REGIONS OF ESCAPE ON THE VELOCITY ELLIPSOID FOR THE PLANAR THREE BODY PROBLEM

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ABSTRACT. The notion of the velocity ellipsoid for the planar three body problem is given. Using the sufficient conditions for escape of one member of a triple system, given by Standish (1971), a region is found on the velocity ellipsoid for which escape is guaranteed.

1. INTRODUCTION

We shall consider planar systems of three bodies P_1, P_2, P_3 with masses m_1, m_2, m_3 and with a certain value L of the total angular momentum with respect to their center of mass G. To describe the motion we shall use a non-inertial frame Oxy with its origin O as the center of mass of the primaries P_1 and P_2, the x-axis permanently directed from P_2 to P_1 and the y axis perpendicular to the x-axis. Let x,y be the coordinates of the body P_3 and let $x_1 \geq 0$ be the ordinate of P_1. The location of the body P_2 on the x-axis is easily found when x_1 and x,y are given. Therefore the problem is of three degrees of freedom. The space $Oxyx_1$, may be taken as the configuration space for the dynamical system.

The distances r_1, r_2 of the body P_3 from the bodies P_1, P_2 respectively are given by

$$r_1 = \sqrt{(x-x_1)^2+y^2} \quad \text{and} \quad r_2 = \sqrt{(x-x_1+\frac{x_1}{\mu})^2+y^2} \tag{1}$$

where

$$\mu = \frac{m_2}{m_1+m_2}$$

and

$$m_1+m_2+m_3 = 1.$$

If ϑ is the angle formed by the moving axis Ox and any fixed axis, say GX, on the plane of motion, the non-constant angular velocity

V. Szebehely (ed.), Dynamics of Planets and Satellites and Theories of Their Motion, 357-369.

of the non-inertial frame will be $\dot{\vartheta} = d\vartheta/dt$. The coordinates of the relative velocities are $\dot{x},\dot{y}$ for the body P_3 and $\dot{x}_1$ for the body P_1.

The system provides the energy integral, given by

$$2E = \frac{(1-\mu)(1-m_3)}{\mu}\dot{x}_1^2 + \frac{R(1-m_3)}{\mu}\dot{\vartheta}^2 + m_3(1-m_3)(\dot{x}^2+\dot{y}^2) + $$
$$+ 2m_3(1-m_3)(x\dot{y}-\dot{x}y)\dot{\vartheta} - Q. \tag{2}$$

As to the angular momentum integral

$$L = m_3(1-m_3)(x\dot{y}-\dot{x}y) + \frac{R(1-m_3)}{\mu}\dot{\vartheta}, \tag{3}$$

this will be used to eliminate $\dot{\vartheta}$. Thus Equation (2) may be written as follows (Bozis, 1976)

$$(1-\mu)\dot{x}_1^2 + \mu m_3(\dot{x}^2+\dot{y}^2) - \frac{\mu^2 m_3^2}{R}(x\dot{y}-\dot{x}y)^2 = \frac{\mu}{1-m_3}\Phi \tag{4}$$

where

$$\Phi = 2E + Q - \frac{\mu L^2}{R(1-m_3)} \tag{5}$$

Both R and Q appearing in the above expressions are functions of the position coordinates x_1 and x,y and they are given by

$$R = (1-\mu)x_1^2 + \mu m_3(x^2+y^2) \tag{6}$$

and

$$Q = 2(1-m_3)\left\{\frac{\mu^2(1-\mu)(1-m_3)}{x_1} + \frac{(1-\mu)m_3}{r_1} + \frac{\mu m_3}{r_2}\right\} \tag{7}$$

A simple calculation shows that the moment of inertia I of the three masses with respect to their center of mass G is given by

$$I = \frac{R(1-m_3)}{\mu}.$$

Also the quantity Q, defined by Equation (7), is related to the potential V of the system by the Equation

$$Q = -2V.$$

2. THE VELOCITY ELLIPSOID

The left member of Equation (4) is a definite positive quantity for all values of the variables involved. The inequality

$$\Phi(x,y,x_1) \geq 0$$

defines in the position space $Oxyx_1$ regions where motion is allowed to take place (Bozis, 1976).

