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BOUNDED INDEX AND SUMMABILITY METHODS

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1. Introduction and preliminary results

An entire function $f(z)$ is of bounded index if there exists a non-negative integer N such that

$$\max_{0 \leqq j \leqq N} \left\{ \frac{|f^{(j)}(z)|}{j!} \right\} \geqq \frac{|f^{(k)}(z)|}{k!} \text{ for all } z \text{ and all } k.$$

The least such integer N is called the index of f (see Lepson (1966)).

A sequence $x = \{x_k\}_0^\infty$ of complex numbers is an entire sequence if $\sum_{k=0}^\infty |x_k| q^k$ converges for every positive integer q. If we denote by $\mathscr{E}$ the set of entire sequences, then we see that $\mathscr{E}$ can be identified with the class of entire functions. An entire sequence $x = \{x_k\}_0^\infty$ is of bounded index if $f(z) = \sum_{k=0}^\infty x_k z^k$ is an entire function of bounded index. We will denote the set of sequences of bounded index by $\mathscr{B}$. Furthermore, let ℓ be the set of all absolutely convergent sequences, that is, $\ell = \{x = \{x_k\}_0^\infty : \sum_{k=0}^\infty |x_k| < \infty\}$.

An infinite matrix $A = (a_{n,k})$ of complex entries which transforms $\mathscr{E}$ into $\mathscr{E}$ ($\mathscr{B}$ into $\mathscr{B}$, ℓ into ℓ) will be called an $\mathscr{E}$-$\mathscr{E}$ method ($\mathscr{B}$-$\mathscr{B}$ method, ℓ-ℓ method).

In Fricke and Powell (1970), the authors have shown

THEOREM 1. *A matrix $A = (a_{n,k})$ is an $\mathscr{E}$-$\mathscr{E}$ method if and only if for each integer $q > 0$, there exist an integer $p > 0$ and a constant $M > 0$ such that*

$$|a_{n,k}| q^n \leqq M p^k \text{ for all } n, k = 0, 1, \cdots.$$

The Taylor matrix, $T(r) = (a_{n,k})$, defined by

$$a_{n,k} = \begin{cases} \binom{k}{n} (1-r)^{n+1} r^{k-n} & \text{if } \quad k \geqq n \\ 0 & \text{otherwise,} \end{cases}$$

where r is a complex number, is an $\mathscr{E}$-$\mathscr{E}$ method for any complex number r (by Theorem 1). We now show:

THEOREM 2. *The Taylor matrix, $T(r) = (a_{n,k})$, is a $\mathcal{B}$-$\mathcal{B}$ method for any complex number r.*

PROOF. Let $x = \{x_k\}_0^\infty \in \mathcal{B}$, that is, $f(z) = \sum_{k=0}^\infty x_k z^k$ is a function of bounded index. Thus, for $y = \{y_n\}_0^\infty = Ax$ (where $y_n = \sum_{k=0}^\infty a_{n,k} x_k$),

$$\begin{aligned} g(z) = \sum_{n=0}^\infty y_n z^n &= \sum_{n=0}^\infty \sum_{k=0}^\infty a_{n,k} x_k z^n \\ &= \sum_{n=0}^\infty \sum_{k=n}^\infty \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k z^n \\ &= (1-r) \sum_{k=0}^\infty \sum_{n=0}^k \binom{k}{n} (1-r)^n z^n r^{k-n} x_k \\ &= (1-r) \sum_{k=0}^\infty x_k [(1-r)z + r]^k \\ &= (1-r) f([1-r]z + r). \end{aligned}$$

From Fricke (to appear) we have that the class of functions of bounded index is closed under translation. Hence, $g(z)$ is of bounded index, that is,

$$Ax = y = \{y_n\}_0^\infty \in \mathcal{B}.$$

It is readily seen that mere growth conditions on the entries of a matrix $A = (a_{n,k})$ are not sufficient for A to be a $\mathcal{B}$-$\mathcal{B}$ method. In fact, given any sequence $x = \{x_k\}_0^\infty$ of bounded index and any sequence $\{d_n\}_0^\infty$ of positive numbers, there exists a matrix $A = (a_{n,k})$ with $a_{n,k} = 0$ for $n \neq k$ and $|a_{n,n}| \leqq d_n$ for $n = 0, 1, \cdots$, such that $Ax \notin \mathcal{B}$.

