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PAPER

The fully parabolic multi-species chemotaxis system in $\mathbb{R}^2$

Ke Lin

School of Mathematics, Southwestern University of Economics and Finance, Chengdu, 610074, P.R. China
Email: linke@swufe.edu.cn

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Abstract

This article is devoted to the analysis of the parabolic–parabolic chemotaxis system with multi-components over $\mathbb{R}^2$. The optimal small initial condition on the global existence of solutions for multi-species chemotaxis model in the fully parabolic situation had not been attained as far as the author knows. In this paper, we prove that under the sub-critical mass condition, any solutions to conflict-free system exist globally. Moreover, the global existence of solutions to system with strong self-repelling effect has been discussed even for large initial data. The proof is based on the modified free energy functional and the Moser–Trudinger inequality for system.

1. Introduction

The well-known classical parabolic–parabolic Keller–Segel model reads as [24]

$$\begin{cases} \partial_t u = \Delta u - \alpha \nabla \cdot (u \nabla v), & x \in \mathbb{R}^2, \quad t > 0, \\ \tau \partial_t v = \Delta v - \beta v + \gamma u, & x \in \mathbb{R}^2, \quad t > 0, \end{cases} \tag{1.1}$$

where $u = u(x, t)$ and $v = v(x, t)$ denote the cell density and the concentration of the chemical substance, respectively. α and γ are positive constants. The constants τ and β are non-negative. The system (1.1) can be regarded as one of the simplest models to describe the overall behaviour of cells under the influence of chemotaxis, that is the motion of cells partially orient their movement towards higher concentration of a certain chemical substance produced by cells themselves. A striking feature of the Keller–Segel system is that the behaviour of solutions is determined by the total mass of cells which remains constant over time, see [5, 16, 31, 34] for instance. Namely, given a non-negative and suitable smooth initial data u_0, any solution with $m = \|u_0\|_{L^1(\mathbb{R}^2)} < 8\pi/(\alpha\gamma)$ exists globally, while blow-up solution appears if $m > 8\pi/(\alpha\gamma)$. Note that the main idea to prove the global existence is based on the following free energy functional,

$$\mathcal{F}_{KS} = \int_{\mathbb{R}^2} u \log u dx + \frac{\alpha}{2\gamma} \int_{\mathbb{R}^2} \left(|\nabla v|^2 + \beta v^2 \right) dx - \alpha \int_{\mathbb{R}^2} uv dx,$$

which is a monotonic non-increasing function with respect to time variable. In view of this fact, Calvez and Corrias use a minimisation principle for entropy functionals and Onofri's inequality to derive a priori estimates under the sub-critical mass $m < 8\pi/(\alpha\gamma)$, where the assumptions $u_0 \log\left(1 + |x|^2\right) \in L^1(\mathbb{R}^2)$ and $u_0 \log u_0 \in L^1(\mathbb{R}^2)$ are necessary [5], while these extra assumptions have been removed by applying a modified free energy functional with the Moser–Trudinger inequality in unbounded domain [31].

For the parabolic–elliptic Keller–Segel system (i.e. taking $\tau = 0$ in $(1.1)_2$)

$$0 = \Delta v - \beta v + \gamma u,$$

the above two-dimensional mass threshold phenomenon also exists. See [4, 8, 33] for the global well-posedness results and [3, 4] for the blow-up arguments. The main feature to prove the global existence of solutions in this simplified chemotaxis system over (1.1) is that v could be expressed by the fundamental solution of the elliptic equation, then it leads to a single parabolic problem for u. For example, if $\beta = 0$, an explicit expression for v takes form like $v = \gamma K * u$, so $(1.1)_1$ becomes

$$\partial_t u = \Delta u - \alpha\gamma \nabla \cdot (u \nabla K * u), \quad x \in \mathbb{R}^2, \quad t > 0,$$

where $K = -(1/2\pi) \log |\cdot|$. A direct application of the logarithmic Hardy–Littlewood–Sobolev inequality (see [2]) on the corresponding free energy yields the global existence of solutions if $m < 8\pi/(\alpha\gamma)$ [4].

Compared with the one-population chemotaxis system (1.1), an interesting and complex question is to derive sharp conditions to recognise global existence and blow-up of solutions for the following multi-species chemotaxis model in $\mathbb{R}^2$,

$$\begin{cases} \partial_t u_i = \Delta u_i - \displaystyle\sum_{j=1}^{m} \alpha_{i,j} \nabla \cdot (u_i \nabla v_j), & i \in \mathcal{I} = \{1, \cdots, n\}, \\ \tau_j \partial_t v_j = \Delta v_j - \beta_j v_j + \displaystyle\sum_{i=1}^{n} \gamma_{i,j} u_i, & j \in \mathcal{J} = \{1, \cdots, m\}, \end{cases} \tag{1.2}$$

where $\tau_j \geq 0$, $j \in \mathcal{J}$. This model was first proposed by Wolansky in [42] to describe the chemotactic movement of n populations with respect to m chemical substances. Here, $u_i = u_i(x, t)$ denotes the density of i-th population, and $v_j = v_j(x, t)$ represents the concentration of j-th chemical signal. The total number of species $|\mathcal{I}| = n \geq 1$ is assumed to be finite. $\boldsymbol{\alpha} = (\alpha_{i,j})_{n\times m}$ and $\boldsymbol{\gamma} = (\gamma_{i,j})_{n\times m}$ define a pair of $n \times m$ matrices for the sensitivity parameter and the production/consumption rate, respectively. $\beta_j \in \mathbb{R}$, $j \in \mathcal{J}$, presents the growth/degradation rate for chemical substance. We introduce $\boldsymbol{\beta} = (\beta_{j,l})_{m\times m}$ with $\beta_{j,l} = \beta_j \delta_{j,l}$ as a $m \times m$ diagonal matrix for convenience, where $\delta_{j,l}$ satisfies

$$\delta_{j,l} = \begin{cases} 1, & \text{if} \quad j = l, \\ 0, & \text{if} \quad j \neq l. \end{cases}$$

It is very important to understand the multi-species chemotaxis in biology, and this phenomenon has been observed in numerous experiments. We take the following two examples. First one is that a system with two different species, reacting on one common chemical, appears in the cell sorting process during the early aggregation state of mound formation [40]. And a two-species chemotaxis system with two chemicals has been proposed in [25] to imitate the breast cancer metastatic process. The readers could see [20, 21] for other biological motivations.

Just recently, some authors have started to look more closely at the parabolic–elliptic case of (1.2) (i.e. $\tau_j = 0$) for n-populations interacting via a self-produced chemical agent. Consider (1.2) with $|\mathcal{I}| = |\mathcal{J}| = n$ is subject to symmetric sensitivity coefficients matrix $\boldsymbol{\alpha} = (\alpha_{i,j})_{n\times n}$ with non-negative entries, zero matrix $\boldsymbol{\beta}$ and unit matrix $\boldsymbol{\gamma}$, that is,

$$\begin{cases} \partial_t u_i = \Delta u_i - \displaystyle\sum_{j=1}^{n} \alpha_{i,j} \nabla \cdot (u_i \nabla v_j), \\ -\Delta v_j = u_j, \quad i, j \in \mathcal{I} = \{1, \cdots, n\}. \end{cases} \tag{1.3}$$

Karmakar and Wolansky [22] had derived the global well-posedness of weak solutions with respect to time in the sub-critical regime

$$\Lambda_{\mathcal{K}}(\boldsymbol{m}) := 8\pi \sum_{i\in\mathcal{K}} m_i - \sum_{i,k\in\mathcal{K}} \alpha_{i,k} m_i m_k > 0, \quad \forall\, \emptyset \neq \mathcal{K} \subset \mathcal{I}, \tag{1.4}$$

where $m_i = \|u_0\|_{L^1(\mathbb{R}^2)}$. Furthermore, the borderline case of critical mass $\Lambda_{\mathcal{I}}(\boldsymbol{m}) = 0$, and $\Lambda_{\mathcal{K}}(\boldsymbol{m}) > 0$, $\forall\, \emptyset \neq \mathcal{K} \subsetneq \mathcal{I}$, has been considered in [23]. It shows that a free energy solution exists globally in time. According to analogous results about (1.1) mentioned above, it is expected that if the condition $\Lambda_{\mathcal{K}}(\boldsymbol{m}) \geq 0$ for some $\emptyset \neq \mathcal{K} \subset \mathcal{I}$ is violated, a finite-time blow-up solution appears. Using the second-moment techniques in [15], some solutions of (1.3) blow-up in finite time provided $\Lambda_{\mathcal{I}}(\boldsymbol{m}) < 0$. While the basic strategy to prove global existence is the logarithmic Hardy–Littlewood–Sobolev inequality for system, see [38] for details. In particular, in the case of parabolic–elliptic system (1.2) with $|\mathcal{I}| = 2$, $|\mathcal{J}| = 1$, Espejo et al. [7, 12] discovered a threshold curve to help us to determine whether the solutions of system are blow-up or global in time. See related works for parabolic–elliptic system (1.2) with $|\mathcal{I}| = |\mathcal{J}| = 2$ in [18, 19]. Moreover, [9–11, 13] could be refereed to characterise the simultaneous or non-simultaneous blow-up results in two-species model.

However, it should be noted that fewer papers have been considered on Cauchy problem of the fully parabolic multi-species (i.e. $\tau_j > 0$ in (1.2)) than the parabolic–elliptic case. For a two-dimensional bounded domain, the author and coauthors have researched the initial boundary problems of (1.2), and we have tried to find optimal conditions on the global existence or blow-up in [27–30]. In this article, under a conflict-free situation given by Definition 1 (*ii*), a sufficient (or possibly optimal) condition on the global solvability of the Cauchy problem for parabolic–parabolic system (1.2) with arbitrary $|\mathcal{I}| = n \geq 1$ and $|\mathcal{J}| = m \geq 1$ has been obtained. For simplicity, we assume $\tau_j = 1$ for all $j \in \mathcal{J}$ in (1.2) and consider

$$\begin{cases} \partial_t u_i = \Delta u_i - \sum_{j=1}^{m} \alpha_{i,j} \nabla \cdot (u_i \nabla v_j), & i \in \mathcal{I}, \\ \partial_t v_j = \Delta v_j - \beta_j v_j + \sum_{i=1}^{n} \gamma_{i,j} u_i, & j \in \mathcal{J}, \\ u_{i0}(x) = u_i(x,0),\, v_{j0}(x) = v_j(x,0), & i \in \mathcal{I}, \quad j \in \mathcal{J}, \end{cases} \tag{1.5}$$

with initial data $\boldsymbol{u}_0 = (u_{10}, \cdots, u_{n0})$ and $\boldsymbol{v}_0 = (v_{10}, \cdots, v_{m0})$ satisfying

$$\begin{cases} u_{i0}(x) \in C^0(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^1\left(\mathbb{R}^2, \log\left(1 + |x|^2\right) dx\right) \cap L^\infty(\mathbb{R}^2), \\ u_{i0} \geq 0 \quad \text{and} \quad u_{i0} \not\equiv 0, \quad i \in \mathcal{I}, \\ v_{j0}(x) \in W^{1,p}(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2) \quad \text{with some } p > 2, \quad j \in \mathcal{J}. \end{cases} \tag{1.6}$$

Before stating our main results, let us go over some notations and definitions in [42].

$$\lambda_{i,k} := \sum_{j=1}^{m} \alpha_{i,j} \gamma_{k,j} = \boldsymbol{\alpha}_i^T \cdot \boldsymbol{\gamma}_k, \quad i, k \in \mathcal{I},$$

is the number to quantify the tendency of a population i towards a population k by taking an accounting of the action of all the agents, where $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \cdots, \alpha_{i,m})^T$ and $\boldsymbol{\gamma}_i = (\gamma_{i,1}, \cdots, \gamma_{i,m})^T$. The condition $\lambda_{i,k} > 0$ means that a population i is attracted by a population k; otherwise, the population i is repelled from the population k if $\lambda_{i,k} < 0$. Especially, a population is self-attracting (resp. self-repelling) if $\lambda_{i,i} > 0$ (resp. $\lambda_{i,i} < 0$). A pair (i, k) of populations $i, k \in I$ is said to be in a conflict if $\lambda_{i,k} \times \lambda_{k,i} < 0$. In general, $\boldsymbol{\lambda} = (\lambda_{i,k})_{i,k\in\mathcal{I}}$ is not symmetric. We assume that there exist nonzero constants $a_1, \cdots, a_n$ satisfying

$$a_i \lambda_{i,k} = a_k \lambda_{k,i}, \quad i, k \in \mathcal{I}, \tag{1.7}$$

then $D_a\boldsymbol{\lambda}$ is symmetric, where $D_a = \mathrm{Diag}\{a_1, \cdots, a_n\}$. If $\boldsymbol{\lambda}$ is non-singular, there exists a $m \times m$ symmetric matrix $\boldsymbol{B}$ which transforms $\boldsymbol{\gamma}_i$ into $a_i\boldsymbol{\alpha}_i$ for all $i \in \mathcal{I}$, i.e.

$$\boldsymbol{B}\boldsymbol{\gamma}_i = a_i\boldsymbol{\alpha}_i, \quad i \in \mathcal{I}. \tag{1.8}$$

In fact, denote

$$\boldsymbol{\alpha} = (\alpha_{i,j})_{n\times m} = \begin{bmatrix} \boldsymbol{\alpha}_1^T \\ \vdots \\ \boldsymbol{\alpha}_n^T \end{bmatrix}, \quad \boldsymbol{\gamma} = (\gamma_{i,j})_{n\times m} = \begin{bmatrix} \boldsymbol{\gamma}_1^T \\ \vdots \\ \boldsymbol{\gamma}_n^T \end{bmatrix}.$$

First, we observe that the ranks $R(\boldsymbol{\alpha}) = R(\boldsymbol{\gamma}) = n$ due to $\boldsymbol{\lambda} = \boldsymbol{\alpha}\boldsymbol{\gamma}^T$ is non-singular. Then, there exists a solution $X = (x_{i,j})_{m\times m}$ to a linear system of equations $\boldsymbol{\gamma}X = D_a\boldsymbol{\alpha}$ since both the ranks of its coefficient matrix and augmented matrix equal to n. Finally, the choice $\boldsymbol{B} = X^T$ fulfils (1.8). Moreover, using the symmetry of $D_a\boldsymbol{\lambda}$, one is able to show that $\boldsymbol{\gamma}\boldsymbol{B}^T\boldsymbol{\gamma}^T = \boldsymbol{\gamma}\boldsymbol{B}\boldsymbol{\gamma}^T$. This implies that $\boldsymbol{B}$ can be symmetric.

Now we give the following definitions throughout this paper.

Definition 1. (i) *A pair (i, k) of populations $i, k \in \mathcal{I}$ is said to be in a conflict if $\lambda_{i,k} \times \lambda_{k,i} < 0$. All populations are mutually attractive if $\lambda_{i,k} > 0$, $\forall\, i, k \in \mathcal{I}$.*

(ii) *System (1.5) is called a conflict-free system if $\lambda_{i,k} \times \lambda_{k,i} > 0$, $\forall\, i, k \in \mathcal{I}$, and if there exist positive constants $a_1, \cdots, a_n$ such that (1.7) is valid.*

The main result of this article is stated as follows.

Theorem 1.1. *Let $\boldsymbol{\gamma} = (\gamma_{i,j})_{n\times m}$, $\boldsymbol{\lambda} = (\lambda_{i,k})_{n\times n}$, $\boldsymbol{\alpha} = (\alpha_{i,j})_{n\times m}$ with full column rank $R(\boldsymbol{\alpha}) = m$, and $\boldsymbol{\beta} = (\beta_{j,l})_{m\times m}$ with $\beta_{j,l} = \beta_j\delta_{j,l}$, $\beta_j \in \mathbb{R}$, $j, l \in \mathcal{J}$. Assume that (1.5) is a conflict-free system with initial data $(\boldsymbol{u}_0, \boldsymbol{v}_0)$ satisfying (1.6). Suppose that there exist positive constants $a_1, \cdots, a_n$ and a positive definite matrix $\boldsymbol{R} = (r_{i,k})_{n\times n}$ with $r_{i,k} \geq 0$, $i, k \in \mathcal{I}$, such that*

$$\boldsymbol{\alpha}^T\boldsymbol{R}^{-1}\boldsymbol{\alpha}\boldsymbol{\gamma}_i = a_i\boldsymbol{\alpha}_i, \quad \forall\; i \in \mathcal{I}. \tag{1.9}$$

Then for any initial data satisfying

$$\Lambda_{\mathcal{K}}^{\boldsymbol{a}}(\boldsymbol{m}) =: 8\pi \sum_{i\in\mathcal{K}} a_i m_i - \sum_{i,k\in\mathcal{K}} a_i a_k r_{i,k} m_i m_k > 0, \quad \forall\; \emptyset \neq \mathcal{K} \subset \mathcal{I}, \tag{1.10}$$

the corresponding initial boundary value problem (1.5) possesses a unique smooth global solution.