Therefore we conclude that, for definite values of E and L, to each point $P(x,y,x_1)$ of the position space there corresponds a "velocity ellipsoid" given by Equation (4) in the velocity space $0'\dot{x}\dot{y}\dot{x}_1$.

As P moves in the permissible region of the configuration space $Oxyx_1$ its velocity vector may have any direction. Its magnitude, however, is such that, in the velocity space $0'\dot{x}\dot{y}\dot{x}_1$, the vector $\overrightarrow{0'V}$ parallel to the velocity of P, terminates on the surface of the velocity ellipsoid of the point P.

One of the main axes of the velocity ellipsoid is along the $0'\dot{x}_1$ axis. The other two main axes are on the plane $0'\dot{x}\dot{y}$. In order to orient the ellipsoid along its main axes let us use the new orthogonal system $0'\dot{\xi}\dot{\eta}\dot{x}_1$.

If $\varphi(0\leq\varphi<\frac{\pi}{2})$ is the angle formed by the positive semi-axes $0'\dot{x}$ and $0'\dot{\xi}$ on the plane $0'\dot{x}\dot{y}$ we have

$$\mathrm{tg}2\varphi = \frac{2xy}{x^2-y^2} . \tag{8}$$

The equation of the velocity ellipsoid in the new system $0'\dot{\xi}\dot{\eta}\dot{x}_1$ is then

$$\frac{\dot{\xi}^2}{a^2} + \frac{\dot{\eta}^2}{b^2} + \frac{\dot{x}_1^2}{c^2} = 1$$

where the semi-axes a,b,c are given by the formulae

$$\begin{aligned} a &= \sqrt{\frac{\Phi}{m_3(1-m_3)}} \\ b &= \frac{1}{x_1}\sqrt{\frac{R\Phi}{m_3(1-m_3)(1-\mu)}} \\ c &= \sqrt{\frac{\mu\Phi}{(1-\mu)(1-m_3)}} \end{aligned} \tag{9}$$

We observe that for all points x,y,x_1 it is

$$a \leq b$$

and that

$$\frac{c}{a} = \sqrt{\frac{\mu m_3}{1-\mu}} = \text{constant.}$$

The quantities a,b and c all vanish for points x,y,x_1 on the surface of zero velocity $\Phi = 0$. The same quantities tend to infinity near collisions.

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The value of a may be smaller or larger than the value of c since the quantity $\mu m_3/1-\mu$ ranges from 0 to ∞. It may be that $a=c$ for values of μ and m_3 on the part of the hyperbola $\mu m_3=1-\mu$ inside the square $0\leq\mu\leq 1$, $0\leq m_3\leq 1$.

It may also be that $a=b\neq 0$ and this actually happens only for points on the x_1-axis of the position space. It is therefore understood that for certain values of the parameters μ and m_3 and for points x_1, $x_3=0$, $y_3=0$ the velocity ellipsoid is reduced to a sphere. In this case the magnitude of the velocity, at these points, is completely defined if E and L are given.

In all other cases the magnitude of the velocity at a point x,y,x_1, depends on its direction and ranges between the maximum and the minimum value of the quantities a, b and c, evaluated at this point.

We shall now compare this statement with the information given by the well known Sundmann's inequality (e.g. Birkhoff, 1927). In our notation this inequality may be written as

$$\left(\frac{dI}{dt}\right)^2 \leq 4I\Phi \tag{10}$$

Its meaning is that for each point (x,y,x_1) of the position space there exists a certain velocity which makes the quantity $(dI/dt)^2$ maximum. The quantity

$$\frac{dI}{dt} = \frac{1-m_3}{\mu}\{2(1-\mu)x_1\dot{x}_1 + 2\mu m_3(x\dot{x}+y\dot{y})\} \tag{11}$$

is a function of $\dot{x}$ and $\dot{y}$ only, since $\dot{x}_1 = \dot{x}_1(\dot{x}_3, \dot{y}_3)$ by Equation (4). The critical values of (11) are then found to be

$$\dot{x}_o = \pm\sqrt{\frac{\Phi}{I}}\,x, \quad \dot{y}_o = \pm\sqrt{\frac{\Phi}{I}}\,y \quad \text{and} \quad \dot{x}_{1,o} = \pm\sqrt{\frac{\Phi}{I}}\,x_1 \tag{12}$$

and they define on the velocity ellipsoid two points symmetric with respect to the origin $0'$.