We now prove a result on functions of bounded index which we will require later on.

THEOREM 3. *Let f be a function of bounded index. If*

$$\lim_{n\to\infty} f^{(k)}(a_n) = 0 \quad \textit{for } k = 0, 1, \cdots,$$

where $\{a_n\}_0^\infty$ is a sequence of complex numbers, then, for all $r > 0$,

$$\lim_{n\to\infty} \max_{|z-a_n|=r} \{|f^{(k)}(z)|\} = 0 \quad \textit{for all } k = 0, 1, \cdots.$$

PROOF. Let f be of index N. It is sufficient to prove Theorem 3 for $r = 1/(2N+2)$. Since f is of index N we have, for each a_n, that there exists an integer $\ell = \ell(a_n)$ with $0 \leqq \ell \leqq N$ such that

$$\max_{|z-a_n|=r} \left\{ \frac{|f^{(\ell)}(z)|}{\ell!} \right\} \geqq \max_{|z-a_n|=r} \left\{ \frac{|f^{(j)}(z)|}{j!} \right\} \text{ for } j = 0, 1, \cdots.$$

Hence,

$$\max_{|z-a_n|=r}\left\{\frac{|f^{(\ell)}(z)|}{\ell\,!}\right\} \geqq \max_{|r-a_n|=r}\left\{\frac{|f^{(\ell+1)}(z)|}{(\ell+1)!}\right\}$$

$$\geqq \frac{1}{(\ell+1)!}\,\frac{\max_{|z-a_n|=r}\{|f^{(\ell)}(z)|\} - |f^{(\ell)}(a_n)|}{r}$$

$$\geqq 2\left(\max_{|z-a_n|=r}\left\{\frac{|f^{(\ell)}(z)|}{\ell\,!}\right\} - \frac{|f^{(\ell)}(a_n)|}{\ell\,!}\right).$$

Thus,

$$\max_{|z-a_n|=r}\left\{\frac{|f^{(\ell)}(z)|}{\ell\,!}\right\} \leqq 2\frac{|f^{(\ell)}(a_n)|}{\ell\,!}.$$

Therefore, for $n = 0, 1, \cdots$ and $j = 0, 1, \cdots$,

$$\max_{|z-a_n|=r}\left\{\frac{|f^{(j)}(z)|}{j\,!}\right\} \leqq 2 \max_{0 \leqq t \leqq N}\left\{\frac{|f^{(t)}(a_n)|}{t\,!}\right\}.$$

Thus, using the hypothesis,

$$\lim_{n\to\infty}\max_{|z-a_n|=r}\{|f^{(j)}(z)|\} = 0.$$

2. The matrix $A(f, z_i)$

For an entire function $f(z)$ and a sequence $\{z_i\}_0^\infty$ of complex numbers define the matrix transformation $A(f, z_i) = (a_{n,k})$ by

$$f(z) = \sum_{k=0}^{\infty} a_{n,k}(z - z_n)^k \quad \text{for } n = 0, 1, \cdots.$$

For this matrix transformation we can express the Silvermann-Toeplitz conditions for regularity as follows.

THEOREM 4. *The matrix transformation $A(f, z_i) = (a_{n,k})$ is regular if and only if*

(i) $\lim_{n\to\infty} f^{(k)}(z_n) = 0$ *for* $k = 0, 1, \cdots$,

(ii) $\lim_{n\to\infty} f(z_n + 1) = 1$,

and

(iii) $\sum_{k=0}^{\infty} |a_{n,k}| \leqq M$ *for some* $M > 0$ *and all* $n = 0, 1, \cdots$.