We would like to give an explanation for assumptions in Theorem 1.1. First, since $\boldsymbol{\alpha} = (\alpha_{i,j})_{n\times m}$ is required to be full column rank, it ensures that $\boldsymbol{B} = (b_{j,l})_{m\times m} = \boldsymbol{\alpha}^T\boldsymbol{R}^{-1}\boldsymbol{\alpha}$ is a positive definite matrix if $\boldsymbol{R}$ is chosen to be positive definition. Condition (1.9) can probably be viewed as one of necessary condition for the existence of energy functional for the conflict-free system (1.5) (see [20, 30, 42]). In order to handle the whole domain case better in this paper, we use a modified free energy functional $\mathcal{F}$ as

$$\mathcal{F}[\boldsymbol{u}, \boldsymbol{v}] = \sum_{i=1}^n a_i \int_{\mathbb{R}^2} (u_i + 1)\log(u_i + 1)dx + \frac{1}{2}\sum_{j=1}^m\sum_{l=1}^m b_{j,l}\int_{\mathbb{R}^2}\left(\nabla v_j \cdot \nabla v_l + \beta_l v_j v_l\right)dx$$
$$- \sum_{i=1}^n\sum_{j=1}^m a_i\alpha_{i,j}\int_{\mathbb{R}^2} u_i v_j dx.$$

Second, condition (1.10) can be regarded as an optimality condition to guarantee the global existence of solution to (1.5). This is because if $D_a\boldsymbol{\lambda}$ is a positive definite matrix with $\lambda_{i,k} \geq 0$, $\forall\, i, k \in \mathcal{I}$, then (1.10) is actually equivalent to the following sub-critical mass condition obtained in [30, 42]

$$8\pi \sum_{i\in\mathcal{K}} a_i m_i - \sum_{i,k\in\mathcal{K}} a_i\lambda_{i,k} m_i m_k > 0, \quad \forall\; \emptyset \neq \mathcal{K} \subset \mathcal{I}.$$

Indeed, let $\boldsymbol{R}=(r_{i,k})_{n\times n}=D_{\boldsymbol{a}}^{-1}\boldsymbol{\lambda}^T$. Then in terms of $\lambda_{i,k}=\boldsymbol{\alpha}_i^T\cdot\boldsymbol{\gamma}_k$ and $R(\boldsymbol{\alpha})=m$, $\boldsymbol{B}=\boldsymbol{\alpha}^T\boldsymbol{R}^{-1}\boldsymbol{\alpha}$ is a positive definite matrix which satisfies $\boldsymbol{B}\boldsymbol{\gamma}_i=a_i\boldsymbol{\alpha}_i$, $\forall\ i\in\mathcal{I}$. Moreover, we obtain

$$8\pi\sum_{i\in\mathcal{K}}a_im_i-\sum_{i,k\in\mathcal{K}}a_i\lambda_{i,k}m_im_k=8\pi\sum_{i\in\mathcal{K}}a_im_i-\sum_{i,k\in\mathcal{K}}a_ia_kr_{i,k}m_im_k>0,\quad\forall\ \emptyset\neq\mathcal{K}\subset\mathcal{I},$$

from $r_{i,k}=\lambda_{k,i}/a_i\geq 0$, $i,k\in\mathcal{I}$.

Theorem 1.1 gives a sharp criterion on the global existence of the general chemotaxis system (1.5). Hence, a large amount of known global existence results are particular cases in our paper. We give several typical examples here. It is obvious that the sub-critical mass condition (1.10) recovers the threshold condition, i.e. $m<8\pi/(\alpha\gamma)$, for global regularity of the Keller–Segel model (1.1). When $|\mathcal{I}|=2$, $|\mathcal{J}|=1$, consider a chemotaxis system involving two species that interact via one-single chemical signal [40]

$$\begin{cases}\partial_tu_i=\Delta u_i-\alpha_{i,1}\nabla\cdot(u_i\nabla v_1),\quad i\in\mathcal{I}=\{1,2\},\\ \partial_tv_1=\Delta v_1-v_1+\gamma_{1,1}u_1+\gamma_{2,1}u_2,\end{cases}\tag{1.11}$$

with $\alpha_{i,1}>0$, $\gamma_{i,1}=1$, $i=1,2$. Note that $\lambda_{i,k}=\alpha_{i,1}>0$, $i,k=1,2$. Then, (1.9) can be satisfied if one takes $a_i=1/\alpha_{i,1}$ and chooses a positive definite matrix

$$\boldsymbol{R}=\begin{bmatrix}\alpha_{1,1}^2(1+\epsilon) & \alpha_{1,1}\alpha_{2,1}(1-\epsilon)\\ \alpha_{1,1}\alpha_{2,1}(1-\epsilon) & \alpha_{2,1}^2(1+\epsilon)\end{bmatrix}$$

for some small $\epsilon\in(0,1)$. Then, (1.10) reads as

$$\begin{cases}8\pi>\alpha_{1,1}(1+\epsilon)m_1,\\ 8\pi>\alpha_{2,1}(1+\epsilon)m_2,\\ 8\pi\left(\dfrac{m_1}{\alpha_{1,1}}+\dfrac{m_2}{\alpha_{2,1}}\right)>(m_1+m_2)^2+\epsilon(m_1-m_2)^2.\end{cases}\tag{1.12}$$

However, (1.12) can be simplified as $m_1<8\pi/\alpha_{1,1}$, $m_2<8\pi/\alpha_{2,1}$ and $(m_1+m_2)^2<8\pi\big(m_1/\alpha_{1,1}+m_2/\alpha_{2,1}\big)$ by letting $\epsilon\to 0$, which coincides with global existence condition for (1.11) in a bounded domain ([27, Theorem 1.1]).

Now suppose that $|\mathcal{I}|=|\mathcal{J}|=n$, $\boldsymbol{\beta}=\boldsymbol{0}$, $\boldsymbol{\alpha}=(\alpha_{i,j})_{n\times n}$ with $\alpha_{i,j}\geq 0$ is positive definite, and $\boldsymbol{\gamma}$ is an unit matrix. Then, (1.5) becomes

$$\begin{cases}\partial_tu_i=\Delta u_i-\sum_{j=1}^{n}\alpha_{i,j}\nabla\cdot(u_i\nabla v_j),\\ \partial_tv_j=\Delta v_j+u_j,\quad i,j\in\mathcal{I}=\{1,\cdots,n\}.\end{cases}\tag{1.13}$$

Taking $a_i=1$ and $\lambda_{i,k}=\alpha_{i,k}$, $i,k\in\mathcal{I}$, one can find that Cauchy problem (1.13) has a global solution if (1.4) is valid. This global result is similar to that of the parabolic–elliptic system (1.3) [15, 22].

The idea to show the global existence is to derive an a prior estimate for modified total entropy

$$\mathcal{S}[\boldsymbol{u}]=\sum_{i=1}^{n}\|(u_i+1)\log(u_i+1)\|_{L^1(\mathbb{R}^2)}.$$

For this purpose, we need to give a lower bound for $\mathcal{F}$. In this situation, the last term consisting of u_i and v_j in $\mathcal{F}$ could be controlled by $\mathcal{S}$ and the last second term under (1.10). For the case of single variable, a common approach to achieve this goal is to use the well-known Moser–Trudinger inequality [32, 39]

$$\frac{1}{2}\int_\Omega|\nabla\rho|^2dx-8\pi\log\left(\int_\Omega e^\rho dx\right)\geq -C,\quad\forall\ \rho\in H_0^1(\Omega),\tag{1.14}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain. For initial Neumann boundary value problem (1.1), Nagai et al. [36] had used a version of (1.14) in the Sobolev space $W^{1,2}$ (see also [6]) to obtain the global existence of solution if $m < 8\pi/(\alpha\gamma)$. Later, Mizoguchi [31] applies (1.14) to get a similar global result for Cauchy problem (1.1) in $\mathbb{R}^2$. Hence, we expect the Moser–Trudinger inequality for vector is valid for our problem. Shafrir and Wolansky [38] had proved the following Moser–Trudinger inequality for system. For $\forall\ \rho_i \in H^1(\mathbb{S}^2)$ satisfying $\int_{\mathbb{S}^2} \rho_i = 0,\ \ i \in \mathcal{I}$, there exists a constant $C > 0$ such that

$$\Phi_{\mathbb{S}^2}(\boldsymbol{\rho}) = \frac{1}{2}\sum_{i,k\in\mathcal{I}} s_{i,k}\int_{\mathbb{S}^2} \nabla\rho_i\cdot\nabla\rho_k - \sum_{i\in\mathcal{I}} M_i \log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2}\exp\left(\sum_{k\in\mathcal{I}} s_{i,k}\rho_k\right)\right) \geq -C$$

if and only if

$$\begin{cases} \Lambda^{S}_{\mathcal{K}}(\boldsymbol{M}) \geq 0, \quad \forall\, \emptyset \neq \mathcal{K} \subset \mathcal{I}, \\ \text{if } \Lambda^{S}_{\mathcal{K}}(\boldsymbol{M}) = 0 \text{ for some } \mathcal{K}, \text{ then } s_{i,i} + \Lambda^{S}_{\mathcal{K}\setminus\{i\}}(\boldsymbol{M}) > 0,\ \forall\, i \in \mathcal{K}, \end{cases}$$

where $\mathbb{S}^2 \subset \mathbb{R}^3$ is the unit sphere, $\mathcal{I} = \{1, 2, \cdots, n\}$, $\boldsymbol{M} := \{M_1, \cdots, M_n\} \in (\mathbb{R}_+)^n$, $\boldsymbol{S} := (s_{i,k})_{n\times n}$ is a positive definite matrix with $s_{i,k} \geq 0$, $i, k \in \mathcal{I}$, and $\Lambda^{S}_{\mathcal{K}}$ is given by

$$\Lambda^{S}_{\mathcal{K}}(\boldsymbol{M}) = 8\pi\sum_{i\in\mathcal{K}} M_i - \sum_{i,k\in\mathcal{K}} s_{i,k}M_iM_k, \quad \forall\ \emptyset \neq \mathcal{K} \subset \mathcal{I}.$$

We will firstly transform the above Moser–Trudinger inequality for system to $\mathbb{R}^2$ via the stereographic projection and next use it to show that $\mathcal{S}$ is bounded under the assumption (1.10). Then, one invokes the Moser iterative to obtain the global existence of solutions to (1.5). Finally, we should point out that such idea has been used to establish the global well-posedness of solutions to initial Neumann boundary value problem for multi-species and chemicals in a two-dimensional bounded domain [30].

Our second object is to show certain conflict-free parabolic system admits a global solution for any initial data. More precisely,

Theorem 1.2. *Let* $\boldsymbol{\alpha} = (\alpha_{i,j})_{n\times m}$, $\boldsymbol{\gamma} = (\gamma_{i,j})_{n\times m}$, $\boldsymbol{\lambda} = (\lambda_{i,k})_{n\times n}$ *and* $\boldsymbol{\beta} = (\beta_{j,l})_{m\times m}$ *with* $\beta_{j,l} = \beta_j\delta_{j,l}$, $\beta_j \in \mathbb{R}$, $j, l \in \mathcal{J}$. *Assume that* (1.5) *is a conflict-free system with initial data* $(\boldsymbol{u}_0, \boldsymbol{v}_0)$ *satisfying* (1.6). *Suppose that there exist positive constants* $a_1, \cdots, a_n$ *and a positive definite matrix* $\boldsymbol{B} = (b_{j,l})_{m\times m}$ *such that*

$$\boldsymbol{B}\boldsymbol{\gamma}_i = -a_i\boldsymbol{\alpha}_i, \quad \forall\ i \in \mathcal{I},$$

then Cauchy problem (1.5) *possesses a unique smooth global solution.*

Remark 1. *In addition, if* $\beta_j > 0$ *for all* $j \in \mathcal{J}$ *in Theorem* 1.2, *then the solution to* (1.5) *is uniformly bounded with respect to time.*

Remark 2. *As mentioned above, the existence of* $\boldsymbol{B}$ *in Theorem* 1.2 *can be ensured if* $D_{\boldsymbol{a}}\boldsymbol{\lambda}$ *is negative definite and* $R(\boldsymbol{\alpha}) = m$. *Hence, one may assert that there exists a unique global smooth solution, provided that the self-repelling effects are strong enough in the sense that* $\lambda_{ii} < 0$, $i \in \mathcal{I}$, *is negative sufficiently large. Following are two prototypical examples. Consider* (1.1) *with* $\alpha < 0, \gamma > 0$, *the local solution can in fact be extended for all times. As an application of Theorem* 1.2 *on two-species system* (1.11) *with* $\alpha_{i,1} < 0$, $\gamma_{i,1} > 0$, $i = 1, 2$, *one would derive the global stability by choosing* $a_1 = -1/(\alpha_{1,1}\gamma_{2,1})$, $a_2 = -1/(\alpha_{2,1}\gamma_{1,1})$ *and* $\boldsymbol{B} = (1/(\gamma_{1,1}\gamma_{2,1}))_{1\times 1}$.

Compared with the proof of Theorem 1.1, the main difference to prove Theorem 1.2 is to derive a prior estimates for the modified total entropy $\mathcal{S}$ through the following modified free energy functional

$$\mathcal{G}[\boldsymbol{u}, \boldsymbol{v}] = \sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}(u_i + 1)\log(u_i + 1)dx + \frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\left(\nabla v_j\cdot\nabla v_l + \beta_l v_j v_l\right)dx.$$

This paper is organised as follows. In Section 2, we would like to establish the local existence of smooth solutions and present some basic inequalities. Theorem 1.1 will be proved in Section 3. The existence of modified free energy functional $\mathcal{F}$ will be shown firstly, then the stepwise bounds on the total entropy, L^2 and L^∞ norms under the condition (1.10) will end the proof of this theorem. In Section 4, we will prove Theorem 1.2 by making use of $\mathcal{G}$. In appendix, the proof of Lemma 2.1 will be contained.

We introduce some notations which will be used later. Let $|\mathcal{I}| = n \geq 1$ denote the total number of species, and $|\mathcal{J}| = m \geq 1$ represent the total number of chemical substances. $|\Omega|$ is the Lebesgue measure of $\Omega \subset \mathbb{R}^2$. For $\alpha_{i,j}$, $\gamma_{i,j}$, $b_{i,j}$, $r_{i,k}$, $t_{i,j}$, a_i, $\beta_j \in \mathbb{R}$, $i, k \in \mathcal{I}$, $j \in \mathcal{J}$, we define

$$\alpha^* = \max_{i\in\mathcal{I},j\in\mathcal{J}} \{|\alpha_{i,j}|\},\ \ \gamma^* = \max_{i\in\mathcal{I},j\in\mathcal{J}} \{|\gamma_{i,j}|\},\ \ b^* = \max_{i\in\mathcal{I},j\in\mathcal{J}} \{|b_{i,j}|\},\ \ r^* = \max_{i,k\in\mathcal{I}} \{|r_{i,k}|\},$$
$$t^* = \max_{i\in\mathcal{I},j\in\mathcal{J}} \{|t_{i,j}|\},\ \ a^* = \max_{i\in\mathcal{I}} \{|a_i|\},\ \ \beta^* = \max_{j\in\mathcal{J}} \{|\beta_j|\},\ \ \beta_* = \min_{j\in\mathcal{J}} \{\beta_j\} \text{ if } \beta_j > 0.$$

2. Preliminaries

In this section, we list some lemmas which will be frequently used throughout this paper. Under some certain assumption on the initial data, we assert that Cauchy problem (1.5) admits a local classical solution. A number of fundamental properties, such as uniqueness, positivity, and regularity, have also been obtained in the following lemma.

Lemma 2.1. *Suppose that* $\boldsymbol{u}_0 \in [C^0(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)]^n$ *and* $\boldsymbol{v}_0 \in [W^{1,p}(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2)]^m$ *for some* $p > 2$. *Then, there exists a positive constant* $T_{\max} \in (0, \infty]$ *such that the Cauchy problem* (1.5) *has a unique solution* $(\boldsymbol{u}, \boldsymbol{v})$ *satisfying*

$$\begin{cases} \boldsymbol{u} \in \left[C^0([0, T_{\max}]; L^1(\mathbb{R}^2)) \cap C^{2,1}(\mathbb{R}^2 \times (0, T_{\max}))\right]^n, \\ \boldsymbol{v} \in \left[C^0([0, T_{\max}]; W^{1,p}(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2)) \cap C^{2,1}(\mathbb{R}^2 \times (0, T_{\max}))\right]^m. \end{cases} \tag{2.1}$$

Moreover, it holds that

(i) $(\boldsymbol{u}, \boldsymbol{v})$ *solves* (1.5) *classically in* $\mathbb{R}^2 \times (0, T_{\max})$;

(ii) If $T_{\max} < \infty$, *then*

$$\limsup_{t\to T_{\max}} \left(\sum_{i=1}^n \|u_i(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \right) = \infty;$$

(iii) $u_i > 0$ *in* $\mathbb{R}^2 \times (0, T_{\max})$, $i \in \mathcal{I}$;

(iv) For $t \in (0, T_{\max})$, $\|u_i(\cdot, t)\|_{L^1(\mathbb{R}^2)} = \|u_{i0}\|_{L^1(\mathbb{R}^2)} = m_i$, $i \in \mathcal{I}$;

(v) For $q \geq 1$, $T \in (0, T_{\max})$, *then there exists a constant* $A_q = A_q(q, \beta^*, \gamma^*, \|\boldsymbol{u}_0\|_{L^1(\mathbb{R}^2)}, \|\boldsymbol{v}_0\|_{L^q(\mathbb{R}^2)}, T) > 0$ *such that*

$$\sum_{j=1}^m \|v_j(\cdot, t)\|_{L^q(\mathbb{R}^2)} \leq A_q, \quad t \in (0, T).$$

Moreover, A_q *is independent of* T *if* $\beta_j > 0$ *for all* $j \in \mathcal{J}$.

