Canad. Math. Bull. Vol. **38** (2), 1995 pp. 182–186

THE CONTRAGREDIENT ISOTYPIC COMPONENT OF THE REGULAR REPRESENTATION OF PSEUDOREFLECTION GROUPS

To Louis Solomon on his 65th birthday

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ABSTRACT. For the regular representation of a pseudoreflection group G we characterize the occurrences of the contragredient representation as the gradient spaces of a set of Chevalley generators of the invariants of G.

1. **The statements.** In what follows V will be a finite dimensional vector space over a field $\mathbb{K}$ and $G \subset \mathrm{GL}(V)$ will be a group. We shall denote the symmetric algebra of V over $\mathbb{K}$ by S and uniquely extend the elements of G to automorphisms of S. We then define the fixed point algebra

$$R := \{x \in S \mid gx = x \,\forall g \in G\}.$$

We shall view S as an $\mathbb{N}$-graded algebra and G-module

$$S = \bigoplus_{d \in \mathbb{N}} S^d$$

in the natural way and in what follows, any reference to "graded" or "homogeneous" objects will be with respect to this grading.

We denote by V^* the dual space of V and view this as a G-module via the contragredient action. Each $\omega \in V^*$ extends uniquely to a derivation ∂_ω of S. One easily verifies the following three properties:

(1) $g\partial_\omega(s) = \partial_{g\omega} gs$ for all $g \in G$, $\omega \in V^*$ and $s \in S$.

(2) If $\{v_1, \ldots, v_\ell\}$ is a basis of V and $\{\omega_1, \ldots, \omega_\ell\}$ is the corresponding dual basis of V^*, then for $s \in S^d$ where $d \in \mathbb{N}$ we have (Euler's identity)

$$\sum_{i=1}^{\ell} v_i \partial_{\omega_i} s = ds.$$

(3) The map $\omega \longmapsto \partial_\omega$ is a $\mathbb{K}$-linear map from V^* to the $\mathbb{K}$-space of derivations of S.

If $s \in S$ we define its *gradient space* $\nabla_{\mathbb{K}}(s)$ by

$$\nabla_{\mathbb{K}}(s) := \{\partial_\omega(s) \mid \omega \in V^*\}. \tag{4}$$

Both authors acknowledge the help of NSERC Canada.
Received by the editors January 14, 1994.
AMS subject classification: 17B20, 20C.

https://doi.org/10.4153/CMB-1995-026-4 Published online by Cambridge University Press

It follows from (3) that thus defined $\nabla_{\mathbb{K}}(s)$ is a $\mathbb{K}$-subspace of S.

Next we assume that G is finite, generated by pseudoreflections (recall that a *pseudoreflection* is a non trivial element of $\mathrm{GL}(V)$ which pointwise fixes a hyperplane), and that $|G|$ is invertible in $\mathbb{K}$. (We refer to this as the "Chevalley conditions.") We recall some basic properties of these groups. Our running reference for this will be [Bbk, Chapter 5, number 5]. The ring R is a polynomial ring on a set of homogeneous generators $f_1, \ldots, f_\ell$. That is

$$R = \mathbb{K}[f_1, \ldots, f_\ell].$$

If we set $d_i := \deg f_i$ (the degrees), then $|G| = \prod_{i=1}^{\ell} d_i$. The set of degrees is independent of the choice of Chevalley generators $f_1, \ldots, f_\ell$.

Finally if R_+ denotes the augmentation ideal of R there exist G-stable graded supplements to R_+S in S and any such supplement is isomorphic to the regular representation of G. Fix once and for all such a supplement U. Write

$$S = R_+S \oplus U \tag{5}$$

and let

$$\bar{\ } : S \longrightarrow U$$

be the corresponding projection.

It is natural to ask where in U do the different irreducible representations of G appear. We give a definite answer to this question as far as the occurrence of the contragredient module V^* is concerned.

THEOREM 1. *Let $G \subset \mathrm{GL}(V)$ be a group satisfying Chevalley conditions. Assume V is an irreducible G-module. If $f_1, \ldots, f_\ell$ is a set of Chevalley generators of R and $\bar{\ } : S \longrightarrow U$ is given by (5) above, then*

(i) Each $\overline{\nabla_{\mathbb{K}}(f_i)}$ is a G-module isomorphic to V^*.