PROOF. For the matrix $A(f, z_i) = (a_{n,k})$, we have $a_{n,k} = (f^{(k)}(z_n))/k\,!$ for n, $k = 0, 1, \cdots$, and $f(z_n + 1) = \Sigma_{k=0}^{\infty}\, a_{n,k}$. Hence, conditions (i), (ii) and (iii) are identical to the Silvermann-Toeplitz conditions

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$$\lim_{n\to\infty} a_{n,k} = 0 \text{ for } k = 0, 1, \cdots,$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1,$$

and

$$\sup_n \left\{ \sum_{k=0}^{\infty} |a_{n,k}| \right\} < \infty.$$

THEOREM 5. *If $f(z)$ is an entire function of bounded index then $A(f, z_i) = (a_{n,k})$ is not regular for any sequence $\{z_i\}_0^\infty$.*

PROOF. By Theorem 3, $\lim_{n\to\infty} f^{(k)}(z_n) = 0$ for $k = 0, 1, \cdots$ implies $\lim_{n\to\infty} f(z_n + 1) = 0$ since f is of bounded index. Thus, conditions (i) and (ii) of Theorem 4 cannot be satisfied simultaneously.

It is worth noting, however, that functions of bounded index can give rise to conservative matrices. For example, let $f(z) = e^z$ (f is of bounded index with index 0) and choose $z_n = 2\pi in$. Thus

$$f^{(k)}(z_n) = 1 \text{ for all } k, \; n = 0, 1, \cdots,$$

$$f(z_n + 1) = e \text{ for all } n = 0, 1, \cdots,$$

and

$$\sum_{k=0}^{\infty} |a_{n,k}| = e \text{ for all } n = 0, 1, \cdots.$$

We now examine the matrix $A'(f, z_i) = (b_{n,k})$ which is defined by $f(z) = \sum_{n=0}^{\infty} b_{n,k}(z - z_k)^n$ for $k = 0, 1, \cdots$. The matrix $A'(f, z_i) = (b_{n,k})$ is the transpose of $A(f, z_i) = (a_{n,k})$, that is, $a_{n,k} = b_{k,n}$ for $n, k = 0, 1, \cdots$.

THEOREM 6. *If $f(z)$ is an entire function of bounded index then, for any sequence $\{z_i\}_0^\infty$, $A'(f, z_i) = (b_{n,k})$ is an ℓ-ℓ method if and only if*

$$\sup_n \{|f^{(k)}(z_n)|\} < \infty \textit{ for } k = 0, 1, \cdots.$$

PROOF. Knopp and Lorentz (1949) showed that a necessary and sufficient condition for a matrix $A = (a_{n,k})$ to be an ℓ-ℓ method is

$$\sup_k \left\{ \sum_{n=0}^{\infty} |a_{n,k}| \right\} < \infty.$$

Let $A'(f, z_i) = (b_{n,k})$ be an ℓ-ℓ method. Thus, there exists a constant $M > 0$ such that

$$\sum_{n=0}^{\infty} |b_{n,k}| \leqq M \text{ for } k = 0, 1, \cdots.$$

Hence,

$$|b_{n,k}| = \frac{|f^{(n)}(z_k)|}{n!} \leqq M \text{ for } n, \; k = 0, 1, \cdots.$$

Therefore,

$$\sup_k \{|f^{(n)}(z_k)|\} \leqq n!\ M < \infty \text{ for } k = 0, 1, \cdots.$$

Now, let f be an entire function of bounded index and let $\{z_i\}_0^\infty$ be a sequence such that

$$\sup_n \{|f^{(k)}(z_n)|\} < \infty \text{ for } k = 0, 1, \cdots.$$

Since $f(z)$ is of bounded index we have that $f(2z)$ is of bounded index (see Fricke (to appear)). Let N be the index of $f(2z)$. Thus,

$$\max_{0 \leqq j \leqq N} \left\{\frac{2^j |f^{(j)}(z)|}{j!}\right\} \geqq \frac{2^n |f^{(n)}(z)|}{n!} \text{ for all } z \text{ and all } n.$$