Proof. For the case $|\mathcal{I}| = |\mathcal{J}| = 1$ to system (1.5) in $\mathbb{R}^n$ $(n \geq 3)$, Winkler has proved these properties in [41, Proposition 1.1]. One can apply similar arguments to obtain the desired results. Please see the proof in appendix. □

The following inequalities are very important to derive a prior estimates for solutions.

Lemma 2.2. *Let* $\eta \in (0, 1)$. *Then for any non-negative function* $f \in L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, *one has*

$$\int_{\mathbb{R}^2} (f+1)\log(f+1)dx \leq \eta \left(\int_{\mathbb{R}^2} f dx \right) \left(\int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f+1} dx \right) + c \int_{\mathbb{R}^2} f dx, \tag{2.2}$$

$$\int_{\mathbb{R}^2} f^2 dx \le \frac{1+\eta}{4\pi}\left(\int_{\mathbb{R}^2} f dx\right)\left(\int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f+1} dx\right) + \frac{2}{\eta}\int_{\mathbb{R}^2} f dx, \tag{2.3}$$

$$\int_{\mathbb{R}^2} f^3 dx \le \eta\left(\int_{\mathbb{R}^2} (f+1)\log(f+1) dx\right)\left(\int_{\mathbb{R}^2} |\nabla f|^2 dx\right) + c\int_{\mathbb{R}^2} f dx, \tag{2.4}$$

where $c = c(\eta) \to \infty$ *as* $\eta \to 0$.

Proof. Inequality (2.2) has been shown in [43, Lemma 2.3], and (2.3)–(2.4) have been proved in [37, Lemmas 2.3–2.4]. □

Now, the Gagliardo–Nirenberg inequality will be given as follows.

Lemma 2.3. *Let* $1 \le p < \infty$, $1 \le q, r \le \infty$ *and* $\theta \in [0, 1]$ *such that*

$$\frac{1}{p} = \theta\left(\frac{1}{r} - \frac{1}{2}\right) + (1-\theta)\frac{1}{q}.$$

Then for any $u(x) \in W^{1,r}(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$, *there exists a constant* $c = c(p, q, r) > 0$ *such that*

$$\|u\|_{L^p(\mathbb{R}^2)} \le c\|u\|_{L^q(\mathbb{R}^2)}^{1-\theta}\|\nabla u\|_{L^r(\mathbb{R}^2)}^{\theta}.$$

Proof. The proof of this lemma has been given in [14, Theorem 6.1.1]. □

3. Proof of Theorem 1.1

In this section, the proof of Theorem 1.1 will be divided into several steps.

Now we would like to give an equality for the modified free energy functional $\mathcal{F}$. Notice that the equality for one-single variable can be found in [34, Proposition 4.1].

Lemma 3.1. *Let* $\alpha_{i,j}, \beta_j, \gamma_{i,j} \in \mathbb{R}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$ *and* $T > 0$. *Let* $(\boldsymbol{u}, \boldsymbol{v})$ *be a local solution of* (1.5) *with initial data* $(\boldsymbol{u}_0, \boldsymbol{v}_0)$. *Assume that there exist positive constants* $a_1, \cdots, a_n$ *and a positive definite matrix* $\boldsymbol{B} = (b_{j,l})_{m\times m}$ *such that*

$$\boldsymbol{B}\boldsymbol{\gamma}_i = a_i\boldsymbol{\alpha}_i, \quad \forall\ i \in \mathcal{I}. \tag{3.1}$$

Then,

$$\begin{aligned} &\frac{d}{dt}\mathcal{F}[\boldsymbol{u},\boldsymbol{v}] + \int_{\mathbb{R}^2} (\partial_t\boldsymbol{v})^T \boldsymbol{B}\,(\partial_t\boldsymbol{v})\,dx + \sum_{i=1}^n a_i \int_{\mathbb{R}^2}\left\{u_i\left|\nabla\left(\log(u_i+1) - \sum_{j=1}^m \alpha_{i,j}v_j\right)\right|^2\right.\\ &\qquad\left.+\left|\nabla\left(\log(u_i+1) - \frac{1}{2}\sum_{j=1}^m \alpha_{i,j}v_j\right)\right|^2\right\}dx \\ &=\frac{1}{2}\sum_{j=1}^m\sum_{l=1}^m b_{j,l}\beta_l\int_{\mathbb{R}^2}\left(v_j\partial_t v_l - v_l\partial_t v_j\right)dx + \frac{1}{4}\sum_{i=1}^n a_i\int_{\mathbb{R}^2}\left|\nabla\left(\sum_{j=1}^m \alpha_{i,j}v_j\right)\right|^2 dx \end{aligned} \tag{3.2}$$

for $t \in (0, T)$, *where* $\mathcal{F}$ *is given by*

$$\begin{aligned}\mathcal{F}[\boldsymbol{u},\boldsymbol{v}] = &\sum_{i=1}^n a_i\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx - \sum_{i=1}^n\sum_{j=1}^m a_i\alpha_{i,j}\int_{\mathbb{R}^2} u_i v_j dx\\ &+\frac{1}{2}\sum_{j=1}^m\sum_{l=1}^m b_{j,l}\int_{\mathbb{R}^2}\left(\nabla v_j\cdot\nabla v_l + \beta_l v_j v_l\right)dx.\end{aligned}$$

Moreover, there exists a constant $c > 0$ such that

$$\begin{aligned} &\frac{d}{dt}\mathcal{F}[\boldsymbol{u},\boldsymbol{v}] + \frac{1}{2}\int_{\mathbb{R}^2} (\partial_t \boldsymbol{v})^T \boldsymbol{B}(\partial_t \boldsymbol{v})\, dx \\ &\quad + \sum_{i=1}^n a_i \int_{\mathbb{R}^2} \left\{ u_i \left| \nabla \left(\log(u_i+1) - \sum_{j=1}^m \alpha_{i,j} v_j \right) \right|^2 \right. \\ &\quad \left. + \left| \nabla \left(\log(u_i+1) - \frac{1}{2}\sum_{j=1}^m \alpha_{i,j} v_j \right) \right|^2 \right\} dx \\ &\le \frac{1}{4}\sum_{i=1}^n a_i \int_{\mathbb{R}^2} \left| \nabla \left(\sum_{j=1}^m \alpha_{i,j} v_j \right) \right|^2 dx + c, \quad t \in (0,T). \end{aligned} \tag{3.3}$$

Proof. Multiplying both sides of i-th equation in (1.5) by $a_i \log(u_i+1)$, integrating by parts and summing them with respect to i, we have

$$\begin{aligned} &\frac{d}{dt}\left[\sum_{i=1}^n a_i \int_{\mathbb{R}^2} (u_i+1)\log(u_i+1) dx \right] \\ &= -\sum_{i=1}^n a_i \int_{\mathbb{R}^2} \frac{|\nabla u_i|^2}{u_i+1} dx + \sum_{i=1}^n \sum_{j=1}^m a_i \alpha_{i,j} \int_{\mathbb{R}^2} \frac{u_i}{u_i+1} \nabla u_i \cdot \nabla v_j dx. \end{aligned} \tag{3.4}$$

By the symmetry of $\boldsymbol{B}$, it is clear that

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\sum_{j=1}^m \sum_{l=1}^m b_{j,l} \int_{\mathbb{R}^2} \left(\nabla v_j \cdot \nabla v_l + \beta_l v_j v_l \right) dx \right] \\ &= \frac{1}{2}\sum_{j=1}^m \sum_{l=1}^m b_{j,l} \int_{\mathbb{R}^2} \left[\nabla\left(\partial_t v_j\right) \cdot \nabla v_l + \nabla v_j \cdot \nabla(\partial_t v_l) \right] dx \\ &\quad + \frac{1}{2}\sum_{j=1}^m \sum_{l=1}^m b_{j,l}\beta_l \int_{\mathbb{R}^2} \left(v_l \partial_t v_j + v_j \partial_t v_l \right) dx \\ &= \sum_{j=1}^m \sum_{l=1}^m b_{j,l} \int_{\mathbb{R}^2} \nabla\left(\partial_t v_j\right) \cdot \nabla v_l dx + \frac{1}{2}\sum_{j=1}^m \sum_{l=1}^m b_{j,l}\beta_l \int_{\mathbb{R}^2} \left(v_l \partial_t v_j + v_j \partial_t v_l \right) dx \\ &= -\sum_{j=1}^m \sum_{l=1}^m b_{j,l} \int_{\mathbb{R}^2} \left(\partial_t v_j\right) (\Delta v_l - \beta_l v_l)\, dx \\ &\quad + \frac{1}{2}\sum_{j=1}^m \sum_{l=1}^m b_{j,l}\beta_l \int_{\mathbb{R}^2} \left(v_j \partial_t v_l - v_l \partial_t v_j \right) dx, \end{aligned}$$

and then using (3.1) and the j-th equation in (1.5), it implies that

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\left(\nabla v_j\cdot\nabla v_l+\beta_l v_j v_l\right)dx\right]\\
&=-\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\partial_t v_j\partial_t v_l dx+\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\gamma_{i,l}\int_{\mathbb{R}^2} u_i\partial_t v_j dx\\
&\quad+\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\beta_l\int_{\mathbb{R}^2}\left(v_j\partial_t v_l-v_l\partial_t v_j\right)dx\\
&=-\int_{\mathbb{R}^2}(\partial_t \boldsymbol{v})^T\boldsymbol{B}\,(\partial_t\boldsymbol{v})\,dx+\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2} u_i\partial_t v_j dx\\
&\quad+\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\beta_l\int_{\mathbb{R}^2}\left(v_j\partial_t v_l-v_l\partial_t v_j\right)dx.
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{d}{dt}\left(-\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2} u_i v_j dx\right)&=-\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2} v_j\partial_t u_i dx\\
&\quad-\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2} u_i\partial_t v_j dx
\end{aligned}
$$

and

$$
\begin{aligned}
-\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2} v_j\partial_t u_i dx&=-\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2}\left(\Delta u_i-\sum_{l=1}^{m}\alpha_{i,l}\nabla\cdot(u_i\nabla v_l)\right)v_j dx\\
&=\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2}\nabla u_i\cdot\nabla v_j dx\\
&\quad-\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i\left|\nabla\left(\sum_{j=1}^{m}\alpha_{i,j}v_j\right)\right|^2 dx,
\end{aligned}
$$

one can obtain

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\left(\nabla v_j\cdot\nabla v_l+\beta_l v_j v_l\right)dx-\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2} u_i v_j dx\right]\\
&=-\int_{\mathbb{R}^2}(\partial_t\boldsymbol{v})^T\boldsymbol{B}(\partial_t\boldsymbol{v})\,dx+\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2}\nabla u_i\cdot\nabla v_j dx\\
&\quad-\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i\left|\nabla\left(\sum_{j=1}^{m}\alpha_{i,j}v_j\right)\right|^2 dx\\
&\quad+\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\beta_l\int_{\mathbb{R}^2}\left(v_j\partial_t v_l-v_l\partial_t v_j\right)dx,
\end{aligned}
$$

which together with (3.4) yields that

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}[\boldsymbol{u},\boldsymbol{v}] =& -\int_{\mathbb{R}^2} (\partial_t \boldsymbol{v})^T \boldsymbol{B}(\partial_t \boldsymbol{v})\,dx - \sum_{i=1}^n a_i \int_{\mathbb{R}^2} \frac{|\nabla u_i|^2}{u_i+1} dx \\
&+ \sum_{i=1}^n a_i \int_{\mathbb{R}^2} \frac{2u_i+1}{u_i+1} \nabla u_i \cdot \nabla\left(\sum_{j=1}^m \alpha_{i,j} v_j\right) dx - \sum_{i=1}^n a_i \int_{\mathbb{R}^2} u_i \left|\nabla\left(\sum_{j=1}^m \alpha_{i,j} v_j\right)\right|^2 dx \\
&+ \frac{1}{2}\sum_{j=1}^m \sum_{l=1}^m b_{j,l}\beta_l \int_{\mathbb{R}^2} \left(v_j \partial_t v_l - v_l \partial_t v_j\right) dx \\
=& -\int_{\mathbb{R}^2} (\partial_t \boldsymbol{v})^T \boldsymbol{B}(\partial_t \boldsymbol{v})\,dx - \sum_{i=1}^n a_i \int_{\mathbb{R}^2} u_i \left|\nabla\left(\log(u_i+1) - \sum_{j=1}^m \alpha_{i,j} v_j\right)\right|^2 dx \\
&- \sum_{i=1}^n a_i \int_{\mathbb{R}^2} \left|\nabla\left(\log(u_i+1) - \frac{1}{2}\sum_{j=1}^m \alpha_{i,j} v_j\right)\right|^2 dx \\
&+ \frac{1}{4}\sum_{i=1}^n a_i \int_{\mathbb{R}^2} \left|\nabla\left(\sum_{j=1}^m \alpha_{i,j} v_j\right)\right|^2 dx + \frac{1}{2}\sum_{j=1}^m \sum_{l=1}^m b_{j,l}\beta_l \int_{\mathbb{R}^2} \left(v_j \partial_t v_l - v_l \partial_t v_j\right) dx.
\end{aligned}$$

Hence, we have proved (3.2).

By means of the positivity of $\boldsymbol{B}$, there exists a constant $c_1 > 0$ such that

$$c_1 \sum_{j=1}^m \sum_{l=1}^m b_{j,l} \int_{\mathbb{R}^2} \partial_t v_j \partial_t v_l dx \geq \sum_{j=1}^m \int_{\mathbb{R}^2} \left|\partial_t v_j\right|^2 dx,$$

then we have

$$\begin{aligned}
&\frac{1}{2}\sum_{j=1}^m \sum_{l=1}^m b_{j,l}\beta_l \int_{\mathbb{R}^2} \left(v_j \partial_t v_l - v_l \partial_t v_j\right) dx \\
&\leq \beta^* \sum_{j=1}^m \sum_{l=1}^m |b_{j,l}| \int_{\mathbb{R}^2} \left|\partial_t v_j\right| |v_l| dx \\
&\leq \frac{1}{2c_1} \sum_{j=1}^m \int_{\mathbb{R}^2} \left|\partial_t v_j\right|^2 dx + \frac{c_1 (b^* \beta^* |\mathcal{J}|)^2}{2} \sum_{j=1}^m \int_{\mathbb{R}^2} v_j^2 dx \\
&\leq \frac{1}{2}\sum_{j=1}^m \sum_{l=1}^m b_{j,l} \int_{\mathbb{R}^2} \partial_t v_j \partial_t v_l dx + \frac{c_1 (b^* \beta^* |\mathcal{J}|)^2}{2} \sum_{j=1}^m \int_{\mathbb{R}^2} v_j^2 dx
\end{aligned}$$

due to Young's inequality. Employing (3.2) and the boundedness of v_j in L^2 space, we can obtain a constant $c > 0$ such that (3.3) holds for all $t \in (0, T)$. □

Remark 3. *Let us define*

$$\begin{aligned}
\mathcal{E}[\boldsymbol{u},\boldsymbol{v}] =& \sum_{i=1}^n a_i \int_{\mathbb{R}^2} u_i \log u_i dx - \sum_{i=1}^n \sum_{j=1}^m a_i \alpha_{i,j} \int_{\mathbb{R}^2} u_i v_j dx \\
&+ \frac{1}{2}\sum_{j=1}^m \sum_{l=1}^m b_{j,l} \int_{\mathbb{R}^2} \left(\nabla v_j \cdot \nabla v_l + \beta_l v_j v_l\right) dx.
\end{aligned}$$

Then, $d\mathcal{E}[\boldsymbol{u},\boldsymbol{v}]/dt \leq 0$ for $t \in (0, T)$ if one has (3.1) and $\boldsymbol{B}\boldsymbol{D}_{\boldsymbol{\beta}}$ is symmetric.

We list the Moser–Trudinger inequality for system on the two-dimensional unit sphere [38, Theorem 2 (ii)].