From another point of view Sundmann's inequality implies that for each point (x,y,x_1) of the position space the vector of the velocity $\overrightarrow{0'V}$ in the velocity space $0'\dot{x}\dot{y}\dot{x}_1$ must be selected so that its terminal point V lies somewhere between the two parallel planes

$$(1-\mu)x_1\dot{x}_1 + \mu m_3(x\dot{x}+y\dot{y}) = -\frac{\mu}{1-m_3}\sqrt{I\Phi} \tag{13a}$$

and

$$(1-\mu)x_1\dot{x}_1 + \mu m_3(x\dot{x}+y\dot{y}) = \frac{\mu}{1-m_3}\sqrt{I\Phi}\ . \tag{13b}$$

One could probably think that the planes (13) intersect the surface (4) and that part of the surface is excluded. However, this is not the case.

It can easily be shown that "Sundmann's planes" (13a) and (13b) are always tangent to the velocity ellipsoid (4) at the points

$$P_o\left(\sqrt{\frac{\Phi}{I}}\,x,\quad \sqrt{\frac{\Phi}{I}}\,y,\quad \sqrt{\frac{\Phi}{I}}\,x_1\right)$$

and

$$P_o^*\left(-\sqrt{\frac{\Phi}{I}}\,x,\quad -\sqrt{\frac{\Phi}{I}}\,y,\quad -\sqrt{\frac{\Phi}{I}}\,x_1\right) \tag{14}$$

The conclusion is that, as far as the magnitude of the velocity is concerned, more information is given by the velocity ellipsoid than by Sundmann's planes.

3. CONDITIONS FOR ESCAPE

We shall prove in this section that in general to a certain part of the surface of the velocity ellipsoid there correspond escaping orbits.

We shall limit our study to negative values of the energy E. For such values of E the zero velocity surfaces allow for the following types of dissolution: (i). The body P_3 escapes to infinity leaving the bodies P_1 and P_2 in a close binary. (ii). The body P_3 goes to infinity in a close binary with either P_1 or P_2. In either case (for E<0) there is a minimum distance of the bodies bounded by the quantity (Birkhoff, 1927)

$$r_* = \frac{(1-m_3)\{\mu(1-\mu)(1-m_3)+m_3\}}{-E}\,. \tag{15}$$

Let us study case (i). It is exactly for this case that Standish (1971) has given sufficient conditions for escape. In our notation these conditions are the following:

$$\sqrt{x^2+y^2} > r_*$$

$$\frac{d}{dt}\left(\sqrt{x^2+y^2}\right) > 0$$

and

$$\left(\frac{d}{dt}\sqrt{x^2+y^2}\right)^2 \geq 2\left[\frac{1}{\sqrt{x^2+y^2}} + \frac{\mu(1-\mu)r_*^2}{(\sqrt{x^2+y^2}-r_*)(x^2+y^2)}\right]$$

These are rewritten as follows:

$$x^2+y^2 > r_*^2 \tag{16a}$$

$$x\dot{x}+y\dot{y} > 0 \tag{16b}$$

$$x\dot{x}+y\dot{y}-S \geq 0 \tag{16c}$$

where

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$$S = \sqrt{2\left[\sqrt{x^2+y^2} + \frac{\mu(1-\mu)r_*^2}{\sqrt{x^2+y^2}-r_*}\right]} \tag{17}$$

Only the positive square root in (17) is considered in view of the inequality (16b). Since (16c) is stronger than (16b) we only need consider (16c).

Equation

$$x\dot{x} + y\dot{y} - S = 0 \tag{18}$$

represents, in the $0'\dot{x}\dot{y}\dot{x}_1$ space, a plane parallel to the $\dot{x}_1$ axis. The inequality (16c) holds to the other side of this plane from the side where the origin $0'$ is.

