BULL. AUSTRAL. MATH. SOC. MOS 20E10, 20E15
VOL. 3 (1970), 293-311.

On the classification of just-non-Cross varieties of groups

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Apart from some (insoluble) subvarieties of $\underline{\underline{K}}_5$, the jnC (just-non-Cross) varieties known so far comprise the following list: $\underline{\underline{A}}$, $\underline{\underline{A}}_p\underline{\underline{A}}_p$, $\underline{\underline{A}}_p\underline{\underline{T}}_q$, $\underline{\underline{A}}_p\underline{\underline{A}}_q\underline{\underline{A}}_r$, where p, q and r are any three distinct primes. In a recent paper I gave a partial confirmation of the conjecture that the soluble jnC varieties all appear in this list. Here I show that a jnC variety is reducible if and only if it is soluble of finite exponent; this reduces the problem of classifying jnC varieties to finding the irreducibles of finite exponent. I observe that these fall into three distinct classes, and show that the questions of whether or not two of these classes are empty have some bearing on some apparently difficult problems of group theory.

1. Introduction

It is convenient for our purposes to call a variety (of groups) Cross if it can be generated by a single finite group; we shall not need the celebrated Theorem of Sheila Oates and M.B. Powell [*12*] that this definition is equivalent to the usual one. L.G. Kovács and M.F. Newman [*10*, Theorem 1] have pointed out that a variety is non-Cross if and only

Received 24 July 1970. This work was completed while the author was a Ph.D. student holding a Commonwealth Scholarship and Fellowship Plan award at the Australian National University. The author thanks his supervisor, Dr L.G. Kovács, for all the time and effort he has spent on him.

https://doi.org/10.1017/S0004972700046001 Published online by Cambridge University Press

if it contains a non-Cross variety whose proper subvarieties are all Cross: a so-called *just-non-Cross* variety. From now on, we abbreviate "just-non-Cross" to "jnC", for as our title indicates, this paper is a contribution to the problem of classifying all jnC varieties.

In [9], Kovács and Newman point out that the only jnC variety of infinite exponent is the variety $\underline{\underline{A}}$ of all abelian groups (it is of course also the only abelian jnC variety), and they show that the decomposable (that is, nontrivially factorisable) jnC varieties are precisely the $\underline{\underline{A}}_p\underline{\underline{A}}_p$, $\underline{\underline{A}}_p\underline{\underline{T}}_q$ and $\underline{\underline{A}}_p\underline{\underline{A}}_q\underline{\underline{A}}_r$, where p, q and r are any three distinct primes. In a recent paper [2], I have reduced their conjecture [9, p. 222] that every soluble jnC variety of finite exponent is decomposable by showing that if there exists a soluble jnC variety $\underline{\underline{V}}$ of finite exponent which is not decomposable, then one can find (distinct) primes p and q and an integer n (all three depending on $\underline{\underline{V}}$), such that $\underline{\underline{V}}$ is a subvariety of $\underline{\underline{A}}_p\left(\underline{\underline{V}} \wedge \underline{\underline{B}}_{q^n}\right)$, where $\underline{\underline{V}} \wedge \underline{\underline{B}}_{q^n}$ is nilpotent of class at least three. The main result of the present paper stems from the observation that a crucial property of soluble jnC varieties of finite exponent is that they are all *reducible*; that is, each is contained in a product of proper subvarieties.

THEOREM 1.1. *A* jnC *variety is reducible if and only if it is soluble of finite exponent.*

Thus, apart from my inability to decide Conjecture 1.3 of [2], Theorem 1.2 of [2] and Theorem 1.1 reduce the classification problem to finding the irreducible jnC varieties. In this direction, I merely state the following result, as a proof of it may be obtained by routinely amending the proof of (3) of L.G. Kovács [7, p. 13].

THEOREM 1.2. *A* jnC *variety is irreducible if and only if either*

(a) it is not locally finite, or

(b) it is locally finite and locally nilpotent but insoluble, or

(c) it is locally finite and contains infinitely many (isomorphism classes of) finite simple groups.

The recently announced insolubility of $\underline{\underline{K}}_5$ [1] implies the existence of irreducible jnC varieties of type *(b)*; but I have so far

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made little progress towards their classification. The existence of irreducible jnC varieties of type *(c)* would simultaneously falsify the conjectures that there are only a finite number of finite simple groups of given exponent, and that there is a bound to the number of elements necessary to generate a finite simple group. The existence of a non locally finite jnC variety $\underline{\underline{V}}$ of finite exponent n would have one of two interesting consequences. If $\underline{\underline{V}}$ were generated by its finite groups, then the restricted Burnside conjecture for exponent n would be false; in the other case, $\underline{\underline{V}}$ would have only finitely many subvarieties, contrary to the conjecture that a variety has this property if and only if it is Cross.

2. Some technical lemmas

Throughout this paper, "group" means "finite group", except in certain places, when its meaning will always be clear from the context. We shall follow as far as possible the notation of Hanna Neumann's book [*11*]; however, if G is a group and if $K \trianglelefteq H \leq G$, we shall prefer to call the quotient H/K a section of G. The *socle* $M(G)$ of a group G is the product of the minimal normal subgroups of G, and a group is called *monolithic* if it has only one minimal normal subgroup. The letters p and q will always denote prime numbers. Recall that a subgroup T of a group B is *intravariant* in B if the image of T under every automorphism of B is conjugate to T in B.