(ii) The sum $\sum_{i=1}^{\ell} \overline{\nabla_{\mathbb{K}}(f_i)}$ is direct and this G-space is the isotypic component of V^* in U.

For Weyl groups the theorem is due to Solomon ([Slm]). It can also be found in [BL] (where the occurrence of all irreducible representations in U is studied). Note that for Weyl groups $V \simeq V^*$.

2. **The proofs.** Before turning into the main proof we point out a (probably well known) general fact (Proposition 1 below) from which part (i) of the theorem will follow.

The group G is now allowed to be arbitrary. If H is a subset of S we define its *gradient space* by

$$\nabla_{\mathbb{K}}(H) := \sum_{s \in H} \nabla_{\mathbb{K}}(s)$$

where $\nabla_{\mathbb{K}}(s)$ is as in (4).

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PROPOSITION 1. *Let V be a finite dimensional $\mathbb{K}$-space and let $G \subset \mathrm{GL}(V)$ be a group. If $H \subset S$ is a G-module, then $\nabla_{\mathbb{K}}(H) \subset S$ is a G-module which is a homomorphic image of the G-module $H \otimes_{\mathbb{K}} V^*$.*

PROOF OF PROPOSITION 1. That $\nabla_{\mathbb{K}}(H)$ is a G-module follows from (1). To see that $\nabla_{\mathbb{K}}(H)$ is a homomorphic image of $H \otimes_{\mathbb{K}} V^*$ we consider the unique $\mathbb{K}$-linear surjection $\psi: H \otimes_{\mathbb{K}} V^* \longrightarrow \nabla_{\mathbb{K}}(H)$ satisfying

$$\psi: h \otimes \omega \longmapsto \partial_\omega h \quad \text{for all } h \in H \text{ and } \omega \in V^*.$$

If we denote by $\cdot$ the action of G on $H \otimes_{\mathbb{K}} V^*$, then for all $g \in G$, $h \in H$, and $\omega \in V^*$ we have

$$\psi\big(g \cdot (h \otimes \omega)\big) := \psi(gh \otimes g\omega) := \partial_{g\omega}(gh) = g\partial_\omega(h) = g\psi(h \otimes \omega)$$

(this penultimate equality by (1)) showing that ψ is a G-module homomorphism. ■

REMARK 1. Let $r \in R \setminus \{0\}$. Then $\mathbb{K}r$ is a trivial one dimensional G-module. Proposition 1 shows that $\nabla_{\mathbb{K}}(r)$ is a homomorphic image of V^*.

REMARK 2. Let $r \in R^d := R \cap S^d$ be nonzero and assume that d is invertible in $\mathbb{K}$. Euler's identity (2) shows that $\nabla_{\mathbb{K}}(r) \neq (0)$.

REMARK 3. Having fixed a basis $\{v_1, \ldots, v_\ell\}$ of V we can for each $g \in G$ write

$$gv_j = \sum_{k=1}^{\ell} a_{kj}(g) v_k$$

where $a_{kj}(g) \in \mathbb{K}$.

Let $d \in \mathbb{N}$ and assume a copy of V^* appears in S^{d-1}. One can ask if this is of the form $\nabla_{\mathbb{K}}(r)$ for some $r \in R^d$. Though the answer in this generality is no, one can still perform the following suggestive calculation. Let $h_1, \ldots, h_\ell$ be a basis of such copy of V^* chosen so that

$$gh_i = \sum_{k=1}^{\ell} a_{ik}(g^{-1}) h_k.$$

Let

$$r := \sum_{i=1}^{\ell} v_i h_i.$$

A straightforward calculation shows that $gr = r$ so that r is invariant. However we do not know if $r = 0$ or, even if $r \neq 0$, whether $\nabla_{\mathbb{K}}(r) \neq (0)$. If $\{\omega_1, \ldots, \omega_\ell\} \subset V^*$ is the basis dual to $\{v_1, \ldots, v_\ell\}$, then h_i need not be a multiple of $\partial_{\omega_i} r$ in general but, remarkably enough, this is the case for irreducible pseudoreflection groups. (See the proof of Theorem 1(ii) below.)

