POINT-FINITE AND LOCALLY FINITE COVERINGS

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1. Introduction. An interesting feature of recent topological developments is the increasingly important role played by locally finite coverings.[1] Point-finite coverings, on the other hand, even though conceptually simpler, have received very little attention. And deservedly so, since they are much less useful. Nevertheless, it sometimes happens (as it did to the author in **(5)**) that one is confronted by a covering which is known to be point-finite, but not necessarily locally finite. When does such a covering have a locally finite refinement? The purpose of this paper is to provide some answers to this question in the following two theorems (which the author happens to need in **(5)**), and to construct some counter-examples to certain related conjectures. It should be pointed out that, while Theorem 2 seems to be new, Theorem 1 is known **(6**, Theorem 3 and Lemma 3**)**, and is stated here only for completeness, and because it is needed in the proof of Theorem 2.

THEOREM 1 (Morita). *Every countable, point-finite covering of a normal space has a locally finite refinement.*

THEOREM 2. *Every point-finite covering of a collectionwise normal space has a locally finite refinement.*

Whether Theorem 1 remains true with "point-finite" omitted is one of the major unsolved problems in point-set topology, and it is equivalent to the problem of whether the cartesian product of a normal space and the closed unit interval is normal **(2**; **4)**. It is of course *not* possible to omit "point-finite" in Theorem 2, since a collectionwise normal space need not be paracompact. And finally, the following two counter-examples show that two other plausible directions for improving Theorem 2 are also barred:

Example 1. There exists a normal space, every point-finite covering of which has a locally finite refinement, but which is not collectionwise normal.

Example 2. There exists a normal space, not every point-finite covering of which has a locally finite refinement.

In §3, where these examples are constructed, it will be shown that they can even be slightly strengthened, and that, in particular, the spaces can be chosen to be perfectly normal.

We conclude this introduction with a quick review of our principal concepts. Let X be a *Hausdorff* space. In this paper, a *covering* of X is a collection of *open* subsets of X whose union is X. A collection $\mathscr{A}$ of subsets of X is *point-*

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1. All terms are defined at the end of this introduction.

finite if every $x \in X$ is an element of only finitely many $A \in \mathscr{A}$; it is *locally finite* if every $x \in X$ has a neighborhood which intersects only finitely many $A \in \mathscr{A}$. If $\mathscr{V}$ and $\mathscr{W}$ are coverings of X, then $\mathscr{W}$ is a *refinement* of $\mathscr{V}$ if every $W \in \mathscr{W}$ is a subset of some $V \in \mathscr{V}$. *Normal* spaces are, of course, familiar. According to Bing **(1)**, X is *collectionwise normal* if, whenever $\{A_\alpha\}$ is a collection of subsets of X which is *discrete* (i.e., locally finite, and with pairwise disjoint closures), there exists a disjoint collection $\{U_\alpha\}$ of open subsets of X such that $A_\alpha \subset U_\alpha$ for every α. Finally, X is *paracompact* if every covering of X has a locally finite refinement. The relations between these three types of spaces, as shown by Bing **(1)**, are that

$$\text{paracompact} \rightarrow \text{collectionwise normal} \rightarrow \text{normal},$$

and that neither arrow can be reversed.

2. Proof of Theorem 2. Let $\mathscr{U}$ be a point-finite covering of the collectionwise normal space X. We are going to construct a sequence $\{\mathscr{W}_i\}$ $(i = 0, 1, \ldots)$ of collections of open subsets of X such that, denoting $\bigcup\{W | W \in \mathscr{W}_i\}$ by W_i, the following conditions are satisfied for all i:

(a) Every $W \in \mathscr{W}_i$ is a subset of some $U \in \mathscr{U}$.

(b) $\mathscr{W}_i$ is locally finite (in fact, discrete).

(c) If $x \in X$ is an element of at most i elements of $\mathscr{U}$, then

$$x \in \bigcup_{k=0}^{i} W_k.$$

(d) Every $x \in W_i$ is an element of at least i elements of $\mathscr{U}$.

Suppose, for a moment, that $\{\mathscr{W}_i\}$ has been constructed, and notice how the theorem follows. In fact, remembering that $\mathscr{U}$ is point-finite, we see that $\{W_i\}$ is a covering of X (by (c)) which is point-finite (by (d)). It then follows from Theorem 1 that $\{W_i\}$ has a locally finite refinement $\{V_i\}$, with $V_i \subset W_i$ for every i, and therefore $\bigcup_{i=0}^{\infty}\{V_i \cap W | W \in \mathscr{W}_i\}$ is a locally finite refinement of $\mathscr{U}$ (by (a) and (b)).

It remains to construct the sequence $\{\mathscr{W}_i\}$. Let $\mathscr{W}_0 = \{\phi\}$ (i.e., the only element of $\mathscr{W}_0$ is the null set); then conditions (a)-(d) are clearly satisfied for $i = 0$. Suppose, therefore, that $\mathscr{W}_0, \ldots, \mathscr{W}_n$ have been constructed to satisfy (a)-(d) for all $i \leqslant n$, and let us construct $\mathscr{W}_{n+1}$.