LEMMA 2.1. *A nonabelian simple group has a non-nilpotent intravariant proper subgroup.*

Proof. Let B be a nonabelian simple group, and let q be any odd prime dividing $|B|$. Sylow's Theorems assert that a Sylow q-subgroup Q of B is intravariant in B. Then $C_B(Z(Q))$ and $N_B(J(Q))$ are intravariant (necessarily proper) subgroups of B, where $J(Q)$ denotes the Thompson subgroup of Q (see [*6*, IV, 6.1]). By a theorem of J.G. Thompson [*6*, IV, 6.2], at least one of $C_B(Z(Q))$ and $N_B(J(Q))$ is not even q-nilpotent. //

LEMMA 2.2. *Let B be a p'-subgroup of a group G, and let T be an intravariant subgroup of B. If P is a p-subgroup of $N_G(B)$, then*

P also normalises some B-conjugate of T.

Proof. Let the distinct B-conjugates of T be $T = T_1, \ldots, T_n$. Since $n = |B : N_B(T)|$, p does not divide n. Now P normalises B, and T is intravariant in B, and so P permutes (by conjugation) $\{T_1, \ldots, T_n\}$. Since the orbits of P have cardinality a power of p, there is a fixed point, say T_i, in $\{T_1, \ldots, T_n\}$; that is, $P \leq N_G(T_i)$. //

The proof of Theorem 1.1 essentially depends upon a close analysis of the following situation. Let G be a group, and let N be a minimal normal subgroup of G. Conjugation by elements of G induces automorphisms of N; in this way G is represented as a subgroup of $\mathrm{Aut}N$ with kernel $C_G(N)$. If N is abelian, say of exponent p, we can think of it as a vector space over the field $\mathrm{F}(p)$ of p elements; thus the representation of G is a group representation in the sense of Curtis and Reiner [4]. If, on the other hand, N is nonabelian, there is a nonabelian simple group, say B, such that N is isomorphic to a direct power of B. We shall show as Corollary 2.4 that in this case we are led to consider permutation representations of G.

LEMMA 2.3. *Let B be a group with trivial centre, and let K be a normal subgroup of the direct product $B_1 \times \ldots \times B_n$, where each B_i is isomorphic to B. Then $K \cap B_i$ is nontrivial if and only if the image of K under its projection into B_i is nontrivial. In particular, if B is a nonabelian simple group, K is the direct product of some subset of $\{B_1, \ldots, B_n\}$.*

Proof. Denote the projection of K into B_i by π_i. If $K\pi_i$ is nontrivial, there is an element, say k, in K with $k\pi_i \neq e$. Since $Z(B_i)$ is trivial, there is an element, say b_i, of B_i which fails to commute with $k\pi_i$. Then $[k, b_i]$ is a nonidentity element of $K \cap B_i$. //

COROLLARY 2.4. *Let the simple direct factors of the nonabelian minimal normal subgroup* N *of* G *be* $B_1, \ldots, B_n$. *Then if* $g \in G$, *and* $1 \le i \le n$, $B_i^g \in \{B_1, \ldots, B_n\}$. *Thus* G *is represented by conjugation as a transitive permutation group on* $\{B_1, \ldots, B_n\}$. //

As far as permutation representations by conjugation are concerned, we shall need a corollary to the following theorem.

THEOREM 2.5. *Let* G *be a group, and suppose that the subgroup* M *of* G *is the direct product of its subgroups* $B_1, \ldots, B_{p^t}$. *Let* P *be a p-subgroup of* G , *and suppose that conjugation by elements of* P *transitively permutes* $\left\{B_1, \ldots, B_{p^t}\right\}$. *If* $P \cap B_1$ *is nontrivial,* P *has class at least* $t+1$.

Proof. The claim is trivial for $t = 0$, so suppose $t > 0$. Put $\Omega = \left\{B_1, \ldots, B_{p^t}\right\}$, and denote $\cap\left\{N_P\left(B_i\right) : 1 \le i \le p^t\right\}$ by N and $N_P\left(B_1\right)$ by P_0 . Then P/N acts (by conjugation) as a transitive permutation group on Ω , the stabiliser of the "point" B_1 being P_0/N . By [*14*, 3.2], $\left|P : P_0\right| = p^t$, and so we can choose subgroups $P_1, \ldots, P_t$ of P such that

$$P_0 \triangleleft P_1 \triangleleft \ldots \triangleleft P_t = P .$$

Since $\left|P_i : P_0\right| = p^i$, an orbit of P_i has cardinality p^i . If S is a subset of P , $\left\langle B_1^{SN/N}\right\rangle$ is the direct product of the elements of $\left\{B_1^g : g \in S\right\}$, and so we may assume that the points of Ω have been numbered so that