PROOF OF THEOREM 1. Since $\prod_{i=1}^{\ell} d_i = |G|$ it follows that each d_i is invertible in $\mathbb{K}$. Remarks 1 and 2 above, together with the assumption that V (hence V^*) is irreducible, gives

$$\nabla_{\mathbb{K}}(f_i) \simeq V^*. \tag{6}$$

Thus if $\overline{\nabla_{\mathbb{K}}(f_i)}$ is not isomorphic to V^*, then $\overline{\nabla_{\mathbb{K}}(f_i)} = (0)$. Now if $\overline{\nabla_{\mathbb{K}}(f_i)} = (0)$, then for any fixed basis $\{\omega_1, \ldots, \omega_\ell\}$ of V^* we have $\partial_{\omega_j} f_i \in R_+S$ for all $1 \leq j \leq \ell$. We show this is not possible.

Recall that a nonzero element $z \in S$ is called *antiinvariant* if

$$gz = (\det g)^{-1} z.$$

An example of such an element is the Jacobian

$$J = \det(\partial_{\omega_j} f_i), \quad 1 \leq i, j \leq \ell.$$

Furthermore any antiinvariant is of the form rJ for some $r \in R$. Now if $\partial_{\omega_j} f_i \in R_+S$ for all $1 \leq j \leq \ell$, then $J \in R_+S$ and therefore *all* nonzero antiinvariants would belong to R_+S. This however cannot be the case, for U contains a copy of the one dimensional G-module affording the character $\det^{-1}$; and hence nonzero antiinvariant elements. The proof of (i) is now complete.

As for (ii) it will suffice to show that the sum

(7) $$\sum_{i=1}^{\ell} \overline{\nabla_{\mathbb{K}}(f_i)}$$

is direct (the statement about the isotypic component then follows from (i)).

To simplify the notation we will write M_i instead of $\overline{\nabla_{\mathbb{K}}(f_i)}$. If the sum (7) is not direct after rearranging the f_i's if necessary, we can find $1 \leq p < q \leq \ell$ so that $d_p = d_{p+1} = \cdots = d_q$ and

(8) $$\text{the sum } \sum_{k=p+1}^{q} M_k \text{ is direct}$$

(9) $$M_p \cap \bigoplus_{k=p+1}^{q} M_k \neq (0).$$

From (9) it follows that $M_p \subset \oplus_{k=p+1}^{q} M_k := M$ and we can therefore for each $1 \leq i \leq \ell$ write

$$\overline{\partial_{\omega_i} f_p} = v_{i,p+1} + \cdots + v_{i,q}$$

where $v_{i,k} \in M_k$.

We claim that for each $p < k \leq q$

(10) there exists $\chi_k \in \mathbb{K}$ such that $v_{i,k} = \chi_k \overline{\partial_{\omega_i} f_k}$ for all $1 \leq i \leq \ell$.

Indeed for $p < k \leq q$ let $\psi_k \colon M \longrightarrow M_k$ be the projection map. By irreducibility $\psi_k(M_p) = (0)$ or $\psi_k(M_p) = M_k$. If $\psi_k(M_p) = (0)$ we simply set $\chi_k = 0$. If $\psi_k(M_p) = M_k$ consider the unique linear map $\varphi_k \colon M_k \longrightarrow M_p$ satisfying $\varphi_k \colon \overline{\partial_{\omega_i} f_k} \longrightarrow \overline{\partial_{\omega_i} f_p}$. By explicitly looking at the proof of part (i) of the theorem we conclude that φ_k is a G-module isomorphism. Thus $\psi_k \circ \varphi_k$ is an automorphism of M_k and hence a homothety (Schur's lemma). But $\psi_k \circ \varphi_k(\overline{\partial_{\omega_i} f_k}) = v_{i,k}$ and hence there exists a $\chi_k \in \mathbb{K}^\times$ so that $v_{i,k} = \chi_k \overline{\partial_{\omega_i} f_k}$ for all $1 \leq i \leq \ell$. Now (10) is established.

Finally consider $f := f_p - \chi_{p+1} f_{p+1} - \cdots - \chi_q f_q$. It is clear that $f_1, \ldots, f_{p-1}, f, f_{p+1}, \ldots, f_\ell$ is a set of Chevalley generators and hence that, as we have already proved, $\overline{\nabla_{\mathbb{K}}(f)} \simeq V^*$. By (10) however $\overline{\partial_{\omega_i} f} = 0$ for all $1 \leq i \leq \ell$. This contradiction finishes the proof of (ii). ■

https://doi.org/10.4153/CMB-1995-026-4 Published online by Cambridge University Press

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