The question now is: Does the plane (18) intersect the velocity ellipsoid (4)? This happens indeed if, on the plane $\dot{x}_1=0$, the straight line (18) intersects the ellipse

$$\mu m_3(\dot{x}^2+\dot{y}^2) - \frac{\mu^2 m_3^2}{R}(x\dot{y}-\dot{x}y)^2 = \frac{\mu\Phi}{1-m_3}\ . \tag{19}$$

Eliminating $\dot{y}$ between (18) and (19) we get, with $y\neq 0$, the quadratic equation

$$\alpha\dot{x}^2 + \beta\dot{x} + \gamma = 0 \tag{20}$$

where

$$\alpha = (1-\mu)\mu m_3 \frac{x^2+y^2}{Ry^2} x_1^2$$

$$\beta = -2(1-\mu)\mu m_3 \frac{Sx}{Ry^2} x_1^2$$

$$\gamma = -\frac{\mu\Phi}{1-m_3} + \mu m_3 \frac{S^2}{Ry^2}\{(1-\mu)x_1^2 + \mu m_3 y^2\}.$$

In order that the Equation (20) has two real roots we must have

$$\frac{\Phi}{m_3(1-m_3)} r^2 - S^2 \geq 0 \tag{21}$$

where Φ is given by (5), S is given by (17) and

$$r^2 = x^2+y^2 \tag{22}$$

We shall replace the inequality (21) by a weaker but much simpler inequality.

We have, in view of Equations (6), (7) and (22),

$$R \geq R_o = \mu m_3 r^2 \tag{23}$$

and

$$Q \geq Q_o = \frac{2\mu^2(1-\mu)(1-m_3)^2}{x_1} . \tag{24}$$

Therefore, in view of Equation (5),

$$\Phi \geq \Phi_o = 2E + Q_o - \frac{\mu L^2}{R_o(1-m_3)} . \tag{25}$$

We thus come to the conclusion that the inequality (21) is always true provided that

$$\frac{\Phi_o r^2}{m_3(1-m_3)} - S^2 \geq 0. \tag{26}$$

The inequality (26) is equivalent to the inequality.

$$0 < x_1 \leq f(r) \tag{27}$$

where

$$f(r) = \frac{2\mu^2(1-\mu)(1-m_3)^2 r^2}{m_3(1-m_3)S^2 - (2Er^2 - \frac{L^2}{m_3(1-m_3)})} \tag{28}$$

The meaning of the last inequality (27) is the following: Suppose that the initial conditions (i.e. coordinates of position and velocity) of the problem are given. Also suppose that the angular momentum L is given. The value of E is then known and from Equation (15) the value of r_* may be found. Then from Equation (17) and (28) the values of S and $f(r)$ corresponding to the given initial conditions are found. Now if the inequalities

$$r > r_* \quad \text{and} \quad x_1 \leq f(r) \tag{29}$$

are satisfied we understand that the orbit corresponding to the given initial conditions may be escaping. This is because these two inequalities are sufficient conditions for the plane (18) and the ellipsoid (4) to intersect each other, thus forming on the velocity ellipsoid a patch to the points of which there correspond velocities leading to escaping orbits.

Therefore it will depend on the direction of the velocity $\dot{x},\dot{y},\dot{x}_1$ whether or not the orbit will be escaping.

Another way of interprating inequality (27) is the following: Suppose that the third body is brought to a distance $r>r_*$ from the center of mass of the primaries P_1 and P_2. The question is: With a certain negative value of the energy E (by means of which r_* was determined) and of the angular momentum L are there any velocities which make the body P_3 escape to infinity? The answer may definetely be affirmative provided that the distance x_1/μ between the primaries P_1 and P_2 is sufficiently small as to satisfy the inequality (27).