$$\left\langle B_1^{P_i/N}\right\rangle = B_1 \times \ldots \times B_{p^i} ,$$

$0 \le i \le t$. Let x_i be an element of $P_i - P_{i-1}$, $0 < i \le t$. Observe that since $P_{i+1} = \langle P_i, x_{i+1} \rangle$, $B_1^{P_i/N} \cap B_1^{P_i x_{i+1} N/N}$ is empty. Let x_0 be an element of B_1 , $e \neq x_0$. We shall prove by induction on i that (for $0 \leq i \leq t$) $e \neq [x_0, x_1, \ldots, x_i] \in \langle B_1^{P_i/N} \rangle$. In case $i = 0$, this claim reduces to $e \neq x_0 \in B_1$. Suppose that $i < t$, and that $e \neq [x_0, \ldots, x_i] \in \langle B_1^{P_i/N} \rangle$. Since $\langle P_i, x_{i+1} \rangle = P_{i+1}$, it follows that $[x_0, \ldots, x_{i+1}] \in \langle B_1^{P_{i+1}/N} \rangle$. Now

$$[x_0, \ldots, x_{i+1}] = [x_0, \ldots, x_i]^{-1}[x_0, \ldots, x_i]^{x_{i+1}} ,$$

and $e \neq [x_0, \ldots, x_i]^{x_{i+1}} \in \langle B_1^{P_i x_{i+1} N/N} \rangle$. Since $B_1^{P_i/N} \cap B_1^{P_i x_{i+1} N/N}$ is empty, the claim is established. In particular, if $P \cap B_1$ is nontrivial, we can choose $x_0 \in P$. Since $[x_0, \ldots, x_t] \neq e$, it follows that P has class at least $t + 1$. //

COROLLARY 2.6. *Let the minimal normal subgroup M of G be isomorphic to a direct product of p^t copies of a nonabelian simple group. If P is a Sylow p-subgroup of G , and $M \cap P$ is nontrivial, then P has class at least $t + 1$.*

Proof. By Corollary 2.4 and Theorem 2.5. //

We shall need a lemma and two theorems about group representations.

THEOREM 2.7. *Let B be a nonabelian group, E a field, and V a faithful EB-module. In case E has (nonzero) characteristic p , suppose that B' is not a p-group. Then there is an abelian-by-cyclic subgroup S of B such that V_S has an irreducible submodule of dimension at least two.*

Proof. Suppose E has characteristic p , and choose any prime q unequal to p which divides $|B|$. (Thus if p is zero, q may be any prime dividing $|B|$.) Let Q be any abelian q-subgroup of B , and let $g \in N_B(Q)$. Denote $\langle Q, g \rangle'$ by R ; then $R \leq Q$, and so by Maschke's

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Theorem, V_R is completely reducible; say

$$V_R = V_1 \oplus \ldots \oplus V_k$$

is a decomposition of V_R into irreducible submodules. Let W_1 be the sum of those V_i that are trivial, and W_2 the sum of those that are not. Observe that both W_1 and W_2 admit $\langle Q, g\rangle$, and hence that

$$V_{\langle Q,g\rangle} = W_1 \oplus W_2 \ .$$

If W_2 is nonzero for some choice of Q and g , let U be any irreducible $E\langle Q, g\rangle$-submodule of it. Now no nonzero element of U is fixed by the whole of R , and so R is nontrivial and the kernel N of U avoids R . It follows that $\langle Q, g\rangle/N$ is nonabelian and is faithfully and irreducibly represented on U ; thus U has E-dimension at least two. In this case, therefore, we can choose S equal to $\langle Q, g\rangle$.

Suppose on the other hand, that for all possible choices of Q and g , W_2 is zero. Since V is faithful, it follows that R is always trivial, and hence that $N_B(Q) = C_B(Q)$ for all abelian q-subgroups Q of G . But in any q-group, the subgroups maximal with respect to being abelian and normal are self-centralising [6, III, 7.3]. Since we could have chosen g to be a q-element, we must conclude that every q-subgroup of B is abelian. By a theorem of Burnside [6, IV, 2.6], B is q-nilpotent for all q unequal to p . If p is zero, it follows that B is nilpotent, and hence abelian, a contradiction. If p is not zero, then since the normal q-complement $B(q)$ of B is complemented in B by an abelian Sylow q-subgroup of B , $B(q) \geq B'$. It follows that $B' \leq \cap\{B(q) : q \neq p\}$, and hence that B' is a p-group. Again we have a contradiction, and the Theorem is proved. //

In Section 4 we shall need to use a rather detailed version of Clifford's Theorem [6, V, 17.3], and so we shall find it useful to have available the following abbreviation: if G is a group, N is a normal subgroup of G , and V is an irreducible EG-module, for some field E , we shall say

$$V_N = \bigoplus_{i=1}^{k} V_i \ ; \quad V_i = \bigoplus_{j=1}^{l} V_{ij} \quad \textit{is a Clifford decomposition of } V_N$$

https://doi.org/10.1017/S0004972700046001 Published online by Cambridge University Press

to indicate that $V_1, \ldots, V_k$ are the homogeneous components of V_N, and that

$$V_i = \bigoplus_{j=1}^{l} V_{ij}$$

is a direct decomposition of V_i into (isomorphic) irreducible submodules.

