_1, A_2, then M splits nontrivially in $\mathcal{C}$ into summands M_1, M_2. However, it is not apparent from the construction that M_i is an exponent of A_i.

In order to proceed, we must formally upgrade this last point to a stronger statement. Namely, let M be an object of $\mathcal{C}$ with an exponent admitting a Liouville partition A_1, A_2. We will show that M admits a unique splitting $M_1 \oplus M_2$ in $\mathcal{C}$ for which M_i admits an exponent weakly equivalent to A_i. (Note that this does not guarantee that A_i itself occurs as an exponent for M_i.)

We first verify uniqueness. Suppose that M splits as $M_1' \oplus M_2'$, $M_1'' \oplus M_2''$ with M_i', M_i'' having exponents A_i', A_i'' weakly equivalent to A_i (and hence to each other). Let f be the composition of the inclusion $M_1' \to M$ with the projection $M \to M_2''$. Since $\mathcal{C}$ is an abelian category, we may form the exact sequences

$$0 \to \ker(f) \to M_1' \to \mathrm{image}(f) \to 0, \quad 0 \to \mathrm{image}(f) \to M_2'' \to \mathrm{coker}(f) \to 0$$

and choose exponents E_1, E_2, E_3 for $\ker(f), \mathrm{image}(f), \mathrm{coker}(f)$. By Remark 3.4.14 and Theorem 3.4.16(b), A_1' is weakly equivalent to $E_1 \cup E_2$ and A_2'' is weakly equivalent to $E_2 \cup E_3$. By Proposition 3.4.5(a), A_1', A_2'' constitute a Liouville partition, so we must have $E_2 = \emptyset$, which is to say $f = 0$. Similarly, the composition $M_2' \to M \to M_1''$ equals zero, so we have $M_1' = M_1''$, $M_2' = M_2''$ as desired.

We next verify existence, proceeding by induction on the rank of M. There is nothing to check unless A_1, A_2 are both nonempty; in this case, we know that there exists a nontrivial splitting $M \cong M_1' \oplus M_2'$ in $\mathcal{C}$. Choose exponents A_1', A_2' for M_1', M_2'; then $A_1' \cup A_2'$ is an exponent for M. By Theorem 3.4.16(b), $A_1' \cup A_2'$ is weakly equivalent to $A_1 \cup A_2$, so, by Proposition 3.4.5(b), $A_1' \cup A_2'$ admits a Liouville partition A_1'', A_2'' with A_i'' being weakly equivalent to A_i. In particular, A_1'', A_2'' are disjoint.

Identify finite multisubsets of $\mathbb{Z}_p$ with finitely supported functions $\mathbb{Z}_p \to \mathbb{Z}_{\geqslant 0}$, and put

$$A_{ij}''(x) = \min\{A_i'(x), A_j''(x)\} \quad (x \in \mathbb{Z}_p).$$

Since A_1'', A_2'' are disjoint, we have

$$(A_1''(x), A_2''(x)) \in \{(A_1'(x) + A_2'(x), 0), (0, A_1'(x) + A_2'(x))\}.$$

We thus have

$$A''_{i1}(x) + A''_{i2}(x) = \min\{A'_i(x), A''_1(x)\} + \min\{A'_i(x), A''_2(x)\} = A'_i(x)$$

and hence A''_{i1}, A''_{i2} form a Liouville partition of A'_i. On the other hand,

$$A''_{1i}(x) + A''_{2i}(x) = \min\{A'_1(x), A''_i(x)\} + \min\{A'_2(x), A''_i(x)\} = A''_i(x).$$

By the induction hypothesis, we may split M'_i as $M'''_{i1} \oplus M'''_{i2}$ with M'''_{ij} admitting an exponent A'''_{ij} weakly equivalent to A''_{ij}. Now define $M_i = M'''_{1i} \oplus M'''_{2i}$, so that $M \cong M_1 \oplus M_2$. By construction, M_i admits the exponent $A'''_{1i} \cup A'''_{2i}$ which is weakly equivalent to A''_i and hence to A_i, as desired.