Lemma 3.2. *Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere. Assume $\boldsymbol{S} = (s_{i,k})_{n\times n}$ is a positive definite matrix with $s_{i,k} \geq 0$, $i, k \in \mathcal{I}$, and $\boldsymbol{M} \in (\mathbb{R}_+)^n$. Then for $\rho_i \in H^1(\mathbb{S}^2)$ satisfying $\int_{\mathbb{S}^2} \rho_i = 0,\ \ i \in \mathcal{I}$,*

$$\begin{cases} \Lambda^{\boldsymbol{S}}_{\mathcal{K}}(\boldsymbol{M}) \geq 0, \quad \forall\, \emptyset \neq \mathcal{K} \subset \mathcal{I}, \\ \text{if } \Lambda^{\boldsymbol{S}}_{\mathcal{K}}(\boldsymbol{M}) = 0 \text{ for some } \mathcal{K}, \ \text{ then } s_{i,i} + \Lambda^{\boldsymbol{S}}_{\mathcal{K}\setminus\{i\}}(\boldsymbol{M}) > 0, \ \forall\, i \in \mathcal{K}, \end{cases} \tag{3.5}$$

is the necessary and sufficient condition for the existence of a constant $B > 0$ such that

$$\Phi_{\mathbb{S}^2}(\boldsymbol{\rho}) = \frac{1}{2}\sum_{i,k\in\mathcal{I}} s_{i,k} \int_{\mathbb{S}^2} \nabla\rho_i \cdot \nabla\rho_k - \sum_{i\in\mathcal{I}} M_i \log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} \exp\left(\sum_{k\in\mathcal{I}} s_{i,k}\rho_k\right)\right) \geq -B. \tag{3.6}$$

Now we use the stereographic projection $\mathcal{S}$ to transform the inequality for system in Lemma 3.2 to $\mathbb{R}^2$. In fact, we associate with each $\rho_i : \mathbb{S}^2 \to \mathbb{R}^2$ a function $\widetilde{\rho}_i : \mathbb{R}^2 \to \mathbb{S}^2$ via the transformation

$$\begin{cases} \widetilde{\rho}_i \leftrightarrow \quad \rho_i = \widetilde{\rho}_i \circ \mathcal{S}, \\ \rho_i \leftrightarrow \quad \widetilde{\rho}_i = \rho_i \circ \mathcal{S}^{-1}, \quad i \in \mathcal{I}. \end{cases} \tag{3.7}$$

By a simple calculation, we have

Lemma 3.3. *Let $\boldsymbol{S} = (s_{i,k})_{n\times n}$ be a positive definite matrix with $s_{i,k} \geq 0$, $i, k \in \mathcal{I}$. Then for $\widetilde{\rho}_i \in H^1(\mathbb{R}^2)$, $i \in \mathcal{I}$, condition (3.5) is the necessary and sufficient condition for the existence of a constant $B > 0$ such that*

$$\begin{aligned} &\frac{1}{2}\sum_{i,k\in\mathcal{I}} s_{i,k} \int_{\mathbb{R}^2} \nabla\widetilde{\rho}_i \cdot \nabla\widetilde{\rho}_k dx - \sum_{i\in\mathcal{I}} M_i \log\left(\frac{1}{4\pi}\int_{\mathbb{R}^2} \exp\left(\sum_{k\in\mathcal{I}} s_{i,k}\widetilde{\rho}_k\right) H(x)dx\right) \\ &\quad + \frac{1}{4\pi}\sum_{i,k\in\mathcal{I}} M_i s_{i,k} \int_{\mathbb{R}^2} \widetilde{\rho}_k H(x)dx \geq -B, \end{aligned} \tag{3.8}$$

where $H(x) = 4/\left(1 + |x|^2\right)^2$.

Proof. Let $\rho_i \in H^1(\mathbb{S}^2)$, $i \in \mathcal{I}$. Then, we take

$$\rho_i - \frac{1}{4\pi}\int_{\mathbb{S}^2} \rho_i, \quad i \in \mathcal{I},$$

for ρ_i in (3.6), and obtain that

$$\begin{aligned} &\frac{1}{2}\sum_{i,k\in\mathcal{I}} s_{i,k} \int_{\mathbb{S}^2} \nabla\rho_i \cdot \nabla\rho_k - \sum_{i\in\mathcal{I}} M_i \log\left(\frac{1}{4\pi}\int_{\mathbb{S}^2} \exp\left(\sum_{k\in\mathcal{I}} s_{i,k}\rho_k\right)\right) \\ &\quad + \frac{1}{4\pi}\sum_{i,k\in\mathcal{I}} M_i s_{i,k} \int_{\mathbb{S}^2} \rho_k \geq -B \end{aligned} \tag{3.9}$$

with some constant $B > 0$ if and only if (3.5) is valid. Using the transformation (3.7), one derives that $\int_{\mathbb{S}^2} \rho_k = \int_{\mathbb{R}^2} \widetilde{\rho}_k H(x)dx$, and

$$\int_{\mathbb{S}^2} \nabla\rho_i \cdot \nabla\rho_k = \int_{\mathbb{R}^2} \nabla\widetilde{\rho}_i \cdot \nabla\widetilde{\rho}_k dx,$$

as well as

$$\int_{\mathbb{S}^2} \exp\left(\sum_{k\in\mathcal{I}} s_{i,k}\rho_k\right) = \int_{\mathbb{R}^2} \exp\left(\sum_{k\in\mathcal{I}} s_{i,k}\widetilde{\rho}_k\right) H(x)dx.$$

Hence, we obtain (3.8) from (3.9). □

Under the condition (1.10), utilising the above Moser–Trudinger inequality for system in the whole space, we can give an estimate on the interaction term consisting of u_i and v_j in $\mathcal{F}$ at the first step. The idea is mainly combined with some of the work in [5] and [31] and has been applied to one-species or two-species chemotaxis system with two chemicals [19, 35].

Lemma 3.4. *Suppose* $\boldsymbol{\alpha} = (\alpha_{i,j})_{n\times m}$ *with* $R(\boldsymbol{\alpha}) = m$, $\beta_j \in \mathbb{R}$, $j \in \mathcal{J}$, $\boldsymbol{\gamma} = (\gamma_{i,j})_{n\times m}$ *and* $T > 0$. *Assume that* $(\boldsymbol{u}, \boldsymbol{v})$ *is a local solution of Cauchy problem* (1.5) *with initial data* $(\boldsymbol{u}_0, \boldsymbol{v}_0)$. *Suppose that there exist positive constants* $a_1, \cdots, a_n$ *and a positive definite matrix* $\boldsymbol{R} = (r_{i,k})_{n\times n}$ *with* $r_{i,k} \geq 0$, $i, k \in \mathcal{I}$, *such that*

$$\boldsymbol{\alpha}^T \boldsymbol{R}^{-1} \boldsymbol{\alpha} \boldsymbol{\gamma}_i = a_i \boldsymbol{\alpha}_i, \quad \forall\ i \in \mathcal{I}.$$

Then for any

$$8\pi \sum_{i\in\mathcal{K}} a_i m_i > \sum_{i,k\in\mathcal{K}} a_i a_k r_{i,k} m_i m_k, \quad \forall\ \emptyset \neq \mathcal{K} \subset \mathcal{I}, \tag{3.10}$$

there exist a small $\epsilon > 0$ *and a constant* $c > 0$ *such that*

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2} u_i v_j dx \leq{}& \frac{1}{2(1+\epsilon)}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\nabla v_j\cdot\nabla v_l dx\\ &+\frac{1}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx\\ &-\frac{1}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}u_i(x)\log H(x)dx + c,\end{aligned}$$

where $\boldsymbol{B} = (b_{j,l})_{m\times m} = \boldsymbol{\alpha}^T \boldsymbol{R}^{-1} \boldsymbol{\alpha}$ *and* $H(x) = 4/\big(1+|x|^2\big)^2$.

Proof. From our assumption, $\boldsymbol{B}$ is a positive definite matrix satisfying $\boldsymbol{B}\boldsymbol{\gamma}_i = a_i\boldsymbol{\alpha}_i$, $\forall\, i \in \mathcal{I}$. Define $\boldsymbol{T} := (t_{i,j})_{n\times m} = \boldsymbol{R}^{-1}\boldsymbol{\alpha}$. Then it is easy to find that

$$\alpha_{i,j} = \sum_{k=1}^{n} r_{i,k} t_{k,j}$$

and

$$b_{j,l} = \sum_{i=1}^{n}\sum_{k=1}^{n} t_{i,j} r_{i,k} t_{k,l}.$$

Moreover, one can pick a positive definite matrix $\boldsymbol{S} = (s_{i,k})_{n\times n}$ with $s_{i,k} = (1+\epsilon) r_{i,k} \geq 0$, $i, k \in \mathcal{I}$, such that

$$\sum_{k=1}^{n} s_{i,k} t_{k,j} = (1+\epsilon)\alpha_{i,j}$$

and

$$\sum_{i=1}^{n}\sum_{k=1}^{n} t_{i,j} s_{i,k} t_{k,l} = (1+\epsilon) b_{j,l}.$$

Note that Lemma 2.1 (v) will help us to find a constant $A_1 > 0$ such that

$$\sum_{j=1}^{m} \|v_j(\cdot, t)\|_{L^1(\mathbb{R}^2)} \leq A_1 \quad \text{for } t \in (0, T).$$

Chosen $\epsilon > 0$ small enough, (3.10) implies

$$\sum_{i\in\mathcal{K}} a_i m_i \left[8\pi - (1+\epsilon)^2 \sum_{k\in\mathcal{K}} a_k r_{i,k} m_k\right] > 0, \quad \forall\ \emptyset \neq \mathcal{K} \subset \mathcal{I}. \tag{3.11}$$

Let

$$\overline{\rho}_i(t) := \left(\sum_{j=1}^m \alpha_{i,j} v_j(t) - s\sum_{k=1}^n r_{i,k}\right)_+, \quad \widetilde{\rho}_i(t) := \left(\sum_{j=1}^m t_{i,j} v_j(t) - s\right)_+, \quad i \in \mathcal{I},$$

$$\overline{\Omega}_i(t) := \left\{x \in \mathbb{R}^2 : \sum_{j=1}^m \alpha_{i,j} v_j(t) > s\sum_{k=1}^n r_{i,k}\right\}, \quad \widetilde{\Omega}_i(t) := \left\{x \in \mathbb{R}^2 : \sum_{j=1}^m t_{i,j} v_j(t) > s\right\}, \quad i \in \mathcal{I},$$

and

$$\Omega(t) := \bigcup_{i=1}^n \widetilde{\Omega}_i(t), \quad m_i(t) := \int_{\overline{\Omega}_i(t)} u_i dx \le m_i, \quad i \in \mathcal{I}.$$

We claim some facts in the following. First, since $\widetilde{\rho}_i(t) \in H_0^1(\widetilde{\Omega}_i(t))$ and $\nabla\widetilde{\rho}_i(t) = \sum_{j=1}^m t_{i,j}\nabla v_j(t)$ in $\Omega(t)$, then $\widetilde{\rho}_i(t) \in H_0^1(\Omega(t))$ for all $i \in \mathcal{I}$. Second, the Lebesgue measure of $\Omega(t)$, denoted by $|\Omega(t)|$, is finite. This is because

$$s \cdot |\Omega(t)| \le s\sum_{i=1}^n |\widetilde{\Omega}_i(t)| \le \sum_{i=1}^n \left\|\sum_{j=1}^m t_{i,j} v_j(\cdot, t)\right\|_{L^1(\mathbb{R}^2)} \le c_1$$

implies that $|\Omega(t)| \le c_1/s$. Third, $|\overline{\Omega}_i(t)| \le c_1/s$ holds out due to

$$\overline{\Omega}_i(t) \subset \Omega(t), \quad \forall\ i \in \mathcal{I}.$$

Finally, without loss of generality, we assume $|\Omega(t)| \ge 1$ and $|\overline{\Omega}_i(t)| > 0$ for all $i \in \mathcal{I}$. If $|\overline{\Omega}_i(t)| = 0$ for some $i \in \mathcal{I}$, classical techniques are sufficient to analyse this case.

Fixing $i \in \mathcal{I}$, it is obvious that

$$\begin{aligned} a_i \sum_{j=1}^m \alpha_{i,j} \int_{\mathbb{R}^2} u_i v_j dx &= a_i \int_{\overline{\Omega}_i(t)} u_i \left(\sum_{j=1}^m \alpha_{i,j} v_j\right) dx + a_i \int_{\mathbb{R}^2\setminus\overline{\Omega}_i(t)} u_i \left(\sum_{j=1}^m \alpha_{i,j} v_j\right) dx \\ &\le a_i \int_{\overline{\Omega}_i(t)} u_i \overline{\rho}_i dx + s a_i m_i \left(\sum_{k=1}^n r_{i,k}\right) \\ &\le a_i \int_{\mathbb{R}^2} u_i \overline{\rho}_i dx + s a_i m_i \left(\sum_{k=1}^n r_{i,k}\right). \end{aligned}$$

Denote $u_i^* = m_i \exp((1+\epsilon)\overline{\rho}_i(x,t))H(x)\left(\int_{\mathbb{R}^2} \exp((1+\epsilon)\overline{\rho}_i(x,t))H(x)dx\right)^{-1}$. Then, $\|u_i^*\|_1 = m_i$ and a classical entropy minimisation in [5, Lemma 2.1] implies that the function

$$\mathcal{E}(u_i; \psi) = \int_{\mathbb{R}^2} (u_i(x)\log u_i(x) - u_i(x)\psi(x))dx \quad \text{with any} \ \exp(\psi) \in L^1(\mathbb{R}^2),$$

satisfies

$$\begin{aligned} \mathcal{E}(u_i; (1+\epsilon)\overline{\rho}_i + \log H) &\ge \mathcal{E}\left(u_i^*; (1+\epsilon)\overline{\rho}_i + \log H\right) \\ &= m_i \log m_i - m_i \log\left(\frac{1}{4\pi}\int_{\mathbb{R}^2} \exp((1+\epsilon)\overline{\rho}_i(x,t))H(x)dx\right) - m_i \log(4\pi). \end{aligned}$$

Combining the aforementioned findings, we arrive at the following

$$
\begin{aligned}
&(1+\epsilon)a_i\sum_{j=1}^{m}\alpha_{i,j}\int_{\mathbb{R}^2}u_iv_jdx-a_i\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx\\
&\le(1+\epsilon)a_i\int_{\mathbb{R}^2}u_i\overline{\rho}_idx-a_i\int_{\mathbb{R}^2}u_i\log u_idx+(1+\epsilon)sa_im_i\left(\sum_{k=1}^{n}r_{i,k}\right)\\
&\le a_im_i\log\left\{\frac{1}{4\pi}\int_{\mathbb{R}^2}\exp\big[(1+\epsilon)\overline{\rho}_i(x,t)\big]H(x)dx\right\}-a_i\int_{\mathbb{R}^2}u_i(x)\log H(x)dx\\
&\quad+(1+\epsilon)sa_im_i\left(\sum_{k=1}^{n}r_{i,k}\right)+a_im_i\log\frac{4\pi}{m_i}\\
&\le a_im_i\log\left\{\frac{1}{4\pi}\int_{\mathbb{R}^2}\exp\left[(1+\epsilon)\left(\sum_{j=1}^{m}\alpha_{i,j}v_j(t)-s\sum_{k=1}^{n}r_{i,k}\right)_+\right]H(x)dx\right\}\\
&\quad-a_i\int_{\mathbb{R}^2}u_i(x)\log H(x)dx+(1+\epsilon)sa_im_i\left(\sum_{k=1}^{n}r_{i,k}\right)+a_im_i\log\frac{4\pi}{m_i},
\end{aligned}
$$

where the choice of matrix $\boldsymbol{S}$ allows one to conclude that

$$
\begin{aligned}
&(1+\epsilon)a_i\sum_{j=1}^{m}\alpha_{i,j}\int_{\mathbb{R}^2}u_iv_jdx-a_i\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx\\
&\le a_im_i\log\left\{\frac{1}{4\pi}\int_{\mathbb{R}^2}\exp\left[\sum_{k=1}^{n}s_{i,k}\left(\sum_{j=1}^{m}t_{k,j}v_j-s\right)_+\right]H(x)dx\right\}\\
&\quad-a_i\int_{\mathbb{R}^2}u_i(x)\log H(x)dx+c_{2i}\\
&=a_im_i\log\left[\frac{1}{4\pi}\int_{\mathbb{R}^2}\exp\left(\sum_{k=1}^{n}s_{i,k}\widetilde{\rho}_k\right)H(x)dx\right]-a_i\int_{\mathbb{R}^2}u_i(x)\log H(x)dx+c_{2i},
\end{aligned}
$$

where $c_{2i}=(1+\epsilon)sa_im_i\left(\sum_{k=1}^{n}r_{i,k}\right)+a_im_i\log(4\pi/m_i)$, $i\in\mathcal{I}$. Then, summing it with respect to i from $i=1$ to $i=n$, we get

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{m}a_i\alpha_{i,j}\int_{\mathbb{R}^2}u_iv_jdx\le&\frac{1}{1+\epsilon}\sum_{i=1}^{n}a_im_i\log\left[\frac{1}{4\pi}\int_{\mathbb{R}^2}\exp\left(\sum_{k=1}^{n}s_{i,k}\widetilde{\rho}_k\right)H(x)dx\right]\\
&+\frac{1}{1+\epsilon}\sum_{i=1}^{n}a_i\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx\\
&-\frac{1}{1+\epsilon}\sum_{i=1}^{n}a_i\int_{\mathbb{R}^2}u_i(x)\log H(x)dx+\frac{1}{1+\epsilon}\sum_{i=1}^{n}c_{2i}.
\end{aligned}
\tag{3.12}
$$

Choose

$$
M_i=a_im_i(1+\epsilon),\qquad i\in\mathcal{I}.
$$

Since (3.11) implies that

$$\begin{aligned}\Lambda^{S}_{\mathcal{K}}(\boldsymbol{M}) &= 8\pi \sum_{i\in\mathcal{K}} M_i - \sum_{i,k\in\mathcal{K}} s_{i,k}M_iM_k \\ &= (1+\epsilon)\left[8\pi \sum_{i\in\mathcal{K}} a_im_i - (1+\epsilon)^2 \sum_{i,k\in\mathcal{K}} a_ia_kr_{i,k}m_im_k\right] \\ &> 0, \quad \forall\ \emptyset \neq \mathcal{K} \subset \mathcal{I},\end{aligned}$$

then the Moser–Trudinger inequality for system in Lemma 3.3 helps us to get that

$$\begin{aligned}&\sum_{i=1}^{n} M_i \log\left[\frac{1}{4\pi}\int_{\mathbb{R}^2} \exp\left(\sum_{k=1}^{n} s_{i,k}\widetilde{\rho}_k\right) H(x)dx\right] \\ &= (1+\epsilon)\sum_{i=1}^{n} a_im_i \log\left[\frac{1}{4\pi}\int_{\mathbb{R}^2} \exp\left(\sum_{k=1}^{n} s_{i,k}\widetilde{\rho}_k\right) H(x)dx\right] \\ &\le \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} s_{i,k}\int_{\mathbb{R}^2} \nabla\widetilde{\rho}_i\cdot\nabla\widetilde{\rho}_k dx + \frac{1}{4\pi}\sum_{i,k\in\mathcal{I}} M_is_{i,k}\int_{\mathbb{R}^2}\widetilde{\rho}_k H(x)dx + B \\ &\le \frac{1}{2}(1+\epsilon)\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\nabla v_j\cdot\nabla v_l dx + c_3,\end{aligned}$$

where we have used the bound of $\|v_j\|_{L^1(\mathbb{R}^2)}, j \in \mathcal{J}$, and $\int_{\mathbb{R}^2} H(x)dx = 4\pi$. This together with (3.12) and the positivity of $\boldsymbol{B}$ implies that

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2} u_iv_j dx \le{}& \frac{1}{2(1+\epsilon)}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\nabla v_j\cdot\nabla v_l dx \\ &+ \frac{1}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx \\ &- \frac{1}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i(x)\log H(x)dx + \frac{1}{1+\epsilon}\sum_{i=1}^{n} c_{2i} + \frac{c_3}{(1+\epsilon)^2}.\end{aligned}$$

Therefore, we have finished the proof of this lemma. □

As a consequence of Lemma 3.4, the bound on modified total entropy $\mathcal{S}$ could be obtained.