LEMMA 2.8. *Let* G *be a group,* E *a field, and* V *an irreducible* EG*-module. Let* N *be a normal subgroup of* G *, and suppose that* V_N *has* k *homogeneous components. If* Q *is a Sylow* q*-subgroup of* G *, and* U *is an irreducible submodule of* V_{NQ} *, then the number of homogeneous components of* U_N *is at least the* q*-share of* k *. In particular, the* E*-dimension of* U *is at least the* q*-share of* k *.*

Proof. Suppose that $V_1, \ldots, V_k$ are the homogeneous components of V_N. Let H_i be the inertia group of V_i, $1 \le i \le k$, and denote $\bigcap_{i=1}^{k} H_i$ by H. By Clifford's Theorem, G/H acts as a transitive permutation group on $\{V_1, \ldots, V_k\}$, the stabiliser of V_i being H_i/H. Since $N \lhd NQ$, it follows from Clifford's Theorem that U_N is completely reducible, and that

$$U_N = \left(U_N \cap V_1\right) \oplus \ldots \oplus \left(U_N \cap V_k\right)$$

is the decomposition of U_N into its homogeneous components (although we allow for the possibility that some of the $\left(U_N \cap V_i\right)$ will be zero). We may suppose without loss of generality that the V_i have been numbered so that $\left(U_N \cap V_1\right)$ is not zero; then since U is irreducible it is spanned by $\left(U_N \cap V_1\right)Q$. Hence the number of homogeneous components of U_N is the cardinality of the orbit of QH/H containing V_1, and this is at least the q-share of k [*14*, 3.4]. //

The "outer tensor product theorem" ([*6*, V, 10.3] or better still [*3*, 1.3.15]) is usually stated for direct products. The statement of it which we give here without proof incorporates the observation that it remains true when "direct" is replaced by "central".

THEOREM 2.9. *Let* G *be a finite group given as a central product of its subgroups* $G_1, \ldots, G_n$; *let* H_i *be a subgroup of* G_i $(i = 1, \ldots, n)$, *and* H *the (central) product of* $H_1, \ldots, H_n$ *in* G . *Let* E *be a field and* U *an irreducible* EG*-module. If, for each* i , W_i *is an absolutely irreducible submodule of* U_{H_i} , *then* U_H *has an (absolutely) irreducible submodule* W *isomorphic to* $W_1 \# \ldots \# W_n$. //

We conclude this section by recalling Kovács and Newman's version of the Oates-Powell Theorem, as it is the more convenient for our purposes. For positive integers e, m and c , denote by $\underline{\underline{C}}(e, m, c)$ the class of all (not necessarily finite) groups of exponent dividing e whose chief-sections have order (at most) m , and whose nilpotent sections have class (at most) c . They prove [8]:

THEOREM 2.10. *For all positive integers* e, m *and* c , $\underline{\underline{C}}(e, m, c)$ *is a Cross variety. Furthermore, a variety* $\underline{\underline{V}}$ *is Cross if and only if there exist positive integers* e, m *and* c *such that* $\underline{\underline{V}}$ *is a subclass of* $\underline{\underline{C}}(e, m, c)$. //

3. Various varietal results

In this section we deduce a number of facts about varieties which will later be used in the proof of Theorem 1.1. The most substantial of these, Theorem 3.1, arose from attempts to generalise Lemma 5 of [13]. I am indebted to Dr L.G. Kovács for suggesting it to me. The statement of Theorem 3.1 which we give here serves also to introduce some notation.

THEOREM 3.1. *Let* $\underline{\underline{V}}$ *be a variety of finite exponent* n *in which the nilpotent groups have class at most* c , *and let* B *be a nonabelian simple group. Suppose that* $\underline{\underline{V}}$ *contains an infinite set* Γ *of pairwise nonisomorphic monolithic groups, such that the monolith* $M(G)$ *of each group* G *in* Γ *is isomorphic to a direct power, say* $B^{\alpha(G)}$, *of* B . *(In this way we define a function* α *from* Γ *to the set* P *of positive integers.) Then* $\underline{\underline{V}}$ *is non-Cross, and it has a non-Cross* *subvariety to which* B *does not belong.*

The proof of Theorem 3.1 falls naturally into three steps, the first two of which we isolate as lemmas. First, the claim that $\underline{\underline{V}}$ is non-Cross is easily established.

LEMMA 3.2. $\alpha(\Gamma)$ *is an infinite subset of* P *, and* $\mathrm{var}\Gamma$ *is a non-Cross subvariety of* $\underline{\underline{V}}$ *. In particular,* $\underline{\underline{V}}$ *is non-Cross.*

Proof. Suppose to the contrary that $\alpha(\Gamma)$ is a finite subset of P , say $\alpha(\Gamma) < a$ for all $G \in \Gamma$. Then $\{|M(G)| : G \in \Gamma\}$ is bounded by $|B|^a$. Now $G/C_G(M(G))$ is isomorphic to a subgroup of $\mathrm{Aut}M(G)$, and $C_G(M(G))$ is trivial for $G \in \Gamma$. Hence $\{|G| : G \in \Gamma\}$ is bounded by $(|B|^a)!$, and so Γ is a finite set. This contradiction establishes the first claim; all the others follow from it and Theorem 2.10. //

LEMMA 3.3. *There is a prime* p *and an infinite set* Δ *of monolithic groups in* $\underline{\underline{V}}$ *, such that*

(i) $\mathrm{var}\Delta$ *is a non-Cross subvariety of* $\underline{\underline{V}}$ *;*

(ii) the monolith of each group H *in* Δ *is isomorphic to* $B^{p^{\beta(H)}}$ *, and* $\beta(\Delta)$ *is an infinite subset of* P *;*

(iii) if $H \in \Delta$ *,* $M(H)$ *is supplemented in* H *by a Sylow* p*-subgroup.*