We now observe that

(i) As $r \to r_*$ we have

$$\lim_{r\to r_*} S^2=\infty \qquad \text{and} \qquad \lim_{r\to r_*} f(r) = 0$$

(ii) As $r\to\infty$ we have

$$\lim_{r\to\infty} f(r) = \frac{\mu^2(1-\mu)(1-m_3)^2}{-E}$$

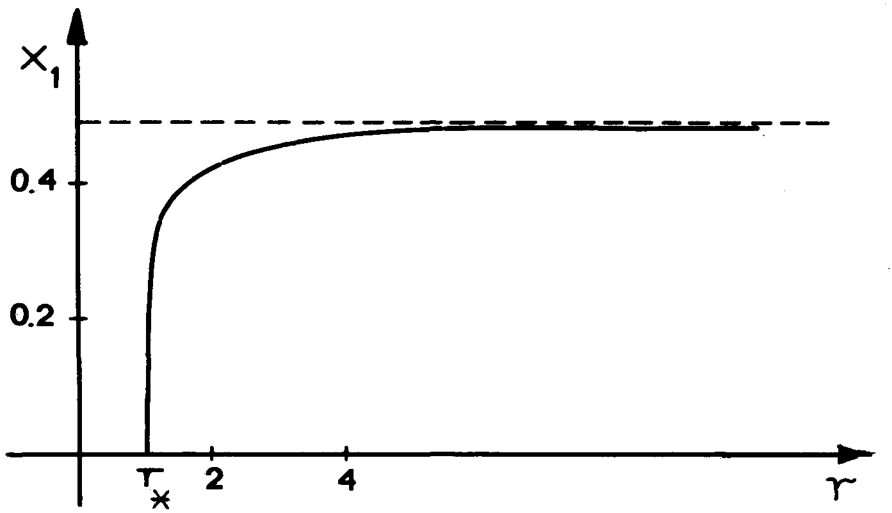

Fig. 1: The function $x_1=f(r)$ as given by Equation (28), drawn for $\mu=0.50$, $m_3=0.01$, $E=-0.25$ and $L=0.05$. For a certain $r>r_*$ and for $x_1<f(r)$ there is always a region of escaping orbits on the velocity ellipsoid.

It is worth noticing that the value of the $\lim_{r\to\infty} f(r)$ is the same with the value k in the position space of the plane $x_1=k=\mu^2(1-\mu)(1-m_3)^2/-E$ which is asymptotically tangent to the zero velocity surfaces (Bozis, 1976).

(iii). A direct calculation gives

$$\frac{df(r)}{dr} = \frac{4}{r^3}\left[f(r)\right]^2\mu^2(1-\mu)(1-m_3)^2\{m_3(1-m_3)r + \frac{L^2}{m_3(1-m_3)} +$$

$$+ m_3(1-m_3)\frac{\mu(1-\mu)r_*^2}{(r-r_*)^2}(3r-r_*)\}.$$

Since $r \geq r_*$, we have $df(r)/dr>0$, i.e. the function $f(r)$ is increasing in the interval $[r_*,\infty)$ (Figure 1).

For a given value of $r>r_*$ we have from Figure 1 the values of x_1 for which the inequality (27) is valid.

Inequality (27) may also be interpreted as follows: The Equation

$$x_1 - f(\sqrt{x^2+y^2}) = 0 \tag{30}$$

represents in the position space a surface of revolution around the $0x_1$ axis. Figure 1 may also serve to visualize the intersection of this surface with either of the planes $y=0$ or $x=0$. The surface of revolution is asymptoticaly tangent to the plane

$$x_1 = \frac{\mu^2(1-\mu)(1-m_3)^2}{-E}.$$

We thus come to the conclusion that for given values of μ, m_3, E and L escape of the third body is not guaranteed in the part of the position space underneath the zero velocity surfaces, above the surface of revolution (30) and outside the cylinder

$$x^2+y^2 = r_*^2 \tag{31}$$

Obviously, inside the cylinder (31) escape is not guaranteed since the inequality (16a) is not valid.