Lemma 3.5. *Let $T>0$. Under the same assumptions in Lemma 3.4, there exists a constant $c = c(T) > 0$ such that*

$$\sum_{i=1}^{n}\int_{\mathbb{R}^2}(u_i(x,t)+1)\log(u_i(x,t)+1)dx \le c \tag{3.13}$$

and

$$\sum_{j=1}^{m}\int_0^t\int_{\mathbb{R}^2}\left|\partial_t v_j\right|^2 dxd\tau \le c \tag{3.14}$$

hold out for $t \in (0,T)$.

Proof. Notice that positive definite matrix $\boldsymbol{B} = (b_{j,l})_{m\times m} = \boldsymbol{\alpha}^T\boldsymbol{R}^{-1}\boldsymbol{\alpha}$ satisfies

$$\boldsymbol{B}\boldsymbol{\gamma}_i = a_i\boldsymbol{\alpha}_i, \quad \forall\ i \in \mathcal{I}.$$

Then thanks to Lemma 3.1, there exists a modified free energy functional $\mathcal{F}$ given by

$$\mathcal{F}[\boldsymbol{u},\boldsymbol{v}] = \sum_{i=1}^{n} a_i \int_{\mathbb{R}^2} (u_i+1)\log(u_i+1)dx - \sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2} u_i v_j dx + \frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\left(\nabla v_j\cdot\nabla v_l + \beta_l v_j v_l\right)dx \tag{3.15}$$

satisfying

$$\begin{aligned}\frac{d}{dt}\mathcal{F}[\boldsymbol{u},\boldsymbol{v}] &+ \frac{1}{2}\int_{\mathbb{R}^2}(\partial_t\boldsymbol{v})^T\boldsymbol{B}(\partial_t\boldsymbol{v})\,dx + \sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}\left\{u_i\left|\nabla\left(\log(u_i+1) - \sum_{j=1}^{m}\alpha_{i,j}v_j\right)\right|^2 + \left|\nabla\left(\log(u_i+1) - \frac{1}{2}\sum_{j=1}^{m}\alpha_{i,j}v_j\right)\right|^2\right\}dx \\ &\le \frac{1}{4}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}\left|\nabla\left(\sum_{j=1}^{m}\alpha_{i,j}v_j\right)\right|^2 dx + c_1 \\ &\le \frac{a^*\alpha^*|\mathcal{I}||\mathcal{J}|}{4}\sum_{j=1}^{m}\int_{\mathbb{R}^2}\left|\nabla v_j\right|^2 dx + c_1\end{aligned} \tag{3.16}$$

with some constant $c_1 > 0$. Moreover, Lemma 3.4 implies the existence of small $\epsilon > 0$ such that

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2} u_i v_j dx \le& \frac{1}{2(1+\epsilon)}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\nabla v_j\cdot\nabla v_l dx \\ &+ \frac{1}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx \\ &- \frac{1}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i(x)\log H(x)dx + c_2\end{aligned} \tag{3.17}$$

is true with some $c_2 > 0$. On the one hand, one has

$$\begin{aligned}\mathcal{F}[\boldsymbol{u},\boldsymbol{v}] \ge& \frac{\epsilon}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx + \frac{\epsilon}{2(1+\epsilon)}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\nabla v_j\cdot\nabla v_l dx \\ &+ \frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\beta_l\int_{\mathbb{R}^2} v_j v_l dx + \frac{1}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i(x)\log H(x)dx - c_2\end{aligned} \tag{3.18}$$

from (3.15) and (3.17). On the other hand, we have

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\nabla v_j\cdot\nabla v_l dx =& \mathcal{F}[\boldsymbol{u},\boldsymbol{v}] - \sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx \\ &- \frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\beta_l\int_{\mathbb{R}^2} v_j v_l dx + \sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2} u_i v_j dx,\end{aligned}$$

which together with (3.17) ensures that

$$\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\nabla v_j\cdot\nabla v_l dx \le \frac{1+\epsilon}{\epsilon}\mathcal{F}[\boldsymbol{u},\boldsymbol{v}] - \sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx$$
$$-\frac{1+\epsilon}{2\epsilon}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\beta_l\int_{\mathbb{R}^2} v_j v_l dx$$
$$-\frac{1}{\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i(x)\log H(x)dx + \frac{(1+\epsilon)c_2}{\epsilon}.$$

Further, using the positivity of $\boldsymbol{B}$ and the bound on the $\|v_j\|_{L^2(\mathbb{R}^2)}$ by Lemma 2.1 (v), there exists a constant $c_3 > 0$ such that

$$\sum_{j=1}^{m}\int_{\mathbb{R}^2}|\nabla v_j|^2 dx \le \frac{(1+\epsilon)c_3}{\epsilon}\left(\mathcal{F}[\boldsymbol{u},\boldsymbol{v}] - \frac{1}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i(x)\log H(x)dx + 1\right).$$

Applying above inequalities and from (3.16), it follows that

$$\frac{d}{dt}\mathcal{F}[\boldsymbol{u},\boldsymbol{v}] + \frac{1}{2}\int_{\mathbb{R}^2}(\partial_t\boldsymbol{v})^T\boldsymbol{B}(\partial_t\boldsymbol{v})\,dx + \sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i\left|\nabla\left(\log(u_i+1) - \sum_{j=1}^{m}\alpha_{i,j}v_j\right)\right|^2 dx$$
$$\le \frac{(1+\epsilon)c_3 a^*\alpha^*|\mathcal{I}||\mathcal{J}|}{4\epsilon}\left(\mathcal{F}[\boldsymbol{u},\boldsymbol{v}] - \frac{1}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i(x)\log H(x)dx + 1\right) + c_1 \tag{3.19}$$

for $t \in (0, T)$. To estimate the second term on the right side of (3.19), we first observe that

$$-\frac{1}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i(x)\log H(x)dx = \frac{2}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i(x)\log\left(1+|x|^2\right)dx$$
$$-\frac{2\log 2}{1+\epsilon}\sum_{i=1}^{n} a_i m_i$$
$$\le \frac{2}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i(x)\log\left(1+|x|^2\right)dx,$$

where we take the derivative of the right term to see that

$$\frac{d}{dt}\left(\frac{2}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i(x)\log\left(1+|x|^2\right)dx\right)$$
$$= -\frac{2}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2} u_i\nabla\left[\log(u_i+1) - \sum_{j=1}^{m}\alpha_{i,j}v_j\right]\cdot\nabla\log\left(1+|x|^2\right)dx$$
$$-\frac{2}{1+\epsilon}\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}\frac{\nabla u_i}{u_i+1}\cdot\nabla\log\left(1+|x|^2\right)dx$$

$$\leq \frac{1}{2}\sum_{i=1}^{n} a_i \int_{\mathbb{R}^2} u_i \left|\nabla\left(\log(u_i+1) - \sum_{j=1}^{m}\alpha_{i,j}v_j\right)\right|^2 dx$$

$$+ \frac{2}{(1+\epsilon)^2}\sum_{i=1}^{n} a_i \int_{\mathbb{R}^2} u_i \left|\nabla \log\left(1+|x|^2\right)\right|^2 dx$$

$$+ \frac{2}{1+\epsilon}\sum_{i=1}^{n} a_i \int_{\mathbb{R}^2} \log(u_i+1)\Delta \log\left(1+|x|^2\right) dx$$

$$\leq \frac{1}{2}\sum_{i=1}^{n} a_i \int_{\mathbb{R}^2} u_i \left|\nabla\left(\log(u_i+1) - \sum_{j=1}^{m}\alpha_{i,j}v_j\right)\right|^2 dx + 10\sum_{i=1}^{n} a_i m_i \quad \text{for } t\in(0,T),$$

by Young's inequality, since

$$\left|\nabla \log\left(1+|x|^2\right)\right| = \left|\frac{2x}{1+|x|^2}\right| \leq 1, \quad \left|\Delta \log\left(1+|x|^2\right)\right| = \left|\frac{4}{\left(1+|x|^2\right)^2}\right| \leq 4,$$

and $\log(s+1) \leq s$ for all $s>0$. Thereby, denoting

$$y(t) := \mathcal{F}[\boldsymbol{u},\boldsymbol{v}] + \frac{2}{1+\epsilon}\sum_{i=1}^{n} a_i \int_{\mathbb{R}^2} u_i(x)\log\left(1+|x|^2\right) dx \quad \text{for } t\in(0,T),$$

one derives that

$$y'(t) + \frac{1}{2}\int_{\mathbb{R}^2} (\partial_t \boldsymbol{v})^T \boldsymbol{B}(\partial_t \boldsymbol{v})\, dx \leq \frac{(1+\epsilon)c_3 a^* \alpha^* |\mathcal{I}||\mathcal{J}|}{4\epsilon}(y(t)+1) + c_4 \quad \text{for } t\in(0,T),$$

with $c_4 = 10\sum_{i=1}^{n} a_i m_i + c_1$, where the Gronwall argument means that

$$y(t)+\frac{1}{2}\int_0^t \int_{\mathbb{R}^2} (\partial_t \boldsymbol{v})^T \boldsymbol{B}(\partial_t \boldsymbol{v})\, dxd\tau \leq (y(0)+1)\, e^{\frac{(1+\epsilon)c_3 a^* \alpha^* |\mathcal{I}||\mathcal{J}|}{4\epsilon}T} + c_4 \quad \text{for } t\in(0,T).$$

Hence, we have proved (3.13)–(3.14) due to (3.18), the choices of positive a_i, $i\in\mathcal{I}$, the positivity of $\boldsymbol{B}$ and the bound on the $\|v_j\|_{L^2(\mathbb{R}^2)}$, $j\in\mathcal{J}$. □

A straightforward argument [36, Lemma 3.6] could be indeed used to obtain L^2 estimates for the solutions by the bound on $\mathcal{S}$.

Lemma 3.6. *For $T>0$, there exists a constant $c=c(T)>0$ such that*

$$\sum_{i=1}^{n} \|u_i(\cdot,t)\|_{L^2(\mathbb{R}^2)} \leq c \quad \textit{for} \quad t\in(0,T).$$

Proof. From Lemma 3.5, there exists a constant $c_1 = c_1(T) > 0$ such that

$$\sum_{i=1}^{n}\int_{\mathbb{R}^2} (u_i+1)\log(u_i+1)dx \leq c_1$$

and

$$\sum_{j=1}^{m}\int_0^T \int_{\mathbb{R}^2} \left|\partial_t v_j\right|^2 dxd\tau \leq c_1. \tag{3.20}$$

We multiply the i-the equation in (1.5) by u_i and integrate them over $\mathbb{R}^2$ to have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\sum_{i=1}^{n}\int_{\mathbb{R}^2}u_i^2dx\right) &= -\sum_{i=1}^{n}\int_{\mathbb{R}^2}|\nabla u_i|^2dx - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{i,j}\int_{\mathbb{R}^2}u_i^2\Delta v_j dx \\
&= -\sum_{i=1}^{n}\int_{\mathbb{R}^2}|\nabla u_i|^2dx - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{i,j}\int_{\mathbb{R}^2}u_i^2\left(\partial_t v_j + \beta_j v_j - \sum_{k=1}^{n}\gamma_{k,j}u_k\right)dx \\
&= -\sum_{i=1}^{n}\int_{\mathbb{R}^2}|\nabla u_i|^2dx + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{m}\alpha_{i,j}\gamma_{k,j}\int_{\mathbb{R}^2}u_i^2u_kdx \\
&\quad - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{i,j}\beta_j\int_{\mathbb{R}^2}u_i^2v_jdx - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{i,j}\int_{\mathbb{R}^2}u_i^2\partial_t v_j dx \qquad (3.21)
\end{aligned}$$

for $t \in (0, T)$. It is clear that

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{m}\alpha_{i,j}\gamma_{k,j}\int_{\mathbb{R}^2}u_i^2u_kdx - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{i,j}\beta_j\int_{\mathbb{R}^2}u_i^2v_jdx \\
&\le \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{m}|\alpha_{i,j}||\gamma_{k,j}|\int_{\mathbb{R}^2}\left(u_i^3+u_k^3\right)dx + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}|\alpha_{i,j}||\beta_j|\int_{\mathbb{R}^2}\left(u_i^3+v_j^3\right)dx \\
&\le \frac{\alpha^*\left(\beta^*+2\gamma^*|\mathcal{I}|\right)|\mathcal{J}|}{2}\sum_{i=1}^{n}\int_{\mathbb{R}^2}u_i^3dx + \frac{\alpha^*\beta^*|\mathcal{I}|A_3^3}{2} \\
&\le \frac{c_1\alpha^*\left(\beta^*+2\gamma^*|\mathcal{I}|\right)|\mathcal{J}|}{2}\sum_{i=1}^{n}\eta_i\int_{\mathbb{R}^2}|\nabla u_i|^2dx \\
&\quad + \frac{\alpha^*\left(\beta^*+2\gamma^*|\mathcal{I}|\right)|\mathcal{J}|}{2}\sum_{i=1}^{n}c_{2i}m_i + \frac{\alpha^*\beta^*|\mathcal{I}|A_3^3}{2},
\end{aligned}$$

where we have used Young's inequality and the following facts $\sum_{j=1}^{m}\|v_j\|_{L^3(\mathbb{R}^2)} \le A_3$ hold due to Lemma 2.1 (v), and for any $\eta_i \in (0, 1)$, $i \in \mathcal{I}$,

$$\begin{aligned}
\int_{\mathbb{R}^2}u_i^3dx &\le \eta_i\left(\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx\right)\left(\int_{\mathbb{R}^2}|\nabla u_i|^2dx\right) + c_{2i}\int_{\mathbb{R}^2}u_idx \\
&\le c_1\eta_i\int_{\mathbb{R}^2}|\nabla u_i|^2dx + c_{2i}m_i, \quad i \in \mathcal{I},
\end{aligned}$$

exists with $c_{2i} = c_{2i}(\eta_i) > 0$ from (2.4). As for the rightmost integral of (3.21), we first use Hölder's inequality to find that

$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{i,j}\int_{\mathbb{R}^2}u_i^2\partial_t v_jdx \le \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}|\alpha_{i,j}|\|u_i\|_{L^4(\mathbb{R}^2)}^2\|\partial_t v_j\|_{L^2(\mathbb{R}^2)}.$$