Proof. Let $G \in \Gamma$, and suppose that the direct factors of $M(G)$ are $B_1, \ldots, B_{\alpha(G)}$. Denote $N_G(B_i)$ by N_i , and $\cap\{N_i : 1 \leq i \leq \alpha(G)\}$ by N . By Corollary 2.4, G is represented (by conjugation) as a transitive permutation group on $\{B_1, \ldots, B_{\alpha(G)}\}$ with kernel N ; the stabiliser of B_i being N_i/N . Since $|G : N_i| = \alpha(G)$, $1 \leq i \leq \alpha(G)$, the prime divisors of $\alpha(G)$ all divide n . But n is finite, and, by Lemma 3.2, $\alpha(\Gamma)$ is an infinite subset of P ; hence there is a prime, say p , such that $\alpha_p(\Gamma)$ is an infinite subset of P , where $\alpha_p(G)$ is the p-share of $\alpha(G)$. If P is a Sylow p-subgroup of G , the orbits of PN/N have cardinality a power, say $p^{\beta(G)}$, of p , and

$p^{\beta(G)} \geq \alpha_p(G)$ [14, 3.4]. Denote $p^{\beta(G)}$ by $\gamma(G)$, and suppose that the B_i have been numbered so that the orbit of PN/N containing B_1 is $\{B_1, \ldots, B_{\gamma(G)}\}$. Put $\langle B_1, P\rangle$ equal to A , and choose $\Delta = \{A/Z_\infty(A) : G \in \Gamma\}$.

If K is the normal closure of B_1 in A , $K = B_1 \times \ldots \times B_{\gamma(G)}$. By Lemma 2.3, K is a minimal normal subgroup of A ; but A need not be monolithic, as there may be (necessarily central) minimal normal subgroups of A contained in P . Thus $A/Z_\infty(A)$ is monolithic, and since $Z_\infty(A)$ avoids K , the monolith of $A/Z_\infty(A)$ is isomorphic to K . An application of Theorem 2.10 completes the proof. //

COROLLARY 3.4. *B is a p'-group.*

Proof. Since $\beta(\Delta)$ is an infinite subset of P , there is a group, say G_1 , in Δ with $\beta(G_1) > c$. Let P_1 be a Sylow p-subgroup of G_1 , and suppose B_1 is a simple direct factor of $M(G_1)$. If p divides $|B|$, $P_1 \cap B_1$ is nontrivial, and so by Corollary 2.6, P_1 has class greater than c . //

We are now ready to prove that $\underline{\underline{V}}$ has a non-Cross subvariety to which B does not belong.

Let $G \in \Delta$, and let P be a Sylow p-subgroup of G ; by Lemma 3.3 and Corollary 3.4, G is a split-extension of $M(G)$ by P . Denote the simple direct factors of $M(G)$ by $B_1, \ldots, B_{\gamma(G)}$ (where, as before, $\gamma(G) = p^{\beta(G)}$) , $N_G(B_i)$ by N_i and $\cap\{N_i : 1 \leq i \leq \gamma(G)\}$ by N . By Lemmas 2.1 and 2.2, B_1 has a non-nilpotent, proper, intravariant subgroup, say T_1 , such that $N_p(T_1)$ contains (and hence equals) $P \cap N_1$. Denote $\langle T_1, P\rangle$ by H , and the normal closure of T_1 in H by T . If $T \cap B_i$ is T_i , $T = T_1 \times \ldots \times T_{\gamma(G)}$. Suppose that $Z_\infty(T)$ is Y and $Z_\infty(T_j)$ is Y_j ; then $Y = Y_1 \times \ldots \times Y_{\gamma(G)}$, and since T_i is non-nilpotent, $Y_i < T_i$, $1 \leq i \leq \gamma(G)$. Observe that Y is normal in H , being characteristic in T . Denote H/Y by $\bar{H}$, T/Y by $\bar{T}$, T_iY/Y

by $\overline{T}_i$, PY/Y by $\overline{P}$ and $\{\overline{H} : H \in \Delta\}$ by Λ . Then $\overline{H}$ is a split-extension of $\overline{T}$ by $\overline{P}$, and $\overline{T} = \overline{T}_1 \times \dots \times \overline{T}_{\gamma(G)}$. Since $Z(\overline{T}_i)$ is trivial, and $\overline{P}$ connects $\{\overline{T}_i : 1 \le i \le \gamma(G)\}$ transitively, Lemma 2.3 implies that a minimal normal subgroup $\overline{L}$ of $\overline{H}$ contained in $\overline{T}$ intersects each $\overline{T}_i$ nontrivially. Thus $|\overline{L}| \ge \gamma(G) = p^{\beta(B)}$, and so by Theorem 2.10, varΛ is a non-Cross subvariety of $\underline{\underline{V}}$.

Observe that varΛ is a subvariety of $(\mathrm{var}T_1).\underline{\underline{N}}$, where $\underline{\underline{N}}$ is the variety of nilpotent groups in $\underline{\underline{V}}$. Since B is critical [*11*, 51.34], it does not belong to varT_1 . Hence B does not belong to varΛ . //

COROLLARY 3.5. *Let* $\underline{\underline{V}}$ *be a* jnC *variety of finite exponent in which the nilpotent groups form a subvariety, and let* B *be a nonabelian simple group. Then* $\underline{\underline{V}}$ *contains only finitely many (isomorphism classes of) monolithic groups whose monoliths have a direct factor isomorphic to* B . //

We conclude this section with two lemmas which describe some important properties of reducible jnC varieties.