A final remark concerns the coefficients of Equation (20). These coefficients have no meaning for $y=0$. This case, however, may be studied separately. In fact the Equation of the straight line (18) becomes

$$x\dot{x} = S$$

whereas the Equation of the ellipse (19) becomes

$$\frac{\dot{x}^2}{a^2} + \frac{\dot{y}^2}{b^2} = 1$$

These two curves intersect each other if

$$\frac{S}{x} \leq a$$

or, since $x=r$, if

$$r \geq \frac{S}{a}$$

Again taking into account (23), (24), (25) we can easily prove that the inequality $r \geq S/a$ is equivallent to the inequality (27). Thus (27) covers the case $y=0$ as well.

4. NUMERICAL EXAMPLE

Figure 1 was drawn for the following values of the parameters:

$$\mu = 0.50, \quad m_3 = 0.01 \quad E = -0.25 \quad L = 0.05.$$

From Equation (15) we find

$$r_* = 1.0197.$$

We now select a distance $r=\sqrt{5} > r_*$ and, for this r, we find from Equation (17)

$$S = 2.2135.$$

Then, either from the diagram of Figure 1 or from Equation (28) we find

$$f(r=\sqrt{5}) = 0.4374.$$

Let us now select $x=1$, $y=2$ (so that $x^2+y^2=5$) and $x_1=0.30$ (so that the inequality (27) is satisfied).

For this position of the triple system (with given masses and E, L) there will be a patch on the velocity ellipsoid for all points of which we will have an escaping orbit for the body P_3.

In Figure 2 we give the projection of this patch on the $0'\dot{x}\dot{y}$ plane. This is found as follows:

For the point

$$x = 1, \quad y = 2, \quad x_1 = 0.30$$

of the configuration space we calculate the values of R, Q and Φ from Equations (6), (7) and (5) respectively. Then with the aid of Formulae (9) and (8) we calculate the semi-axes a,b,c of the velocity ellipsoid as well as its orientation.

We have

$$a = 5.5735, \quad b = 6.9514, \quad c = 0.5573$$

and

$$tg\varphi = 2.$$

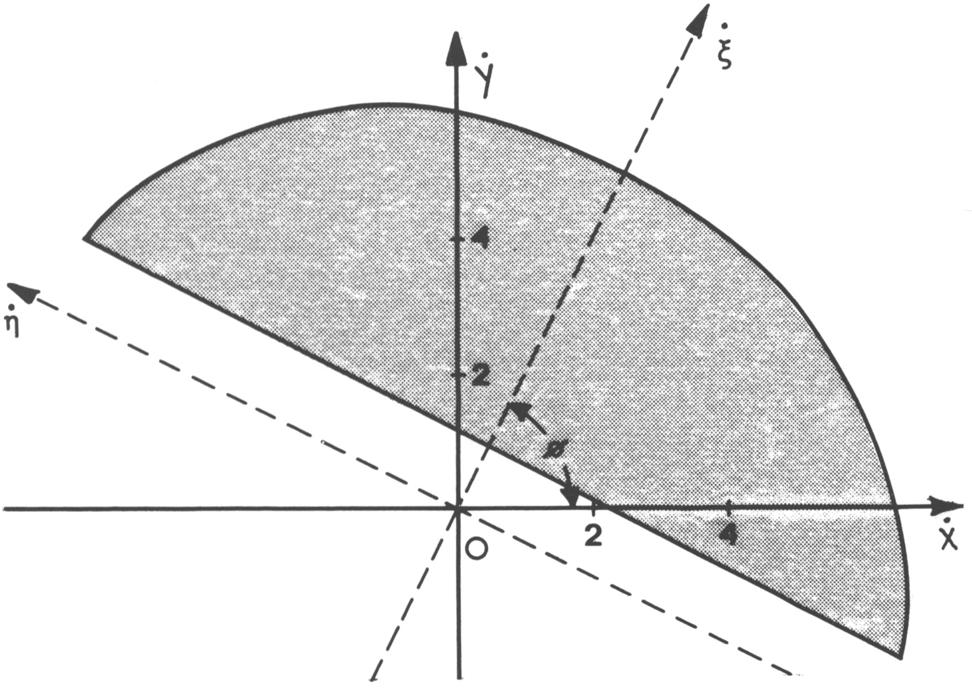

Fig.2 The shaded part of this Figure is the projection of the escaping region of the velocity ellipsoid on the $0'\dot{x}\dot{y}$ plane for μ=0.50, m_3=0.01, E=-0.25, L=0.05. For a certain pair $\dot{x}$, $\dot{y}$ of the shaded region the value of $\dot{x}_1$ is found from Equation (4). The velocities $(\dot{x}, \dot{y}, \dot{x}_1)$ and $(\dot{x}, \dot{y}, -\dot{x}_1)$ are then escaping velocities.