Applying the Gagliardo–Nirenberg inequality with $c_3 > 0$ to have

$$\|u_i\|_{L^4(\mathbb{R}^2)}^2 \le c_3\|\nabla u_i\|_{L^2(\mathbb{R}^2)}\|u_i\|_{L^2(\mathbb{R}^2)},$$

and it infers that

$$-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{i,j}\int_{\mathbb{R}^2}u_i^2\partial_t v_j dx$$

$$\leq \frac{c_3}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}|\alpha_{i,j}|\|\nabla u_i\|_{L^2(\mathbb{R}^2)}\|u_i\|_{L^2(\mathbb{R}^2)}\|\partial_t v_j\|_{L^2(\mathbb{R}^2)}$$

$$\leq \frac{c_3\alpha^*|\mathcal{J}|}{2}\sum_{i=1}^{n}\eta_i\|\nabla u_i\|^2_{L^2(\mathbb{R}^2)}+\frac{c_3\alpha^*}{8}\left(\sum_{i=1}^{n}\frac{1}{\eta_i}\|u_i\|^2_{L^2(\mathbb{R}^2)}\right)\cdot\left(\sum_{j=1}^{m}\|\partial_t v_j\|^2_{L^2(\mathbb{R}^2)}\right)$$

by Young's inequality. Hence, (3.21) gives us that

$$\frac{d}{dt}\left(\sum_{i=1}^{n}\int_{\mathbb{R}^2}u_i^2dx\right)+\sum_{i=1}^{n}\left\{2-\alpha^*\left[c_1\left(\beta^*+2\gamma^*|\mathcal{I}|\right)+c_3\right]|\mathcal{J}|\eta_i\right\}\int_{\mathbb{R}^2}|\nabla u_i|^2dx$$

$$\leq \frac{c_3\alpha^*}{4}\left(\sum_{i=1}^{n}\frac{1}{\eta_i}\|u_i\|^2_{L^2(\mathbb{R}^2)}\right)\left(\sum_{j=1}^{m}\|\partial_t v_j\|^2_{L^2(\mathbb{R}^2)}\right)$$

$$+\alpha^*\left(\beta^*+2\gamma^*|\mathcal{I}|\right)|\mathcal{J}|\sum_{i=1}^{n}c_{2i}m_i+\alpha^*\beta^*|\mathcal{I}|A_3^3 \quad \text{for } t\in(0,T). \tag{3.22}$$

Because

$$\|u_i\|^2_{L^2(\mathbb{R}^2)}\leq c_4\|\nabla u_i\|_{L^2(\mathbb{R}^2)}\|u_i\|_{L^1(\mathbb{R}^2)}=c_4m_i\|\nabla u_i\|_{L^2(\mathbb{R}^2)}$$

$$\leq \|\nabla u_i\|^2_{L^2(\mathbb{R}^2)}+\frac{c_4^2m_i^2}{4}$$

is right for some $c_4>0$, we take $\eta_i=1/\{\alpha^*\left[c_1(\beta^*+2\gamma^*|\mathcal{I}|)+c_3\right]|\mathcal{J}|\}>0$, $i\in\mathcal{I}$, small enough in (3.22) to arrive at

$$y'(t)+\left[1-\frac{c_3(\alpha^*)^2\left[c_1(\beta^*+2\gamma^*|\mathcal{I}|)+c_3\right]|\mathcal{J}|}{4}\left(\sum_{j=1}^{m}\|\partial_t v_j\|^2_{L^2(\mathbb{R}^2)}\right)\right]y(t)$$

$$\leq \alpha^*\left(\beta^*+2\gamma^*|\mathcal{I}|\right)|\mathcal{J}|\sum_{i=1}^{n}c_{2i}m_i+\frac{c_4^2}{4}\sum_{i=1}^{n}m_i^2+\alpha^*\beta^*|\mathcal{I}|A_3^3$$

for $t\in(0,T)$, where $y(t):=\sum_{i=1}^{n}\int_{\mathbb{R}^2}u_i^2dx$. Together with (3.20), the L^2 estimates for the solutions can be obtained by solving this ODE. □

Proof of Theorem 1.1. Let $0<T\leq\infty$. Once one has L^2 estimates on u_i, $i\in\mathcal{I}$, then L^p-L^q estimates for the heat semigroup in Lemma 5.1 ensure that for $r\in(1,\infty)$, we have

$$\sum_{j=1}^{m}\|\nabla v_j(\cdot,t)\|_{L^r(\mathbb{R}^2)}\leq c \quad \text{for} \quad t\in(0,T) \tag{3.23}$$

with some $c>0$. Consequently, applying the Moser iteration technique [1] with (3.23), it means that

$$\sum_{i=1}^{n}\|u_i(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}<\infty \quad \text{for} \quad t\in(0,T).$$

Therefore, the global result in Theorem 1.1 is an immediate consequence of the extensibility criterion in Lemma 2.1.

4. Proof of Theorem 1.2

Under strong self-repelling effect, the global existence of the Cauchy problem (1.5) with arbitrary initial data will be established in this section. As with the above treatment of the proof of Theorem 1.1, the main approach is to give a bound on the modified total entropy $\mathcal{S}$. For this purpose, we would like to create a differential inequality for $\mathcal{G}$.

Lemma 4.1. *Assume that there exist positive constants $a_1, \cdots, a_n$ and a positive definite matrix $\boldsymbol{B} = (b_{j,l})_{m\times m}$ such that*

$$\boldsymbol{B}\boldsymbol{\gamma}_i = -a_i\boldsymbol{\alpha}_i, \quad \forall\ i \in \mathcal{I}. \tag{4.1}$$

Let

$$D_1\boldsymbol{\xi}^T\boldsymbol{\xi} \le \boldsymbol{\xi}^T\boldsymbol{B}\boldsymbol{\xi} \le D_2\boldsymbol{\xi}^T\boldsymbol{\xi}, \quad \forall\ \boldsymbol{\xi} = (\xi_1, \cdots, \xi_m)^T \in \mathbb{R}^m,$$

with some $D_1, D_2 > 0$. Then, there exist a constant $c_1 = c_1(a^*, b^*, \alpha^*, \beta^*, A_2, D_1, D_2, \|\boldsymbol{u}_0\|_{L^1(\mathbb{R}^2)}, |\mathcal{I}|, |\mathcal{J}|) > 0$ such that

$$\frac{d}{dt}\mathcal{G}[\boldsymbol{u}, \boldsymbol{v}] + \frac{1}{2}\sum_{i=1}^n a_i \int_{\mathbb{R}^2} \frac{|\nabla u_i|^2}{u_i + 1}dx \le c_1 \quad \text{for} \quad t \in (0, T),$$

where $\mathcal{G}$ is given by

$$\mathcal{G}[\boldsymbol{u}, \boldsymbol{v}] = \sum_{i=1}^n a_i \int_{\mathbb{R}^2} (u_i + 1)\log(u_i + 1)dx + \frac{1}{2}\sum_{j=1}^m\sum_{l=1}^m b_{j,l}\int_{\mathbb{R}^2} \left(\nabla v_j \cdot \nabla v_l + \beta_l v_j v_l\right) dx.$$

Moreover, if $\beta_j > 0$ for all $j \in \mathcal{J}$, then there exist a constant $c_2 > 0$ independent of T such that

$$\frac{d}{dt}\mathcal{G}[\boldsymbol{u}, \boldsymbol{v}] + \frac{D_1}{D_2}\beta_*\mathcal{G}[\boldsymbol{u}, \boldsymbol{v}] \le c_2 \quad \text{for} \quad t \in (0, T).$$

Proof. Given $a_i > 0$, testing the i-th equation in (1.5) by $a_i \log(u_i + 1)$ and summing the results with respect to i, we get

$$\begin{aligned}
&\frac{d}{dt}\left[\sum_{i=1}^n a_i \int_{\mathbb{R}^2} (u_i + 1)\log(u_i + 1)dx\right] + \sum_{i=1}^n a_i \int_{\mathbb{R}^2} \frac{|\nabla u_i|^2}{u_i + 1}dx \\
&= \sum_{i=1}^n\sum_{j=1}^m a_i\alpha_{i,j}\int_{\mathbb{R}^2} \frac{u_i}{u_i + 1}\nabla u_i \cdot \nabla v_j dx \\
&= \sum_{i=1}^n\sum_{j=1}^m a_i\alpha_{i,j}\int_{\mathbb{R}^2} \nabla u_i \cdot \nabla v_j dx + \sum_{i=1}^n\sum_{j=1}^m a_i\alpha_{i,j}\int_{\mathbb{R}^2} \Delta v_j \log(u_i + 1)dx \quad \text{for} \quad t \in (0, T).
\end{aligned} \tag{4.2}$$

Moreover, we observe that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\sum_{j=1}^m\sum_{l=1}^m b_{j,l}\int_{\mathbb{R}^2} \left(\nabla v_j \cdot \nabla v_l + \beta_l v_j v_l\right)\right] dx \\
&= -\sum_{j=1}^m\sum_{l=1}^m b_{j,l}\int_{\mathbb{R}^2} (\Delta v_l - \beta_l v_l)\partial_t v_j dx + \frac{1}{2}\sum_{j=1}^m\sum_{l=1}^m b_{j,l}\beta_l\int_{\mathbb{R}^2} (v_j\partial_t v_l - v_l\partial_t v_j)dx
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\left(\Delta v_j-\beta_j v_j\right)\left(\Delta v_l-\beta_l v_l\right)dx \\
&\quad+\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\gamma_{i,j}\int_{\mathbb{R}^2}\nabla u_i\cdot\nabla v_l dx+\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\gamma_{i,j}\beta_l\int_{\mathbb{R}^2} u_i v_l dx \\
&\quad+\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\beta_l\int_{\mathbb{R}^2}(v_j\partial_t v_l-v_l\partial_t v_j)dx \quad \text{for} \quad t\in(0,T)
\end{aligned} \tag{4.3}
$$

and that

$$
\begin{aligned}
D_1\sum_{j=1}^{m}\int_{\mathbb{R}^2}\left|\Delta v_j-\beta_j v_j\right|^2 dx &\le \sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\int_{\mathbb{R}^2}\left(\Delta v_j-\beta_j v_j\right)\left(\Delta v_l-\beta_l v_l\right)dx \\
&\le D_2\sum_{j=1}^{m}\int_{\mathbb{R}^2}\left|\Delta v_j-\beta_j v_j\right|^2 dx.
\end{aligned} \tag{4.4}
$$

Then employing $\sum_{j=1}^{m} b_{j,l}\gamma_{i,j}=-a_i\alpha_{i,l}$ by (4.1), it is obvious that

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{G}[\boldsymbol{u},\boldsymbol{v}]+\sum_{i=1}^{n} a_i\int_{\mathbb{R}^2}\frac{|\nabla u_i|^2}{u_i+1}dx+D_1\sum_{j=1}^{m}\int_{\mathbb{R}^2}\left|\Delta v_j-\beta_j v_j\right|^2 dx \\
&\le\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2}\Delta v_j\log(u_i+1)dx-\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\beta_j\int_{\mathbb{R}^2} u_i v_j dx \\
&\quad+\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m} b_{j,l}\beta_l\int_{\mathbb{R}^2}(v_j\partial_t v_l-v_l\partial_t v_j)dx \quad \text{for} \quad t\in(0,T)
\end{aligned} \tag{4.5}
$$

due to (4.2) and (4.3). Since $\log(u_i+1)\le\sqrt{u_i}$, $\|u_i\|_{L^1(\mathbb{R}^2)}=m_i$, $i\in\mathcal{I}$, and

$$
\sum_{j=1}^{m}\|v_j\|_{L^2(\mathbb{R}^2)}\le A_2
$$

hold out, an application of Young's inequality gives that

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2}\Delta v_j\log(u_i+1)dx \\
&=\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\int_{\mathbb{R}^2}(\Delta v_j-\beta_j v_j)\log(u_i+1)dx+\sum_{i=1}^{n}\sum_{j=1}^{m} a_i\alpha_{i,j}\beta_j\int_{\mathbb{R}^2} v_j\log(u_i+1)dx \\
&\le\sum_{i=1}^{n}\sum_{j=1}^{m} a_i|\alpha_{i,j}|\int_{\mathbb{R}^2}|\Delta v_j-\beta_j v_j|\sqrt{u_i}dx+\sum_{i=1}^{n}\sum_{j=1}^{m} a_i|\alpha_{i,j}||\beta_j|\int_{\mathbb{R}^2}|v_j|\sqrt{u_i}dx \\
&\le\frac{D_1}{4}\sum_{j=1}^{m}\int_{\mathbb{R}^2}\left|\Delta v_j-\beta_j v_j\right|^2 dx+\frac{(a^*\alpha^*)^2|\mathcal{I}||\mathcal{J}|}{D_1}\sum_{i=1}^{n} m_i+a^*\alpha^*\beta^*A_2\sum_{i=1}^{n} m_i^{1/2}.
\end{aligned} \tag{4.6}
$$

Note that A_2 is uniformly bounded with respect to time variable if $\beta_j>0$ for all $j\in\mathcal{J}$. However, Lemma 2.2 tells us

$$
\int_{\mathbb{R}^2} u_i^2 dx\le\frac{1}{2\pi}\left(\int_{\mathbb{R}^2} u_i dx\right)\left(\int_{\mathbb{R}^2}\frac{|\nabla u_i|^2}{u_i+1}dx\right)+2\int_{\mathbb{R}^2} u_i dx.
$$

Then, one has

$$-\sum_{i=1}^{n}\sum_{j=1}^{m}a_i\alpha_{i,j}\beta_j\int_{\mathbb{R}^2}u_iv_jdx\le\sum_{i=1}^{n}\frac{\pi a_i}{2m_i}\int_{\mathbb{R}^2}u_i^2dx+\frac{(\alpha^*\beta^*)^2|\mathcal{J}|}{2\pi}\left(\sum_{i=1}^{n}a_im_i\right)\left(\sum_{j=1}^{m}\int_{\mathbb{R}^2}v_j^2dx\right)$$
$$\le\frac{1}{4}\sum_{i=1}^{n}a_i\int_{\mathbb{R}^2}\frac{|\nabla u_i|^2}{u_i+1}dx+\pi\sum_{i=1}^{n}a_i+\frac{(\alpha^*\beta^*A_2|\mathcal{J}|)^2}{2\pi}\left(\sum_{i=1}^{n}a_im_i\right) \tag{4.7}$$

and

$$\frac{1}{2}\sum_{j=1}^{m}\sum_{l=1}^{m}b_{j,l}\beta_l\int_{\mathbb{R}^2}(v_j\partial_tv_l-v_l\partial_tv_j)dx$$
$$\le\beta^*\sum_{j=1}^{m}\sum_{l=1}^{m}|b_{j,l}|\int_{\mathbb{R}^2}|v_j||\partial_tv_l|dx$$
$$\le\beta^*\sum_{j=1}^{m}\sum_{l=1}^{m}|b_{j,l}|\int_{\mathbb{R}^2}|v_j||\Delta v_l-\beta_lv_l|dx+\beta^*\sum_{j=1}^{m}\sum_{l=1}^{m}|b_{j,l}|\int_{\mathbb{R}^2}|v_j|\left|\sum_{i=1}^{n}\alpha_{i,l}u_i\right|dx \tag{4.8}$$
$$\le\frac{1}{4}\sum_{i=1}^{n}a_i\int_{\mathbb{R}^2}\frac{|\nabla u_i|^2}{u_i+1}dx+\frac{D_1}{4}\sum_{j=1}^{m}\int_{\mathbb{R}^2}\left|\Delta v_j-\beta_jv_j\right|^2dx$$
$$+\frac{\left(\alpha^*b^*\beta^*A_2|\mathcal{J}|^2\right)^2}{2\pi}\sum_{i=1}^{n}\frac{m_i}{a_i}+\pi\sum_{i=1}^{n}a_i+\frac{(b^*\beta^*A_2|\mathcal{J}|)^2}{D_1}\quad\text{for}\quad t\in(0,T).$$

Putting (4.5)–(4.8) together, then there exists a constant $c_1>0$ such that $\mathcal{G}$ satisfies

$$\frac{d}{dt}\mathcal{G}[\boldsymbol{u},\boldsymbol{v}]+\frac{1}{2}\sum_{i=1}^{n}a_i\int_{\mathbb{R}^2}\frac{|\nabla u_i|^2}{u_i+1}dx+\frac{D_1}{2}\sum_{j=1}^{m}\int_{\mathbb{R}^2}\left|\Delta v_j-\beta_jv_j\right|^2dx\le c_1\quad\text{for}\quad t\in(0,T).$$

Now if $\beta_j>0$ for all $j\in\mathcal{J}$, combining (2.2) with (4.4) yields that

$$\frac{D_1}{2}\sum_{j=1}^{m}\int_{\mathbb{R}^2}\left|\Delta v_j-\beta_jv_j\right|^2dx\ge\frac{D_1}{2D_2}\sum_{j=1}^{m}\sum_{l=1}^{m}b_{j,l}\int_{\mathbb{R}^2}(\Delta v_j-\beta_jv_j)(\Delta v_l-\beta_lv_l)dx$$
$$\ge\frac{D_1\beta_*}{2D_2}\sum_{j=1}^{m}\sum_{l=1}^{m}b_{j,l}\int_{\mathbb{R}^2}\left(\nabla v_j\cdot\nabla v_l+\beta_lv_jv_l\right)dx$$
$$=\frac{D_1\beta_*}{D_2}\mathcal{G}[\boldsymbol{u},\boldsymbol{v}]-\frac{D_1\beta_*}{D_2}\sum_{i=1}^{n}a_i\int_{\mathbb{R}^2}(u_i+1)\log(u_i+1)dx$$
$$\ge\frac{D_1\beta_*}{D_2}\mathcal{G}[\boldsymbol{u},\boldsymbol{v}]-\frac{D_1\beta_*}{D_2}\sum_{i=1}^{n}a_im_i\eta_i\int_{\mathbb{R}^2}\frac{|\nabla u_i|^2}{u_i+1}dx-c_2\beta_*\sum_{i=1}^{n}a_im_i,$$

where $\eta_i\in(0,1)$, $i\in\mathcal{I}$ and $c_2>0$ are constants. Taking $\eta_i=\frac{D_2}{2D_1m_i\beta_*}$, $i\in\mathcal{I}$, we collect the above two inequalities to obtain

$$\frac{d}{dt}\mathcal{G}[\boldsymbol{u},\boldsymbol{v}]+\frac{D_1\beta_*}{D_2}\mathcal{G}[\boldsymbol{u},\boldsymbol{v}]\le c_1+c_2\beta_*\sum_{i=1}^{n}a_im_i\quad\text{for}\quad t\in(0,T).$$

□

Proof of Theorem 1.2. Thanks to Lemma 4.1, we conclude that $\mathcal{G}$ is bounded. Therefore, there exists a constant $c > 0$ such that

$$\sum_{i=1}^{n} \|(u(\cdot,t)+1)\log(u(\cdot,t)+1)\|_{L^1(\mathbb{R}^2)} \le c \quad \text{for} \quad t \in (0,T),$$
$$\sum_{j=1}^{m} \|\nabla v_j(\cdot,t)\|_{L^2(\mathbb{R}^2)} \le c \quad \text{for} \quad t \in (0,T).$$

Similar to the proof of Theorem 1.1, the global result can be obtained through a classic and standard method.