LEMMA 3.6. *(i) A reducible* jnC *variety is locally finite, and contains only finitely many (isomorphism classes of) finite simple groups.*

(ii) A jnC *variety is reducible and locally nilpotent if and only if it is* $\underline{\underline{A}}_p\underline{\underline{A}}_p$ *for some prime* p .

Proof. *(i)* Suppose that $\underline{\underline{V}}$ is a reducible jnC variety, say $\underline{\underline{V}}$ is a subvariety of $\underline{\underline{V}}_1\underline{\underline{V}}_2$, where the $\underline{\underline{V}}_i$ are proper (and hence Cross) subvarieties of $\underline{\underline{V}}$. Since Cross varieties are locally finite, the first part of *(i)* follows from [*11*, 21.14]. A simple group in $\underline{\underline{V}}$ belongs either to $\underline{\underline{V}}_1$ or to $\underline{\underline{V}}_2$. But simple groups are critical [*11*, 51.34], and Cross varieties contain only finitely many (isomorphism classes of) critical groups.

(ii) If $\underline{\underline{V}}$ is also locally nilpotent, the Oates-Powell Theorem shows that both $\underline{\underline{V}}_1$ and $\underline{\underline{V}}_2$ are nilpotent, and hence that $\underline{\underline{V}}$ is soluble. It then follows from [*10*, Theorem 5] that $\underline{\underline{V}}$ is $\underline{\underline{A}}_p\underline{\underline{A}}_p$ for some prime p . The "if" part of *(ii)* is trivial. //

LEMMA 3.7. *Let* $\underline{\underline{V}}$ *be a locally finite* ȷnC *variety which contains only finitely many (isomorphism classes of) simple groups. If* $\underline{\underline{V}}$ *is not locally nilpotent, there is a prime* p *and a (countably) infinite set* Γ *of monolithic groups in* $\underline{\underline{V}}$ *, such that*

(i) $\underline{\underline{V}} = \mathrm{var}\Gamma$;

(ii) *the monolith of each group in* Γ *is complemented, self-centralising and has exponent* p ;

(iii) $\{|M(G)| : G \in \Gamma\}$ *is an infinite set.*

In particular, the conclusions follow when $\underline{\underline{V}}$ *is a reducible* ȷnC *variety.*

Proof. Since $\underline{\underline{V}}$ is locally finite and not locally nilpotent, there is a bound on the class of the nilpotent groups in $\underline{\underline{V}}$. Now a locally finite variety has finite exponent, and is generated by its finite groups [*11*, 15.63], and so by Theorem 2.10, the orders of the chief-sections of the finite groups in $\underline{\underline{V}}$ form an infinite set. Hence there is a countably infinite set, say Δ , of finite groups in $\underline{\underline{V}}$ such that the orders of the chief-sections of the groups in Δ form an infinite set. Since $\underline{\underline{V}}$ is closed under the operation of taking homomorphic images, we may as well suppose that the orders of the minimal normal subgroups of the groups in Δ form an infinite set. From each $G \in \Delta$, select a minimal normal subgroup $N(G)$ of G , so that $\{|N(G)| : G \in \Delta\}$ is infinite. Let Ω be a (finite) set containing one copy of each of the simple groups in $\underline{\underline{V}}$; then each $G \in \Delta$ determines uniquely an element $B(G)$ in Ω , and a natural number $m(G)$, such that $N(G) \cong B(G)^{m(G)}$. Since Ω is a finite set, it contains an element, say B , such that

$$\{m(G) : B(G) = B,\ G \in \Delta\} \text{ is infinite.}$$

Put $\Delta_1 = \{G : B(G) = B,\ G \in \Delta\}$; since $\underline{\underline{V}}$ is ȷnC , it follows from Theorem 2.10 that $\underline{\underline{V}} = \mathrm{var}\Delta_1$. In case B is nonabelian, put $\Delta_2 = \left\{G/C_G(N(G)) : G \in \Delta_1\right\}$. Observe that every group in Δ_2 is monolithic with monolith isomorphic to a direct power of B . By our choice of B , $\{|M(G)| : G \in \Delta_2\}$ is an infinite set, and we have a contradiction to Corollary 3.5.

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Hence B is abelian, say of order p. Applying [2, 2.2], we replace each group G in Δ_1 by G^*, and put $\Gamma = \{G^* : G \in \Delta_1\}$. Since $N(G^*)$ is similar to $N(G)$, the lemma follows from Theorem 2.10. //

4. The proof of Theorem 1.1

The "if" part of Theorem 1.1 is easy to prove. For if $\underline{\underline{V}}$ is a jnC variety of finite exponent n which is also soluble of length l, then $\underline{\underline{V}}$ is a subvariety of $\left(\underline{\underline{V}} \wedge \underline{\underline{A}}_n\right)^l$. It follows that $\underline{\underline{V}}$ is reducible.