We finally return to the original situation, in which M is a differential module over R_I satisfying the Robba condition, with an exponent admitting the Liouville partition A_1, A_2. By the previous arguments, in $\mathcal{C}$ there is a unique splitting $M \cong M_1 \oplus M_2$ in which M_i admits an exponent weakly equivalent to A_i. For $\lambda \in K^\times$ with $|1-\lambda| < 1$, the substitutions $t \mapsto \zeta t$ and $t \mapsto \lambda t$ commute: consequently, the pullbacks $M_{1,\lambda}$ and $M_{2,\lambda}$ of M_1, M_2 along λ^* are again objects of $\mathcal{C}$ with the same exponents as M_1, M_2. Consequently, the splittings $M_1 \oplus M_2$, $M_{1,\lambda} \oplus M_{2,\lambda}$ of M must coincide, which is to say that M_1 and M_2 are preserved by λ^*. By taking a sequence of values of λ converging to 1, we deduce that M_1 and M_2 are differential submodules, as desired.

Definition 3.7.9. The displayed equation should read

$$D(\mathbf{v}_1) = \mathbf{v}_2, \ldots, D(\mathbf{v}_{e-1}) = \mathbf{v}_e, \quad D(\mathbf{v}_e) = \lambda t^{h/m}\mathbf{v}_1.$$

Lemma 3.7.11. In the fifth line of the proof, the inequality should read $|\alpha - \lambda^{1/e}t^{h/(em)}|_\rho < |\alpha|_\rho$. We give some further justification for why this inequality can be achieved.

There is no issue unless ρ belongs to the divisible closure of $|F^\times|$; we may thus assume without loss of generality that K is algebraically closed and $\rho = 1$. Because of the hypothesis that M is pure, in the notation of Proposition 3.6.3 we must have $s_{\overline{\mu},i}(M) = 0$ for all i and all nonzero $\overline{\mu}$. This means that for each index i such that a_i corresponds to a point (not necessarily a vertex) on the Newton polygon of the polynomial $P(T)$, we can find $\lambda_i \in K$, $m_i \in \mathbb{Z}$ such that $|a_i - \lambda_i t^{m_i}|_\rho < |a_i|$; moreover, $\log|\lambda_i|$ and m_i are determined by the affine function $F_{n-i}(M, r)$ near $r = 0$. From this, we may infer the desired approximation.

Theorem 3.8.16. In light of the modifications to Proposition 1.1.2 and Remark 1.1.3 described above, we must also verify the conditions (iv) and (v) added to Remark 1.1.3.

- To check (iv), we may enlarge K to contain all roots of unity; we may then reduce to checking that if M is of rank 1, M satisfies the Robba condition, and $M^{\otimes \ell}$ is trivial for some prime ℓ, then M becomes trivial after a finite tamely ramified extension (which does not change $G^{0+}(M)$). Apply Theorem 3.4.16 to construct an exponent $A \in \mathbb{Z}_p$ of M; then ℓA is weakly equivalent to 0, so Corollary 3.4.7 implies that $\ell A \in \mathbb{Z}$. If $\ell = p$, then it follows that $A \in \mathbb{Z}$, so Theorem 3.4.20 implies that M is trivial; otherwise M becomes trivial after adjoining $t^{1/\ell}$.
- To check (v), we may apply the lemma stated in the discussion of Theorem 2.3.17; it implies that for any M and any $r > 0$, the filtration of $G^{0+}(M)$ has only finitely many breaks in the interval $[r, pr]$.

References

FH91 W. Fulton and J. Harris, *Representation theory: a first course*, Graduate Texts in Mathematics, vol. 129 (Springer, New York, 1991).

Ked10 K. S. Kedlaya, *p-adic differential equations* (Cambridge University Press, Cambridge, 2010), errata (version of 27 April 2017) available at http://kskedlaya.org/papers/pde-errata.pdf.

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