Hence,

$$\max_{0 \leqq j \leqq N} \left\{\frac{|f^{(j)}(z)|}{j!}\right\} \geqq 2^{n-N} \frac{|f^{(n)}(z)|}{n!} \text{ for all } z \text{ and all } n.$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} |b_{n,k}| &= \sum_{n=0}^{\infty} \frac{|f^{(n)}(z_k)|}{n!} \\ &\leqq \sum_{n=0}^{\infty} 2^{N-n} \max_{0 \leqq j \leqq N} \left\{\frac{|f^{(j)}(z_k)|}{j!}\right\} \\ &\leqq 2^{N+1} N! \max_{0 \leqq j \leqq N} \{|f^{(j)}(z_k)|\}.\end{aligned}$$

Now, since $\sup_k \{\max_{0 \leqq j \leqq N} (|f^{(j)}(z_k)|)\} < \infty$, we have $\sup_k \{\sum_{n=0}^{\infty} |b_{n,k}|\} < \infty$, which shows that $A'(f, z_i)$ is an ℓ-ℓ method.

THEOREM 7. *If $f(z)$ is an entire function of bounded index, then for any sequence $\{z_i\}_0^\infty$, $A'(f, z_i) = (b_{n,k})$ is an $\mathscr{E}$-$\mathscr{E}$ method if and only if for each integer $n > 0$ there exist an integer $p > 0$ and a constant $M > 0$ such that*

$$|f^{(n)}(z_k)| \leqq p^k M \text{ for } k = 0, 1, \cdots.$$

PROOF. If $A'(f, z_i) = (b_{n,k})$ is an $\mathscr{E}$-$\mathscr{E}$ method then, by Theorem 1, for each integer $q > 0$, there exist an integer $t > 0$ and a constant $T > 0$ such that

$$|b_{n,k}|\, q^n \leqq t^k T \text{ for } n,\ k = 0, 1, \cdots.$$

Thus, for $q = 1$ there exist an integer t and a constant T such that

$$|b_{n,k}| = \frac{|f^{(n)}(z_k)|}{n!} \leqq t^k T \text{ for } n,\ k = 0, 1, \cdots.$$

Hence, for each integer n there exist an integer $p = t$ and a constant $M = n!T$ such that

$$|f^{(n)}(z_k)| \leqq t^k n!\, T = p^k M \text{ for } k = 0, 1, \cdots.$$

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Now, let $f(z)$ be an entire function of bounded index and let $\{z_i\}_0^\infty$ be a sequence such that for each integer $n > 0$ there exist an integer $p > 0$ and a constant $M > 0$ such that

$$|f^{(n)}(z_k)| \leqq p^k M \quad \text{for} \quad k = 0, 1, \cdots$$

Let N be the index of $f(z)$. Thus, there exist an integer $r > 0$ and a constant T such that, for $n \leqq N$,

$$|f^{(n)}(z_k)| \leqq r^k T \quad \text{for} \quad k = 0, 1, \cdots.$$

For an integer $q > 0$ we have, by Fricke (to appear), $h(z) = f(qz)$ is of bounded index. Thus, for $q > 0$, let N_q be the index of $h(z) = f(qz)$, that is, for any $z \in \mathfrak{C}$ and $j = 0, 1, \cdots$,

$$\begin{aligned} \frac{|h^{(j)}(z)|}{j!} &= q^j \frac{|f^{(j)}(z)|}{j!} \\ &\leqq \max_{0 \leqq i \leqq N_q} \left\{ q^i \frac{|f^{(i)}(z)|}{i!} \right\} \\ &= \max_{0 \leqq i \leqq N_q} \left\{ \frac{|h^{(i)}(z)|}{i!} \right\}. \end{aligned}$$

Hence, for $j = 0, 1, \cdots$,

$$\begin{aligned} q^j \frac{|f^{(j)}(z_k)|}{j!} &\leqq \max_{i \leqq N_q} \left\{ q^i \frac{|f^{(i)}(z_k)|}{i!} \right\} \\ &\leqq q^{N_q} \max_{i \leqq N} \left\{ \frac{|f^{(i)}(z_k)|}{i!} \right\} \\ &\leqq q^{N_q} \max_{i \leqq N} \{ |f^{(i)}(z_k)| \} \\ &\leqq q^{N_q} r^k T \quad \text{for} \quad k = 0, 1, \cdots. \end{aligned}$$

Therefore, for an integer $q > 0$, there exist an integer $u = r$ and a constant $w = q^{N_q} T$ such that

$$|b_{n,k}| q^n = \frac{|f^{(n)}(z_k)|}{n!} q^n \leqq q^{N_q} r^k T = u^k W \quad \text{for} \quad n, \ k = 0, 1, \cdots.$$

Thus, by Theorem 1, $A'(f, z_i)$ is an $\mathscr{E}$-$\mathscr{E}$ method.