Conversely, let $\underline{\underline{V}}$ be a reducible jnC variety. Then by Lemma 3.6, $\underline{\underline{V}}$ is locally finite, say $\underline{\underline{V}}$ has (finite) exponent n. Moreover, by the same result, $\underline{\underline{V}}$ contains only finitely many simple groups; let Λ be a (finite) set containing one copy of each of them. If $\underline{\underline{V}}$ is locally nilpotent, then by Lemma 3.6 *(ii)*, $\underline{\underline{V}}$ is an $\underline{\underline{A}}_p\underline{\underline{A}}_p$, and so it is soluble of finite exponent. Suppose, therefore, that $\underline{\underline{V}}$ is not locally nilpotent, and consequently that there is a bound, say c, on the class of nilpotent groups in $\underline{\underline{V}}$. Then by Lemma 3.7 (and implicitly Corollary 3.5), there is a prime, say p, and an infinite set Γ of pairwise-nonisomorphic monolithic groups in $\underline{\underline{V}}$ such that

(i) $\underline{\underline{V}} = \operatorname{var}\Gamma$;

(ii) $\{|M(H)| : H \in \Gamma\}$ is an infinite set, and

(iii) the monolith of each group in Γ is complemented, self-centralising, and has exponent p.

By [5, 1.2.2], a soluble group in $\underline{\underline{V}}$ has solubility length at most n_1c^2, where n_1 denotes the number of primes dividing n. Hence the soluble groups in $\underline{\underline{V}}$ form a subvariety, namely $\underline{\underline{V}} \wedge \underline{\underline{A}}^{nc^2}$. For a proof of Theorem 1.1 by contradiction, we assume that $\underline{\underline{V}} \wedge \underline{\underline{A}}^{nc^2}$ is a proper, and hence Cross, subvariety of $\underline{\underline{V}}$. Using Theorem 2.10, we may restate this as follows:

(4.1). The orders of the chief-sections of the soluble groups in $\underline{\underline{V}}$ are bounded, say by d.

Now let $H \in \Gamma$, denote $M(H)$ by V, and let G be a complement for V in H. By [11, 52.24], and properties *(ii)* and *(iii)* of Γ,

$\{|G| : H \in \Gamma\}$ is an infinite set. If we think of V as a faithful irreducible $\mathrm{F}(p)G$-module, we have as a consequence of (4.1):

(4.2). If $A \leq G$, and U is an irreducible submodule of V_A, and $A/\ker U$ is soluble, then U has order at most d.

Let S be the soluble radical of G, and suppose that

$$V_S = \bigoplus_{i=1}^{a(H)} U_i \ ; \quad U_i = \bigoplus_{j=1}^{b(H)} U_{ij}$$

is a Clifford decomposition of V_S; let K_i be the kernel of U_{i1}.

LEMMA 4.3. *The sets* $\{a(H) : H \in \Gamma\}$ *and* $\{|S| : H \in \Gamma\}$ *are finite.*

Proof. Suppose that $\{a(H) : H \in \Gamma\}$ is an infinite set. By Clifford's Theorem, $a(H)$ is the index of the inertia group of U_1 in G, and so the prime divisors of $a(H)$ all divide n. But n is finite, and so there is a prime, say q (which may be p), such that $\{a_q(H) : H \in \Gamma\}$ is infinite, where $a_q(H)$ is the q-share of $a(H)$. In particular, there is an element, say H_1, of Γ such that $a_q(H_1) > d$. Then if Q_1 is a Sylow q-subgroup of G_1, Lemma 2.8 implies that the $\mathrm{F}(p)$-dimension of an irreducible submodule of $V_{1_{S_1Q_1}}$ is at least $a_q(H_1)$. Since S_1Q_1 is a soluble subgroup of G_1, this contradicts (4.2).

Hence $\{a(H) : H \in \Gamma\}$ is finite, say $a(H) < a$ for all $H \in \Gamma$. Since S is irreducibly represented on U_{i1}, it follows from (4.2) that $|U_{i1}| \leq d$, and hence that $|S : K_i| \leq d!$. But V is faithful, and so $\cap\{K_i : 1 \leq i \leq a(H)\}$ is trivial. It follows that $|S| < (d!)^a$. //

Suppose that $|S| < b$, for all $H \in \Gamma$, and denote $C_G(S)$ by C; then $\{|G : C| : H \in \Gamma\}$ is bounded by $b!$, and $\{|C| : H \in \Gamma\}$ is an infinite set. Let $M_1/C\cap S, \ldots, M_{\mu(H)}/C\cap S$ be the minimal normal subgroups of $G/C\cap S$ contained in $C/C\cap S$, and denote $M_1M_2 \ldots M_{\mu(H)}$ by M. Since $C \cap S$ is simultaneously the centre of S, the soluble

https://doi.org/10.1017/S0004972700046001 Published online by Cambridge University Press

radical of C, and the centre of C, $M_i/C\cap S$ is nonabelian, $1 \le i \le \mu(H)$, and so there is a nonabelian simple group, say $B(i, H)$, in Λ such that $M_i/C\cap S$ is isomorphic to a direct power of $B(i, H)$. Observe that $M/C\cap S = M(C/C\cap S)$.

LEMMA 4.4. *$\{|M_i| : 1 \le i \le \mu(H), H \in \Gamma\}$ is a finite set, whereas $\{|M| : H \in \Gamma\}$ is infinite. In particular, $\{\mu(H) : H \in \Gamma\}$ is an infinite set.*

Proof. By Lemma 4.3, $\{|C \cap S| : H \in \Gamma\}$ is a finite set, whereas $\{|C| : H \in \Gamma\}$ is infinite; consequently $\{|C : C \cap S| : H \in \Gamma\}$ is infinite. Now $M/C\cap S$ is the socle of $C/C\cap S$ and is isomorphic to a direct product of nonabelian simple groups; hence $C_{C/C\cap S}(M/C\cap S)$ is trivial. Thus $\{|M : C \cap S| : H \in \Gamma\}$ is infinite.