We now draw the ellipse

$$\frac{\dot{\xi}^2}{a^2} + \frac{\dot{\eta}^2}{b^2} = 1.$$

Next we draw the straight line (18), i.e.

$$\dot{x} + 2\dot{y} = 2.2135$$

The shaded part of Figure 2 corrsponds to velocities for which the third body by all means escapes to infinity. For any pair $\dot{x},\dot{y}$ of the shaded part of Figure 2 the corresponding value of $\dot{x}_1$ must be found from Equation (4). The velocity $(\dot{x},\dot{y},\dot{x}_1)$ as well as the velocity $(\dot{x},\dot{y},-\dot{x}_1)$ is then an escaping velocity.

5. COMMENTS AND CONCLUSIONS

The planar problem of three bodies is of three degrees of freedom in a conveniently selected non-inertial frame. As a consequence, we can think of a representative point P moving in a triaxial space $Oxyx_1$. For given values of the masses of the three bodies and the parameters E and L there exist always, in the position space, regions where motion of the point $P(x,y,x_1)$ is allowed to take place.

On the other hand to each point $P(x,y,x_1)$ of the allowable position space there corresponds, in the velocity space $O'\dot{x}\dot{y}\dot{x}_1$, a velocity ellipsoid, i.e. a surface with Equation (4). The vector of the velocity of the point P may have any direction. However the magnitude of the velocity is found from Equation (4) for any given direction. The range of the velocities which can be used from any point P with certain values of E, L can be found by calculating the semi-axes (9) of the velocity ellipsoid.

Let us look at this from another point of view: Suppose that μ, m_3, E and L are given and also a point $P(x,y,x_1)$ is given. The velocity ellipsoid corresponding to this point P is then determined with the aid of Equations (9) and (8). To each point of this ellipsoid there corresponds a certain orbit through the point P with a definite velocity. The question naturally arises: Do all the orbits through P have any common features? In particular how do they behave as to escaping? The answer is as follows: If the inequalities (29) are satisfied at the point P, then there exists a part on the surface of the velocity ellipsoid to all points of which there correspond escaping orbits. This part is always less than half of its surface. Its projection on the $O'\dot{x}\dot{y}$ plane is defined, as in Fig.2, by the intersection of the ellipse (19) with the straight line (18), which is always parallel to one of the main axes of the ellipse. (See Figure 2).

In case that the inequalities (29) are not simultaneously satisfied there is no velocity for which escape of the third body may be guaranteed. This happens inside the cylinder (31) or outside this cylinder but between the surface of revolution (30) and the zero velocity surface.

If, for instance, in the numerical example worked in §4, we selected $x_1=0.50$ (instead of $x_1=0.30$) the second inequality (29) would not hold and no patch of escaping orbits on the velocity ellipsoid could be found.

Consider now a symmetric periodic orbit. Since at $t=0$ we have $x=x_o$, $y=0$, it follows from (8) that $\varphi=0$ and the velocity ellipsoid is

$$\frac{\dot{x}^2}{a^2} + \frac{\dot{y}^2}{b^2} + \frac{\dot{x}_1^2}{c^2} = 1.$$

Also, since at $t=0$ it is $\dot{x}_o=0$, $\dot{x}_{1,o}=0$ we have $\dot{y}_o=\pm b$.

On the other hand the straight line (8) is, in this case,

$$\dot{x} = \frac{S}{x_o}.$$

Obviously the initial velocity (0,0,b) or (0,0,-b) of the periodic orbit through the point (x_o, 0, x_{10}) corresponds to a point of the velocity ellipsoid outside the shaded region $\dot{x} \geq S/x_o$.

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