We now show that the condition that $f(z)$ is of bounded index cannot be omitted in Theorem 7.

THEOREM 8. *There exists an entire function $f(z)$ of exponential type and of unbounded index and there exists a sequence $\{z_i\}_0^\infty$ such that for each integer n there exist an integer p and a constant M with*

$$|f^{(n)}(z_k)| \leqq p^n M \ \ for \ \ k = 0, 1, \cdots,$$

but $A'(f, z_i) = (b_{n,k})$ *is not an* $\mathscr{E}$-$\mathscr{E}$ *method.*

PROOF. Let $\{a_n\}_1^\infty$ be a sequence of positive numbers such that $a_1 = 1$ and

$$a_{k+1} \geqq \max\{3(k+1)a_k, a_k^{\frac{k+1}{k}}\} \text{ for } k = 1, 2, \cdots.$$

S. M. Shah (1970) showed that $f(z) = \Pi_{n=1}^\infty (1 - z/a_n)^n$ is an entire function of exponential type and of unbounded index. Shah also showed that

$$\lim_{n\to\infty} \frac{|f^{(n)}(a_n)|}{n!} = \infty.$$

Therefore, there exists a sequence $\{n_k\}_{k=1}^\infty$ such that

$$\frac{|f^{(n_k)}(a_{n_k})|}{n_k!} \geqq k! \text{ for } k = 1, 2, \cdots.$$

Choose the sequence $\{z_k\}_{k=0}^\infty$ by $z_0 = 1$ and $z_k = a_{n_k}$ for $k = 1, 2, \cdots$. Thus, for $f(z) = \Pi_{n=1}^\infty (1 - z/a_n)^n$ and the sequence $\{z_k\}_0^\infty$ we have $f^{(n)}(z_k) = 0$ for $k > n$. Hence, for each integer n there exist an integer $p = 1$ and a constant $M = \max_{k\leq n}\{|f^{(n)}(z_k)|\}$ such that

$$|f^{(n)}(z_k)| \leqq p^k M \text{ for } k = 0, 1, \cdots.$$

Now, for $A'(f, z_i) = (b_{n,k})$ and $k = 1, 2, \cdots,$

$$|b_{n_k,k}| = \frac{|f^{(n_k)}(z_k)|}{n_k!} = \frac{|f^{(n_k)}(a_{n_k})|}{n_k!} \geqq k!.$$

Therefore, for any integer $r > 0$ and any constant $T > 0$, there exists k_0 such that

$$|b_{n_k,k}| \geqq k! > r^k T \text{ for } k > k_0.$$

Thus, by Theorem 1, $A'(f, z_i) = (b_{n,k})$ is not an $\mathscr{E}$-$\mathscr{E}$ method.

THEOREM 9. *Let* $f(z)$ *be an entire function of bounded index and* $\{z_i\}_0^\infty$ *be a sequence of complex numbers. If either* $A(f, z_i) = (a_{n,k})$ *or* $A'(f, z_i) = (b_{n,k})$ *is an* ℓ-ℓ *method then* $A'(f, z_i)$ *is an* $\mathscr{E}$-$\mathscr{E}$ *method.*

PROOF. If either $A(f, z_i)$ or $A'(f, z_i)$ is an ℓ-ℓ method then either $\sum_{n=0}^\infty |a_{n,k}| \leqq M$, or

$$\sum_{n=0}^\infty |b_{n,k}| \leqq M \text{ for } k = 0, 1, \cdots.$$

Thus, either

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$$|a_{n,k}| = \frac{|f^{(k)}(z_n)|}{k!} \leqq M,$$

or

$$|b_{n,k}| = \frac{|f^{(n)}(z_k)|}{n!} \leqq M \text{ for } n, \ k = 0, 1, \cdots.$$