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Appendix

In this section, we will prove Lemma 2.1. The following lemma collects some basic facts on the asymptotics of the heat semigroup $(e^{t\Delta})_{t\geq 0}$, given by

$$(e^{t\Delta}\phi)(x) := \int_{\mathbb{R}^2} G(x-y,t)\phi(y)dy, \quad x \in \mathbb{R}^2, \ t > 0,$$

where $\phi \in C^0(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and the Gaussian heat kernel is denoted by $G(z,t) := (4\pi t)^{-1}e^{-\frac{|z|^2}{4t}}$, $z \in \mathbb{R}^2$, $t > 0$.

Lemma 5.1. *Let $(e^{t\Delta})_{t\geq 0}$ be the heat semigroup in $\mathbb{R}^2$. Then, the following properties are true.*

(i) Let $\omega \in \mathbb{N}_0^n$. Then

$$D_x^\omega e^{t\Delta}\phi = e^{t\Delta}D_x^\omega\phi \quad \text{for all } t > 0$$

is valid for all $\phi \in C^{|\omega|}(\mathbb{R}^2) \cap W^{|\omega|,\infty}(\mathbb{R}^2)$.

(ii) If $1 \leq r_1 \leq r_2 \leq \infty$ and $\omega \in \mathbb{N}_0^n$, then there exist a constant $c(r_1, r_2, |\omega|) > 0$ such that

$$\|D_x^\omega e^{t\Delta}\phi\|_{L^{r_2}(\mathbb{R}^2)} \leq c(r_1, r_2, \omega)t^{-\frac{|\omega|}{2}-(\frac{1}{r_1}-\frac{1}{r_2})}\|\phi\|_{L^{r_1}(\mathbb{R}^2)} \quad \text{for all } t > 0$$

holds for all $\phi \in L^{r_1}(\mathbb{R}^2)$*. In particular,* $c(r_1, r_2, |\omega|) = 1$ *if* $|\omega| = 0$ *and* $r_1 = r_2$*.*

Proof. Please see [41, Lemma 2.1] and [14] for details. □

Proof of Lemma 2.1. The proof of Lemma 2.1 will be divided into several steps.

First Step: local existence. The contraction mapping theorem will be used to prove the local existence of mild solutions. Let

$$R := \sum_{i=1}^{n} \left(\|u_{i0}\|_{L^\infty(\mathbb{R}^2)} + \|u_{i0}\|_{L^1(\mathbb{R}^2)}\right) + \sum_{j=1}^{m} \left(\|v_{j0}\|_{L^1(\mathbb{R}^2)} + \|\nabla v_{j0}\|_{L^p(\mathbb{R}^2)} + \|\nabla v_{j0}\|_{L^1(\mathbb{R}^2)}\right) + 1,$$

and let T be a fixed positive number below. Set

$$X := C^0\left([0, T]; \left(C^0(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\right)^n \times \left(W^{1,p}(\mathbb{R}^2) \cap W^{1,1}(\mathbb{R}^2)\right)^m\right)$$

equipped with the norm

$$\begin{aligned}\|(\boldsymbol{u}, \boldsymbol{v})\|_X = \max_{0\le t\le T} \Bigg\{ &\sum_{i=1}^{n} \left[\|u_i(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} + \|u_i(\cdot, t)\|_{L^1(\mathbb{R}^2)}\right] \\ &+ \sum_{j=1}^{m} \left[\|v_j(\cdot, t)\|_{L^1(\mathbb{R}^2)} + \|\nabla v_j(\cdot, t)\|_{L^p(\mathbb{R}^2)} + \|\nabla v_j(\cdot, t)\|_{L^1(\mathbb{R}^2)}\right] \Bigg\}\end{aligned}$$

for $t \in [0, T]$. Moreover, define

$$E := \left\{(\boldsymbol{u}, \boldsymbol{v}) \in X \,\middle|\, (\boldsymbol{u}, \boldsymbol{v})(\cdot, 0) = (\boldsymbol{u}_0, \boldsymbol{v}_0) \quad \text{and} \quad \|(\boldsymbol{u}, \boldsymbol{v})\|_X \le R\right\}.$$

Then, it is easy to see that E is a closed convex subset of X. Consider a nonlinear mapping $\Pi : E \mapsto X$ such that for any $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{v}}) \in E$,

$$(\boldsymbol{u}, \boldsymbol{v}) = \Pi(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{v}}),$$

where $\boldsymbol{u} = (u_1, \cdots, u_n)$ and $\boldsymbol{v} = (v_1, \cdots, v_m)$ satisfy

$$u_i(\cdot, t) = e^{t\Delta}u_{i0} - \sum_{j=1}^{m} \alpha_{i,j} \int_0^t \nabla \cdot e^{(t-s)\Delta}[\widetilde{u}_i(\cdot, s)\nabla\widetilde{v}_j(\cdot, s)]ds, \quad i \in \mathcal{I}, \quad t \in [0, T], \tag{A1}$$

and

$$v_j(\cdot, t) = e^{t(\Delta-\beta_j)}v_{j0} + \sum_{i=1}^{n} \gamma_{i,j} \int_0^t e^{(t-s)(\Delta-\beta_j)}\widetilde{u}_i(\cdot, s)ds, \quad j \in \mathcal{J}, \quad t \in [0, T], \tag{A2}$$

respectively. By the estimates for the heat semigroup in Lemma 5.1 (*ii*) to (A1), there exists a constant $c_1 = c_1(p) > 0$ such that

$$\begin{aligned}\sum_{i=1}^{n} \|u_i(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} &\le \sum_{i=1}^{n} \|e^{t\Delta}u_{i0}\|_{L^\infty(\mathbb{R}^2)} + c_1 \sum_{i=1}^{n}\sum_{j=1}^{m} |\alpha_{i,j}| \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{p}} \|\widetilde{u}_i(\cdot, s)\nabla\widetilde{v}_j(\cdot, s)\|_{L^p(\mathbb{R}^2)}ds, \\ &\le \sum_{i=1}^{n} \|u_{i0}\|_{L^\infty(\mathbb{R}^2)} + c_1\alpha^* \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{p}} \left(\sum_{i=1}^{n} \|\widetilde{u}_i(\cdot, s)\|_{L^\infty(\mathbb{R}^2)}\right) \\ &\quad\cdot \left(\sum_{j=1}^{m} \|\nabla\widetilde{v}_j(\cdot, s)\|_{L^p(\mathbb{R}^2)}\right) ds \\ &\le \sum_{i=1}^{n} \|u_{i0}\|_{L^\infty(\mathbb{R}^2)} + \frac{2c_1p\alpha^*R^2}{p-2}T^{\frac{1}{2}-\frac{1}{p}}, \quad t \in [0, T].\end{aligned}$$

Similarly, one also has

$$\sum_{i=1}^{n}\|u_i(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le \sum_{i=1}^{n}\|u_{i0}\|_{L^1(\mathbb{R}^2)} + c_2\alpha^*\int_0^t (t-s)^{-\frac{1}{2}}\left(\sum_{i=1}^{n}\|\widetilde{u}_i(\cdot,s)\|_{L^\infty(\mathbb{R}^2)}\right) \cdot\left(\sum_{j=1}^{m}\|\nabla\widetilde{v}_j(\cdot,s)\|_{L^1(\mathbb{R}^2)}\right)ds$$
$$\le \sum_{i=1}^{n}\|u_{i0}\|_{L^1(\mathbb{R}^2)} + 2c_2\alpha^* R^2 T^{\frac{1}{2}}, \quad t\in[0,T],$$

with some $c_2>0$. Moreover, one can apply Lemma 5.1 (*ii*) to find constants c_3, $c_4>0$ such that

$$\sum_{j=1}^{m}\|v_j(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le \sum_{j=1}^{m} e^{|\beta_j|t}\|v_{j0}\|_{L^1(\mathbb{R}^2)} + \gamma^*\sum_{j=1}^{m} e^{|\beta_j|t}\int_0^t \left\|\sum_{i=1}^{n}\widetilde{u}_i(\cdot,s)\right\|_{L^1(\mathbb{R}^2)} ds$$
$$\le e^{\beta^* T}\sum_{j=1}^{m}\|v_{j0}\|_{L^1(\mathbb{R}^2)} + \gamma^* e^{\beta^* T} R|\mathcal{J}|T, \quad t\in[0,T],$$

$$\sum_{j=1}^{m}\|\nabla v_j(\cdot,t)\|_{L^p(\mathbb{R}^2)} \le \sum_{j=1}^{m} e^{|\beta_j|t}\|\nabla v_{j0}\|_{L^p(\mathbb{R}^2)} + c_3\sum_{i=1}^{n}\sum_{j=1}^{m}|\gamma_{i,j}|e^{|\beta_j|t}\int_0^t (t-s)^{-\frac{1}{2}}\|\widetilde{u}_i(\cdot,s)\|_{L^p(\mathbb{R}^2)}ds$$
$$\le e^{\beta^* T}\sum_{j=1}^{m}\|\nabla v_{j0}\|_{L^p(\mathbb{R}^2)} + c_3\gamma^* e^{\beta^* t}|\mathcal{J}|\sum_{i=1}^{n}\int_0^t (t-s)^{-\frac{1}{2}} \cdot\|\widetilde{u}_i(\cdot,s)\|_{L^\infty(\mathbb{R}^2)}^{\frac{p-1}{p}}\|\widetilde{u}_i(\cdot,s)\|_{L^1(\mathbb{R}^2)}^{\frac{1}{p}}ds$$
$$\le e^{\beta^* T}\sum_{j=1}^{m}\|\nabla v_{j0}\|_{L^p(\mathbb{R}^2)} + 2c_3\gamma^* e^{\beta^* T}R|\mathcal{J}|T^{\frac{1}{2}}, \quad t\in[0,T], \tag{A3}$$

$$\sum_{j=1}^{m}\|\nabla v_j(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le e^{\beta^* T}\sum_{j=1}^{m}\|\nabla v_{j0}\|_{L^1(\mathbb{R}^2)} + 2c_4\gamma^* e^{\beta^* T}R|\mathcal{J}|T^{\frac{1}{2}}, \quad t\in[0,T]. \tag{A4}$$

Hence, Π maps E into E if we choose T small enough.

We now show that the mapping is a contraction. Indeed, for $(\overline{\boldsymbol{u}},\overline{\boldsymbol{v}})\in E$, $(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{v}})\in E$, we have

$$\|\Pi(\overline{\boldsymbol{u}},\overline{\boldsymbol{v}}) - \Pi(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{v}})\|_X = \max_{0\le t\le T}\{I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)\},$$

where I_i, $i=1,2,\cdots,5$ is introduced as follows,

$$I_1(t) = \sum_{i=1}^{n}\left\|\int_0^t \nabla\cdot e^{(t-s)\Delta}\Big[(\overline{u}_i(\cdot,s)-\widetilde{u}_i(\cdot,s))\Big(\sum_{j=1}^{m}\alpha_{i,j}\nabla\overline{v}_j(\cdot,s)\Big) + \widetilde{u}_i(\cdot,s)\sum_{j=1}^{m}\alpha_{i,j}\Big(\nabla\overline{v}_j(\cdot,s)-\widetilde{v}_j(\cdot,s)\Big)\Big]ds\right\|_{L^\infty(\mathbb{R}^2)}$$
$$\le c_1\sum_{i=1}^{n}\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{p}}\|\overline{u}_i(\cdot,s)-\widetilde{u}_i(\cdot,s)\|_{L^\infty(\mathbb{R}^2)}\left\|\sum_{j=1}^{m}\alpha_{i,j}\nabla\overline{v}_j(\cdot,s)\right\|_{L^p(\mathbb{R}^2)}ds$$
$$+ c_1\sum_{i=1}^{n}\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{p}}\|\widetilde{u}_i(\cdot,s)\|_{L^\infty(\mathbb{R}^2)}\left\|\sum_{j=1}^{m}\alpha_{i,j}\nabla\Big(\overline{v}_j(\cdot,s)-\widetilde{v}_j(\cdot,s)\Big)\right\|_{L^p(\mathbb{R}^2)}ds$$
$$\le \frac{4c_1p\alpha^* R}{p-2}\|(\overline{\boldsymbol{u}},\overline{\boldsymbol{v}})-(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{v}})\|_X, \quad t\in[0,T],$$

$$I_2(t) = \sum_{i=1}^{n} \left\| \int_0^t \nabla \cdot e^{(t-s)\Delta}\Big[(\overline{u}_i(\cdot,s) - \widetilde{u}_i(\cdot,s))\Big(\sum_{j=1}^{m} \alpha_{i,j}\nabla\overline{v}_j(\cdot,s)\Big) + \widetilde{u}_i(\cdot,s)\sum_{j=1}^{m}\alpha_{i,j}\Big(\nabla\overline{v}_j(\cdot,s) - \widetilde{v}_j(\cdot,s)\Big)\Big]ds\right\|_{L^1(\mathbb{R}^2)}$$

$$\le c_2 \sum_{i=1}^{n}\int_0^t (t-s)^{-\frac{1}{2}}\,\|\overline{u}_i(\cdot,s) - \widetilde{u}_i(\cdot,s)\|_{L^\infty(\mathbb{R}^2)}\left\|\sum_{j=1}^{m}\alpha_{i,j}\nabla\overline{v}_j(\cdot,s)\right\|_{L^1(\mathbb{R}^2)} ds$$

$$+ c_2 \sum_{i=1}^{n}\int_0^t (t-s)^{-\frac{1}{2}}\,\|\widetilde{u}_i(\cdot,s)\|_{L^\infty(\mathbb{R}^2)}\left\|\sum_{j=1}^{m}\alpha_{i,j}\nabla\Big(\overline{v}_j(\cdot,s) - \widetilde{v}_j(\cdot,s)\Big)\right\|_{L^1(\mathbb{R}^2)} ds$$

$$\le 4c_2\alpha^* R T^{\frac{1}{2}}\,\|(\overline{\boldsymbol{u}},\overline{\boldsymbol{v}}) - (\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{v}})\|_X, \quad t\in[0,T],$$

$$I_3(t) = \sum_{j=1}^{m}\left\|\int_0^t e^{(t-s)(\Delta-\beta_j)}\left(\sum_{i=1}^{n}\gamma_{i,j}\,(\overline{u}_i(\cdot,s) - \widetilde{u}_i(\cdot,s))\right)ds\right\|_{L^1(\mathbb{R}^2)}$$

$$\le \gamma^* e^{|\beta^*|T}|\mathcal{J}|\sum_{i=1}^{n}\int_0^t \|\overline{u}_i(\cdot,s) - \widetilde{u}_i(\cdot,s)\|_{L^1(\mathbb{R}^2)}\,ds$$

$$\le \gamma^* e^{|\beta^*|T}|\mathcal{J}|T\|(\overline{\boldsymbol{u}},\overline{\boldsymbol{v}}) - (\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{v}})\|_X, \quad t\in[0,T],$$

$$I_4(t) = \sum_{j=1}^{m}\left\|\nabla\left[\int_0^t e^{(t-s)(\Delta-\beta_j)}\left(\sum_{i=1}^{n}\gamma_{i,j}\,(\overline{u}_i(\cdot,s) - \widetilde{u}_i(\cdot,s))\right)\right]\right\|_{L^p(\mathbb{R}^2)}$$

$$\le 2c_3\gamma^* e^{\beta^* T}|\mathcal{J}|R T^{\frac{1}{2}}\,\|(\overline{\boldsymbol{u}},\overline{\boldsymbol{v}}) - (\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{v}})\|_X, \quad t\in[0,T],$$

$$I_5(t) = \sum_{j=1}^{m}\left\|\nabla\left[\int_0^t e^{(t-s)(\Delta-\beta_j)}\left(\sum_{i=1}^{n}\gamma_{i,j}\,(\overline{u}_i(\cdot,s) - \widetilde{u}_i(\cdot,s))\right)\right]\right\|_{L^1(\mathbb{R}^2)}$$

$$\le 2c_4\gamma^* e^{\beta^* T}|\mathcal{J}|R T^{\frac{1}{2}}\,\|(\overline{\boldsymbol{u}},\overline{\boldsymbol{v}}) - (\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{v}})\|_X, \quad t\in[0,T].$$

So Π is a contraction if T is sufficiently small. Thus from Banach's fixed point theorem, Π has a fixed point in the sense that $(\boldsymbol{u},\boldsymbol{v}) = \Pi(\boldsymbol{u},\boldsymbol{v})$. Since the choice of above T depends only on R, α^*, β^*, γ^*, p and $|\mathcal{I}|$, $|\mathcal{J}|$, a standard argument implies that $(\boldsymbol{u},\boldsymbol{v})$ can be extended up to some $T_{\max}$, and

$$\limsup_{t\to T_{\max}}\left\{\sum_{i=1}^{n}\left[\|u_i(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} + \|u_i(\cdot,t)\|_{L^1(\mathbb{R}^2)}\right] + \sum_{j=1}^{m}\left[\|v_j(\cdot,t)\|_{L^1(\mathbb{R}^2)} + \|\nabla v_j(\cdot,t)\|_{L^p(\mathbb{R}^2)} + \|\nabla v_j(\cdot,t)\|_{L^1(\mathbb{R}^2)}\right]\right\} = \infty \tag{A5}$$

holds if $T_{\max} < \infty$.

