# ON THE SUM OF THE RECIPROCALS OF THE FERMAT NUMBERS AND RELATED IRRATIONALITIES

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In **(1)**, P. Erdös showed that the function

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n}$$

takes on irrational values whenever $z = 1/t$, $t = 2, 3, 4, 5, \ldots$. The method of proof uses Lambert's identity,

$$f(z) = \sum_{n=1}^{\infty} d(n) z^n,$$

where $d(n)$ is the number of divisors of $n$; and it is shown that

$$f(1/t) = \sum_{n=1}^{\infty} \frac{d(n)}{t^n}$$

as a number written to the base $t$ has arbitrarily long finite runs of zeros. (This depends on deep arithmetic properties of $d(n)$.) Since the expansion to the base $t$ is accordingly not periodic, $f(1/t)$ is irrational.

In this paper, a conceptually similar approach will be used to show that both

$$F(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{1 + z^{2^n}} \quad \text{and} \quad G(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{1 - z^{2^n}}$$

take on irrational values whenever $z = 1/t$, $t = 2, 3, 4, 5, \ldots$. In particular,

$$F(\tfrac{1}{2}) = \sum_{n=0}^{\infty} \frac{1}{2^{2^n} + 1},$$

the sum of the reciprocals of the Fermat numbers, is irrational. The rapid growth of the Fermat numbers is *not* sufficient, in itself, to guarantee irrationality, as the sequence $b_1 = 2$, $b_{n+1} = (b_n - \frac{1}{2})^2 + \frac{3}{4}$ for $n \geqslant 1$, demonstrates. That is, the sequence 2, 3, 7, 43, 1807, ... grows as fast as the Fermat sequence (the terms of the two sequences alternate with one another in size), but

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \ldots = 1,$$

which is not an irrational sum.

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To begin with,

$$G(z) - F(z) = \sum_{n=0}^{\infty} \frac{2z^{2^{n+1}}}{1 - z^{2^{n+1}}} = 2\left(\frac{z}{1-z} + G(z)\right),$$

whereby

$$F(z) + G(z) = \frac{2z}{1-z}.$$

Since $2z/(1-z)$ is rational whenever $z$ is rational ($z \neq 1$), $F(1/t)$ and $G(1/t)$ are irrational for the same values of $t$. If we define

$$G_1(z) = \sum_{n=1}^{\infty} \frac{z^{2^n}}{1 - z^{2^n}},$$

we have

$$F(z) + G_1(z) = \frac{z}{1-z}, \qquad \text{or} \qquad F(1/t) + G_1(1/t) = \frac{1}{t-1}.$$

Next,

$$G_1(z) = \sum_{n=1}^{\infty} i(n) z^n,$$

where $i(n)$ is the exponent of 2 in the prime decomposition of $n$, that is, the number of zeros at the end of the binary representation of $n$. This is simply a Lambert identity, since

$$i(n) = \sum_{a|n} j(a),$$

where

$$j(a) = \begin{Bmatrix} 1 & \text{if } a = 2^k,\ k > 0 \\ 0 & \text{otherwise} \end{Bmatrix},$$

whence

$$\sum_{n=1}^{\infty} i(n) z^n = \sum_{n=1}^{\infty} \frac{j(n)\, z^n}{1 - z^n} = \sum_{n=1}^{\infty} \frac{z^{2^n}}{1 - z^{2^n}},$$

as asserted.

A curious identity for $F(z)$, which looks similar but is not simply a Lambert identity, is

$$F(z) = \sum_{n=0}^{\infty} w(n) z^n,$$

where $w(n)$ is the number of *ones* in the binary representation of $n$. Since this identity is not needed for the irrationality proof which follows, it will not be proved here.

We now proceed to show that $F(1/t)$ is irrational. We have

$$F(1/t) = \frac{1}{t-1} - \sum_{n=1}^{\infty} \frac{i(n)}{t^n}.$$

In arithmetic to the base $t$,

$$\frac{1}{t-1} = 0.1111111111111111\ldots$$

(which is valid even if $t = 2$), while

$$\sum_{n=1}^{\infty} \frac{i(n)}{t^n} = 0.i(1)\, i(2)\, i(3)\, i(4)\, i(5)\ldots = 0.0102010301020104\ldots,$$

where this last representation is *improper* in that $i(n)$ will sometimes exceed $t - 1$. We group the improper representation of

$$\sum_{1}^{\infty} \frac{i(n)}{t^n}$$

into segments, as follows:

$$0\,.\,(0)(1)(02)(0103)(01020104)(0102010301020105)\ldots.$$

The 0th segment is (0), the 1st segment is (1), the 2nd segment is (02), etc. In general, the $k$th segment ($k > 0$) has $2^{k-1}$ terms, and *ends* in $k$. For $k > 0$, the average of all the terms in the $k$th segment is 1, and the running average, starting from the left of the segment, is *less than* 1 until the very end. This means, for $k > 1$, that the $k$th segment,

$$\sum_{n=2^{k-1}+1}^{2^k} \frac{i(n)}{t^n},$$

is numerically *less* than

$$\sum_{n=2^{k-1}+1}^{2^k} \frac{1}{t^n},$$

though of course positive. Hence the expansion of

$$F(\tfrac{1}{2}) = \frac{1}{t-1} - \sum_{1}^{\infty} \frac{i(n)}{t^n}$$

may be computed segment by segment, without carrying or borrowing between segments, and with the result for the $k$th segment, in the base $t$, being strictly between 0 and

$$\sum_{2^{k-1}+1}^{2^k} \frac{1}{t^n},$$

for all $k > 1$. The subtraction therefore looks like:

$$\begin{array}{l} \phantom{-}0.(1)(1)(11)(1111)(11111111)\ldots(111\ldots111\ \ 1\ \ )\ldots \\ -0.(0)(1)(02)(0102)(01020103)\ldots(010\ldots010k{+}1)\ldots \\ \hline \phantom{-}0.(1)(0)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\ldots\quad 100\ \ 0\ \ )\ldots \end{array}$$

That is, the "first" segment (which follows the "zeroth" segment) in the result is (0), and the $(t + 1)$st segment ends in 000. Similarly, the $(t^3 + t+1)$st

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segment ends in 0000, and the $(t^5 - t^4 + t^3 + t + 1)$st segment ends in 000000. It is clear that for any specified number of zeros, $Z$, there is an $r(Z)$ such that the $r(Z)$th segment ends in $Z$ zeros. Specifically, $r$ is any number such that the improper representation $(\ldots 1030102010r)$ in the base $t$ ends in at least $Z$ *ones*, when rewritten as a proper representation. Thus we see that there are arbitrarily long runs of zeros in the expansion of $F(1/t)$ to the base $t$, so that $F(1/t)$ is irrational for $t = 2, 3, 4, 5, \ldots$. (The representation cannot degenerate to "all zeros," because in fact every segment past the first one is positive.) This proof is "constructive" in the sense that it facilitates the computation of $F(1/t)$ to the base $t$, and predicts in advance where runs of zeros of specified minimum length will be found.

As an example, in binary notation,

$$\sum_{n=0}^{\infty} \frac{1}{2^{2^n}+1} = \underset{(0)}{\underline{.1}}\ \underset{(1)}{\underline{0}}\ \underset{(2)}{\underline{01}}\ \underset{(3)}{\underline{1000}}\ \underset{(4)}{\underline{10010111}}\ \underset{(5)}{\underline{1001100010010110}}$$

$$\underset{(6)}{\underline{10011000100101111001100010010101}}$$

$$\underset{(7)}{\underline{1001100010010111100110001001011010011000100101111001100010010100}} \ldots$$

As shown previously, taking $t = 2$, the 1st segment is 0, the 3rd segment ends in 000, and the 11th segment will end in 0000. In base 16 notation, this becomes

$$\sum_{n=0}^{\infty} \frac{1}{2^{2^n}+1} = \frac{9}{15} - \sum_{n=1}^{\infty} \frac{i(n)}{16^n} = \underset{(0\text{–}2)}{\underline{.9}}\ \underset{(3)}{\underline{8}}\ \underset{(4)}{\underline{97}}\ \underset{(5)}{\underline{9896}}\ \underset{(6)}{\underline{98979895}}$$

$$\underset{(7)}{\underline{9897989698979894}}\ \underset{(8)}{\underline{98979896989798959897989698979893}} \ldots,$$

and the 11th segment will end in sexadecimal 0, which is binary 0000. From the end of one segment to the end of the next, there is a successive "counting down" by one, which guarantees that the *ends* of the segments go through all possible states. This is *not* equivalent to saying that the numbers are *normal*, because the ends of the segments form a minute fraction of the total expansion. (Thus, in the base 16 representation just exhibited, the symbol 9 occurs nearly half the time.)

### Reference

1. P. Erdös, *On arithmetical properties of Lambert Series*, J. Indian Math. Soc., *12* (1948), 63–66.

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