Hence,

$$\frac{|f^{(n)}(z_k)|}{n!} \leqq M \text{ for } n, \ k = 0, 1, \cdots.$$

Therefore, for each n there exist $p = 1$ and $T = n!M$ such that

$$|f^{(n)}(z_k)| \leqq n!M = p^k T \text{ for } k = 0, 1, \cdots.$$

Thus, by Theorem 1, $A'(f, z_i)$ is an $\mathscr{E}$-$\mathscr{E}$ method.

3. Application to analytic continuation

Assume now that

$$\lim_{n \to \infty} f^{(k)}(z_n) = 0 \text{ for } k = 0, 1, \cdots$$

and

$$\lim_{n \to \infty} f(z_n + 1) = 1.$$

Thus, for the matrix $A(f, z_i) = (a_{n,k})$, we have

$$\lim_{n \to \infty} a_{n,k} = 0 \text{ for } k = 0, 1, \cdots \text{ and } \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{n,k} = 1.$$

THEOREM 10. *The $A(f, z_i)$ transform continues the sequence of partial sums of the geometric series, $\sum_{k=0}^{\infty} z^k$, analytically into $\{z : \lim_{n \to \infty} f(z_n + z) = 0\}$.*

PROOF. Let $A(f, z_i) = (a_{n,k})$ and $\{S_k(z)\}_0^{\infty}$ be the sequence of kth partial sums of geometric series. So, the $A(f, z_i)$ transform of $\{S_k(z)\}_0^{\infty}$ is given by

$$\begin{aligned} \sigma_n(z) &= \sum_{k=0}^{\infty} a_{n,k} S_k(z) \\ &= \sum_{k=0}^{\infty} a_{n,k} \frac{1 - z^{k+1}}{1 - z} \\ &= \frac{1}{1-z} \sum_{k=0}^{\infty} a_{n,k} - \frac{z}{1-z} \sum_{k=0}^{\infty} a_{n,k} z^k \\ &= \frac{1}{1-z} f(z_n + 1) - \frac{z}{1-z} f(z_n + z). \end{aligned}$$

So $\lim_{n\to\infty}\sigma_n(z)=\dfrac{1}{1-z}\Leftrightarrow\lim_{n\to\infty}f(z_n+z)=0$.

As an example, for Theorem 10, let α be a complex number with $\operatorname{Im}\alpha>0$, $f(z)=\exp\{i(z/\alpha)^2\}$, and $z_n=\alpha\sqrt{2\pi n}-1$. Now, for fixed k,

$$\lim_{n\to\infty}|f^{(k)}(z_n)|=\lim_{n\to\infty}|P_k(z_n)|\ \exp\left(\operatorname{Re}\left\{i\left(\frac{z_n}{\alpha}\right)^2\right\}\right)=0$$

since $\lim_{n\to\infty}\exp(\operatorname{Re}\{i(z_n/\alpha)^2\})=0$ and P_k is a polynomial of degree k. Also, $\lim_{n\to\infty}f(z_n+1)=1$. Therefore,

$$\left\{z:\lim_{n\to\infty}f(z_n+z)=0\right\}=\left\{z:\operatorname{Im}\left(\frac{z-1}{\alpha}\right)>0\right\},$$

that is, the open half plane containing the origin whose boundary is $\{z: z-1=\alpha x,\ x \text{ real}\}$. In particular, if $\alpha=i$ then $f(z)=\exp(-iz^2)$ and $z_n=\sqrt{2\pi n}i-1$. Thus,

$$\{z:\lim_{n\to\infty}f(z_n+z)=0\}=\{z:\operatorname{Re}z<1\}$$

and, since $\{z:|z|<1\}\subseteq\{z:\lim_{n\to\infty}f(z_n+z)=0\}$, a minimal region into which the $A(f,z_i)$ transform provides the analytic continuation of an arbitrary Taylor series may be determined by use of the Okada theorem (Powell and Shah (1972; pages 155–162)).

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