Second Step: Regularity. Since $e^{t\Delta}$ and $\nabla\cdot$ commute on $C^1(\mathbb{R}^2;\mathbb{R}^2)\cap L^1(\mathbb{R}^2;\mathbb{R}^2)$, a straightforward regularity argument in [17, Lemma 3.3] which includes standard semigroup techniques and bootstrap procedure, and the parabolic Schauder estimates [26] imply that $(\boldsymbol{u},\boldsymbol{v})\in\left[C^{2,1}(\mathbb{R}^2\times(0,T_{\max}))\right]^{m+n}$. In fact, abbreviating $F_i(x,t) = \sum_{j=1}^{m}\alpha_{i,j}u_i(x,t)\nabla v_j(x,t)$ for some $i\in\mathcal{I}$ and from the regularity for the mild solution, we rewritten $(1.5)_i$ as

$$\partial_t u_i = \Delta u_i - \nabla \cdot F_i \quad \text{in } \mathbb{R}^2 \times (0, T) \tag{A6}$$

with continuous and bounded $\mathcal{F}_i$ in $\mathbb{R}^2 \times [0, T]$. Then the Step 2 in [17, Lemma 3.3] tells that u_i is a very weak solution to (A6), i.e.

$$-\int_0^T \int_{\mathbb{R}^2} u_i \phi_t - \int_{\mathbb{R}^2} u_{i0}\phi(\cdot, 0) = \int_0^T \int_{\mathbb{R}^2} u_i \Delta\phi$$
$$+ \int_0^T \int_{\mathbb{R}^2} F_i \cdot \nabla\phi \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^2 \times [0, T)).$$

Moreover, one can improve the regularity of very weak solution by introducing another solution $\overline{u}_i$ to the following initial boundary problem

$$\begin{cases} \partial_t \overline{u}_i = \Delta \overline{u}_i - \nabla \cdot F_i, \quad x \in B_R, \ t \in (\tau, T), \\ \overline{u}_i|_{\partial B_R} = u_i, \quad t \in (\tau, T), \\ \overline{u}(x, \tau) = u(x, \tau), \quad x \in B_R, \end{cases}$$

where $0 < \tau < T$, $R > 0$. Then a similar way in the Step 3 in [17, Lemma 3.3] makes sure that $\nabla u_i \in L^2_{loc}(\mathbb{R}^2 \times (\tau, T))$ and

$$-\int_\tau^T \int_{\mathbb{R}^2} u_i \phi_t - \int_{\mathbb{R}^2} u_i(\cdot, \tau)\phi(\cdot, \tau) = -\int_\tau^T \int_{\mathbb{R}^2} \nabla u_i \cdot \nabla\phi$$
$$+ \int_\tau^T \int_{\mathbb{R}^2} F_i \cdot \nabla\phi \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^2 \times [\tau, T)).$$

Hence, $\nabla \cdot F_i \in L^2_{loc}(\mathbb{R}^2 \times [\tau, T])$, which together with parabolic regularity theory [25] asserts that $u_i \in L^2((\tau, T); W^{2,2}_{loc}(\mathbb{R}^2))$ and $u_i \in L^p((\tau, T); W^{2,p}_{loc}(\mathbb{R}^2))$ for all $p \in (1, \infty)$ by the embedding theorem. Then invoking parabolic Schauder theory, we have $u_i \in C^{2+\gamma, 1+\frac{\gamma}{2}}_{loc}(\mathbb{R}^2 \times (\tau, T))$ with some $\gamma \in (0, 1)$, and u_i solves the i-th equation in (1.5) classically in $\mathbb{R}^2 \times (\tau, T)$. The proof is complete due to the arbitrary choice of τ.

Third Step: Uniqueness, positivity and mass conservation. Construct a non-increasing cut-off function $h(x) \in C^\infty(\mathbb{R})$ to fulfil $h(x) \equiv 1$ in $(-\infty, 0]$ and $h(x) \equiv 0$ in $[1, \infty)$. And for $K > 0$, set $\xi_K(x) := h(|x| - K)$, $x \in \mathbb{R}^2$. Under the help of cut-off function ξ_K, one can utilise localisation arguments to prove uniqueness, positivity and mass conservation of solutions to (1.5). Let us point out that such results were already obtained by Winkler in single-species case. We just describe the following main steps of the proof and refer to [41, Lemmas 2.4–2.7] for more details.

Now we prove the uniqueness. Proceeding as in [41, Lemma 2.4], given $T > 0$ and two solutions $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{v}})$ and $(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{v}})$ in $\mathbb{R}^2 \times (0, T)$, we let $\boldsymbol{w} = \overline{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}$ and $\boldsymbol{z} = \overline{\boldsymbol{v}} - \widetilde{\boldsymbol{v}}$ and obtain by applying straightforward procedure to (1.5) that

$$\partial_t w_i = \Delta w_i - \sum_{j=1}^m \alpha_{i,j} \nabla \cdot (w_i \nabla \overline{v}_j) - \sum_{j=1}^m \alpha_{i,j} \nabla \cdot (\widetilde{u}_i \nabla z_j), \quad i \in \mathcal{I}, \quad t \in [0, T],$$

$$\partial_t z_j = \Delta z_j - \beta_j z_j + \sum_{i=1}^n \gamma_{i,j} w_i, \quad j \in \mathcal{J}, \quad t \in [0, T].$$

With the help of cut-off function ξ_K and Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\xi_K^2 w_i^2 dx \le -\frac{1}{4}\int_{\mathbb{R}^2}\xi_K^2|\nabla w_i|^2 dx + 4\int_{\mathbb{R}^2}|\nabla\xi_K|^2 w_i^2 dx$$
$$+(\alpha^*)^2|\mathcal{J}|\int_{\mathbb{R}^2}\xi_K^2 w_i^2\left(\sum_{j=1}^m|\nabla\overline{v}_j|^2\right)dx$$
$$+(\alpha^*)^2|\mathcal{J}|\int_{\mathbb{R}^2}\xi_K^2(\widetilde{u}_i)^2\left(\sum_{j=1}^m|\nabla z_j|^2\right)dx$$
$$+2\sum_{j=1}^m\alpha_{i,j}\int_{\mathbb{R}^2}\xi_k w_i^2\nabla\xi_K\cdot\nabla\overline{v}_j dx$$
$$+2\sum_{j=1}^m\alpha_{i,j}\int_{\mathbb{R}^2}\xi_K\widetilde{u}_i w_i\nabla\xi_K\cdot\nabla z_j dx$$

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\xi_K^2|\nabla z_j|^2 dx \le \int_{\mathbb{R}^2}|\nabla\xi_K|^2\|\nabla z_j|^2 dx + (\beta^*+(\gamma^*)^2)\int_{\mathbb{R}^2}\xi_K^2|\nabla z_j|^2 dx$$
$$+\frac{|\mathcal{I}|}{4}\int_{\mathbb{R}^2}\xi_K^2\left(\sum_{i=1}^n|\nabla w_i|^2\right)dx.$$

By Hölder's, Young's and the Gagliardo–Nirenberg inequalities,

$$(\alpha^*)^2|\mathcal{J}|\int_{\mathbb{R}^2}\xi_K^2 w_i^2\left(\sum_{j=1}^m|\nabla\overline{v}_j|^2\right)dx \le c_5(\alpha^*)^2|\mathcal{J}|\|\xi_K w_i\|^2_{L^{\frac{2q}{q-2}}(\mathbb{R}^2)}$$
$$\le c_5c_6(\alpha^*)^2|\mathcal{J}|\|\nabla(\xi_K w_i)\|^{\frac{4}{q}}_{L^2(\mathbb{R}^2)}\|\xi_K w_i\|^{\frac{2(q-2)}{q}}_{L^2(\mathbb{R}^2)}$$
$$\le\frac{1}{8}\int_{\mathbb{R}^2}\xi_K^2|\nabla w_i|^2 dx + \frac{1}{8}\int_{\mathbb{R}^2}|\nabla\xi_K|^2 w_i^2 dx$$
$$+c_7\int_{\mathbb{R}^2}\xi_K^2 w_i^2 dx, \tag{A7}$$

$$(\alpha^*)^2|\mathcal{J}|\int_{\mathbb{R}^2}\xi_K^2(\widetilde{u}_i)^2\left(\sum_{j=1}^m|\nabla z_j|^2\right)dx \le c_8^2(\alpha^*)^2|\mathcal{J}|\sum_{j=1}^m\int_{\mathbb{R}^2}\xi_K^2|\nabla z_j|^2 dx,$$

where we set $c_5=\sup_{t\in(0,T)}\sum_{j=1}^m\|\nabla v_j(\cdot,t)\|^2_{L^q(\mathbb{R}^2)}$ and $c_8=\sup_{t\in(0,T)}\sum_{i=1}^n\|\widetilde{u}_i\|_{L^\infty(\mathbb{R}^2)}$. By the finiteness $\|(\nabla\overline{v}_j,\nabla z_j)\|_{L^q(\mathbb{R}^2)}$ for $j\in\mathcal{J}$, $\|(\widetilde{u}_i,w_i)\|_{L^\infty(\mathbb{R}^2)}$ for $i\in\mathcal{I}$ and *supp* $\xi_K\subset(-K,K)$, we find that $y_K(t):=\frac{1}{2}\sum_{i=1}^n\int_{\mathbb{R}^2}\xi_K^2 w_i^2 dx+\frac{1}{4|\mathcal{I}|}\sum_{j=1}^m\int_{\mathbb{R}^2}\xi_K^2|\nabla z_j|^2 dx$ satisfies

$$y_K'(t)\le c_9y_K(t)+c_9\sum_{i=1}^n\|w_i\|^{\frac{q-1}{q}}_{L^1(B_{R+1}\setminus B_R)}+c_9\sum_{i=1}^n\|w_i\|_{L^1(B_{R+1}\setminus B_R)}$$
$$+c_9\sum_{j=1}^m\|\nabla z_j\|^{\frac{q-2}{q-1}}_{L^1(B_{R+1}\setminus B_R)}\quad\text{for } t\in(0,T),$$

where due to $y_K(0)=0$ and (2.1), an integration over $(0,T)$ shows that $y_K(t)\to 0$ as $K\to\infty$. Hence $\overline{\boldsymbol{u}}=\widetilde{\boldsymbol{u}}$ and $\overline{\boldsymbol{v}}=\widetilde{\boldsymbol{v}}$ in $\mathbb{R}^2\times(0,T)$.

To prove the positivity of u_i, $i \in \mathcal{I}$, it is sufficient to make sure that u_i is non-negative in $\mathbb{R}^2 \times (0, T)$ for each $T \in (0, T_{\max})$ by the strong maximum principle and (1.6). Denote $u_i^- = \max\{-u_i, 0\}$. A direct computation shows that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \xi_K^2 (u_i^-)^2 dx \le -\frac{1}{2}\int_{\mathbb{R}^2} \xi_K^2 |\nabla u_i^-|^2 dx + (\alpha^*)^2 |\mathcal{J}| \int_{\mathbb{R}^2} \xi_K^2 |u_i^-|^2 \left(\sum_{j=1}^m |\nabla v_j|^2\right) dx$$

$$+ 4\int_{\mathbb{R}^2} |u_i^-|^2 |\nabla \xi_K|^2 dx + 2\sum_{j=1}^m \alpha_{i,j} \int_{\mathbb{R}^2} \xi_K (u_i^-)^2 \nabla \xi_K \cdot \nabla v_j dx.$$

Since $\nabla v_j \in L^\infty((0, T); L^p(\mathbb{R}^2))$ with $p > 2$, using a similar approach in (A7) shows the existence of $c_{10} > 0$ such that

$$\begin{aligned}
&(\alpha^*)^2 |\mathcal{J}| \int_{\mathbb{R}^2} \xi_K^2 |u_i^-|^2 \left(\sum_{j=1}^m |\nabla v_j|^2\right) dx \\
\le & c_5 (\alpha^*)^2 |\mathcal{J}| \|\xi_K u_i^-\|^2_{L^{\frac{2q}{q-2}}(\mathbb{R}^2)} \\
\le & \frac{1}{2}\int_{\mathbb{R}^2} \xi_K^2 |\nabla u_i^-|^2 dx + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla \xi_K|^2 (u_i^-)^2 dx + c_{10}\int_{\mathbb{R}^2} \xi_K^2 (u_i^-)^2 dx.
\end{aligned}$$

On the other hand, we follow a procedure in proving uniqueness and conclude that $g_K(t) := \int_{\mathbb{R}^2} \xi_K^2 (u_i^-)^2 dx$ fulfils $g_K(t) \to 0$ as $K \to \infty$. Hence non-negativity of u_i, $i \in \mathcal{I}$, has been proved.

Fourth Step: L^q estimates for v_j, $j \in \mathcal{J}$. Integrating j-th equation in (1.5) over $\mathbb{R}^2 \times (0, T)$ directly, it results in

$$\|v_j(\cdot, t)\|_{L^1(\mathbb{R}^2)} \le \begin{cases} \|v_{j0}\|_{L^1(\mathbb{R}^2)} + t\sum_{i=1}^n |\gamma_{i,j}| \|u_{i0}\|_{L^1(\mathbb{R}^2)}, & \text{if} \quad \beta_j = 0, \\ e^{-\beta_j t}\|v_{j0}\|_{L^1(\mathbb{R}^2)} + \frac{1}{|\beta_j|}\left|1 - e^{-\beta_j t}\right| \sum_{i=1}^n |\gamma_{i,j}| \|u_{i0}\|_{L^1(\mathbb{R}^2)}, & \text{if} \quad \beta_j \neq 0. \end{cases}$$

For $q > 1$, applying Lemma 5.1 to (A2) we infer that

$$\begin{aligned}
\|v_j(\cdot, t)\|_{L^q(\mathbb{R}^2)} \le & e^{|\beta_j| t}\|v_{j0}\|_{L^q(\mathbb{R}^2)} + e^{|\beta_j| t}\sum_{i=1}^n |\gamma_{i,j}| \int_0^t (t-s)^{-1+\frac{1}{q}} \|\widetilde{u}_i(\cdot, s)\|_{L^1(\mathbb{R}^2)} ds \\
\le & e^{\beta^* T}\|v_{j0}\|_{L^q(\mathbb{R}^2)} + q\gamma^* e^{\beta^* T} T^{\frac{1}{q}} \sum_{i=1}^n \|u_{i0}\|_{L^1(\mathbb{R}^2)}, \quad j \in \mathcal{J}, \quad t \in [0, T].
\end{aligned}$$

Hence, we have obtained L^q estimates for v_j, $j \in \mathcal{J}$, and found that the upper bound is independent of time variable if $\beta_j > 0$, $\forall\, j \in \mathcal{J}$.

Fifth Step: Criterion. If $T_{\max} < \infty$ and there exists a constant $c_{11} > 0$ such that

$$\sum_{i=1}^n \|u_i(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \le c_{11}.$$

Then from the mass conservation $\|u_i(\cdot, t)\|_{L^1(\mathbb{R}^2)} = \|u_{i0}\|_{L^1(\mathbb{R}^2)}$, $i \in \mathcal{I}$, and the boundedness of L^1 estimate for $v_j(\cdot, t)$, $j \in \mathcal{J}$, for all $t \in (0, T_{\max})$ and the following fact

$$\sum_{j=1}^m \left(\|\nabla v_j(\cdot, t)\|_{L^p(\mathbb{R}^2)} + \|\nabla v_j(\cdot, t)\|_{L^1(\mathbb{R}^2)}\right) \le c_{12}, \quad t \in (0, T_{\max}),$$

is right with some $c_{12} > 0$ because of (A3)–(A4), we claim that (A5) implies $T_{\max}$ cannot be finite.

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