If $\{|M_i| : 1 \le i \le \mu(H), H \in \Gamma\}$ is an infinite set, then so is $\{|M_i : C \cap S| : 1 \le i \le \mu(H), H \in \Gamma\}$. But Λ is a finite set, and so it contains an element, say B, such that

$$\Omega = \{|M_i : C \cap S| : B(i, H) = B, 1 \le i \le \mu(H), H \in \Gamma\}$$

is an infinite set. (Observe that B is nonabelian.) Then if $C_{C/C\cap S}(M_i/C\cap S)$ is denoted by D_i,

$$\Delta = \{(C/C\cap S)/D_i : B(i, H) = B, 1 \le i \le \mu(H), H \in \Gamma\}$$

is a set of monolithic groups in $\underline{\underline{V}}$, and the monolith of each group in Δ is isomorphic to a direct power of B. Since Ω is infinite, we contradict Corollary 3.5. Hence $\{|M_i| : 1 \le i \le \mu(H), H \in \Gamma\}$ is a finite set. //

Using Lemma 4.4, we choose $H \in \Gamma$ such that $2^{\mu(H)} > d$.

LEMMA 4.5. *Each of M' and M_i', $1 \le i \le \mu(H)$ are perfect. Furthermore, M' is a central product of $M_1', \ldots, M_{\mu(H)}'$.*

Proof. The proof that M' and the M_i' are perfect is easy, and is omitted. For the second part, we have to show that if $i \ne j$, then

$M'_j \leq C_{M'}(M'_i)$, and also that $M' = M'_1 M'_2 \ldots M'_{\mu(H)}$. Let $g \in M'_j$, and observe that the map $\alpha(g) : M'_i \to S \cap C$ defined by $h\alpha(g) = [h, g]$ is a homomorphism (since $Z(C) = S \cap C$) . But M'_i is perfect, and $C \cap S$ is abelian, and so M'_i is the kernel of $\alpha(g)$; that is, g centralises M'_i . Since $M = M_1 M_2 \ldots M_{\mu(H)}$, and $M_i = M'_i S \cap C$, it follows that $M' = M'_1 M'_2 \ldots M'_{\mu(H)}$. //

Now M' is normal in G , and V is a faithful irreducible $\mathsf{F}(p)G$-module, and so by Clifford's Theorem, if L is the kernel of an irreducible submodule X of $V_{M'}$, $\cap\{L^g : g \in G\}$ is trivial. Since M'_i is also normal in G , $L \not\geq M'_i$, $1 \leq i \leq \mu(H)$. Let E be the field obtained from $\mathsf{F}(p)$ by adjoining to it all the primitive n-th roots of unity. Since the exponent of G divides n , it follows from [4, 70.24] that E is a splitting field for G . Moreover, E is a finite normal extension of (the perfect field) $\mathsf{F}(p)$, and so by [4, 70.15], X^{E} is completely reducible, and the irreducible components of X^{E} are all Galois conjugate. Thus if U is an irreducible component of X^{E} , the kernel of U is L . By Theorem 2.9, $U \cong U_1 \# U_2 \ldots \# U_{\mu(H)}$, where U_i is an irreducible submodule of $U_{M'_i}$. Since $L \not\geq M'_i$, the kernel L_i of U_i is a proper normal subgroup of M'_i and so M'_i/L_i is nontrivial perfect. In particular, M'_i/L_i is not a p-group, $1 \leq i \leq \mu(H)$. It follows from Theorem 2.7 that there is a subgroup, say A_i , of M'_i containing L_i , and an irreducible submodule, say W_i , of $(U_i)_{A_i}$, such that A_i/L_i is soluble and W_i has E-dimension at least two. Since the kernel N_i of W_i contains L_i , A_i/N_i is soluble also. If $A = \langle A_i : 1 \leq i \leq \mu(H) \rangle$, then it follows from Theorem 2.9 that $W_1 \# \ldots \# W_{\mu(H)}$ is isomorphic to an irreducible submodule, say W , of U_A . Observe that the E-dimension of W is at least $2^{\mu(H)}$, and that the kernel N of W contains $\left\langle N_i : 1 \le i \le \mu(H)\right\rangle$. But $A/\left\langle N_i : 1 \le i \le \mu(H)\right\rangle$ is a homomorphic image of $\prod\{A_i/N_i : 1 \le i \le \mu(H)\}$, and so A/N is soluble. Now W is an irreducible submodule of V^E_A, which is the same thing as $V_A{}^E$, and so by the Jordan-Hölder Theorem, there is a composition factor, say W_1, of V_A such that W is isomorphic to a composition factor of W_1^E. By [4, 70.15] though, W_1^E is completely reducible, and its irreducible components are all Galois conjugate; hence N is also the kernel of W. But

$$|W_1| \ge |W| \ge 2^{\mu(H)} > d ,$$

and A/N is soluble. This contradicts (4.2), and the proof of Theorem 1.1 is complete.

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