Let $\mathfrak{R}$ be the family of all $\mathscr{R} \subset \mathscr{U}$ such that $\mathscr{R}$ has exactly $n + 1$ elements. For every $\mathscr{R} \in \mathfrak{R}$, let

$$A(\mathscr{R}) = \left(X - \bigcup_{k=0}^{n} W_k\right) \cap \left(X - \bigcup\{U \in \mathscr{U} | U \notin \mathscr{R}\}\right)$$

Clearly every $A(\mathscr{R})$ is closed. Let us show that $\{A(\mathscr{R}) | (\mathscr{R}) \in \mathfrak{R}\}$ is discrete, by showing that every $x \in X$ has a neighborhood which intersects at most one $A(\mathscr{R})$. We consider three cases: if x is in $> n + 1$ elements of $\mathscr{U}$, then the intersection of any $n+2$ of these does not intersect any $A(\mathscr{R})$; if x is in $< n+1$ elements of $\mathscr{U}$, then (by (c))

$$x \in \bigcup_{k=0}^{n} W_k,$$

which does not intersect any $A(\mathscr{R})$; and if, finally, x is in exactly $n+1$ elements of $\mathscr{U}$, say in $U_1, \ldots, U_{n+1}$, then

$$\bigcap_{k=1}^{n+1} U_k$$

is a neighborhood of x which does not intersect $A(\mathscr{S})$ for $\mathscr{S} \neq \{U_1, \ldots, U_{n+1}\}$ (since then at least one U_k is not an element of $\mathscr{S}$, and this U_k cannot intersect $A(\mathscr{S})$).

Since $\{A(\mathscr{R}) | \mathscr{R} \in \mathfrak{R}\}$ is thus a discrete collection of closed subsets of the collectionwise normal space X, there exists a disjoint collection $\{V(\mathscr{R}) | \mathscr{R} \in \mathfrak{R}\}$ of open subsets of X such that $A(\mathscr{R}) \subset V(\mathscr{R})$ for every $\mathscr{R} \in \mathfrak{R}$; by a result of Dowker (**3**, p. 308), we can even pick $\{V(\mathscr{R}) | \mathscr{R} \in \mathfrak{R}\}$ to be discrete. Now notice that $A(\mathscr{R}) \subset U$ for every $U \in \mathscr{R}$, since otherwise some $x \in A(\mathscr{R})$ would be an element of $\leqslant n$ elements of $\mathscr{U}$, which is impossible by (c) and the definition of $A(\mathscr{R})$. Hence, if we let

$$P(\mathscr{R}) = V(\mathscr{R}) \cap \bigcap \{U | U \in \mathscr{R}\},$$

then $A(\mathscr{R}) \subset P(\mathscr{R})$ for every $\mathscr{R} \in \mathfrak{R}$. We now define $\mathscr{W}_{n+1} = \{P(\mathscr{R}) | \mathscr{R} \in \mathfrak{R}\}$. Let us check that conditions (a)-(d) are satisfied for $i = n+1$. That (a), (b), and (d) are satisfied follows directly from the definition of $\mathscr{W}_{n+1}$. To check (c), let $x \in X$ be an element of $\leqslant n+1$ elements of $\mathscr{U}$; then clearly there exists an $\mathscr{R} \in \mathfrak{R}$ such that $x \in (X - \bigcup\{U \in \mathscr{U} | U \notin \mathscr{R}\})$. But then either

$$x \in \left(X - \bigcup\{U \in \mathscr{U} | U \notin \mathscr{R}\}\right) \cap \left(X - \bigcup_{k=0}^{n} W_k\right) = A(\mathscr{R}) \subset P(\mathscr{R}) \subset W_{n+1},$$

or

$$x \in \bigcup_{k=0}^{n} W_k;$$

thus in either case

$$x \in \bigcup_{k=0}^{n+1} W_k.$$

This completes the proof.

3. The Counter-examples In this section we shall describe the spaces of Examples 1 and 2 in the introduction, and show that they have the required properties:

Example 1. As a space with the required properties, we submit the normal, but not collectionwise normal, space F of Bing (**1**, Example G). We refer the reader to Bing's paper for the definition of F, and for the related notation.

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We shall use Bing's notation, adding one additional piece of notation of our own: If $p \in P$, and if r is a finite subcollection of Q, then

$$\langle p, r\rangle = \{f \in F \mid f(q) = f_p(q) \quad \text{for all } q \in r\}.$$

We must show that every point-finite covering of F has a locally finite refinement. So let $\mathscr{U}$ be a point-finite covering of F. Let $\mathscr{V} = \{U \in \mathscr{U} | U \cap F_p \neq \phi\}$. There are now two possibilities:

(a) $\mathscr{V}$ is countable. Let $V = \bigcup\{V | V \in \mathscr{V}\}$. Then V is an open and closed subset of F, and is therefore normal. Hence $\mathscr{V}$ is a countable, point-finite covering of the normal space V, and hence (by Theorem 1) has a locally finite refinement $\mathscr{R}$. If we now let $\mathscr{S} = \mathscr{R} \cup \{\{f\} | f \in (F - V)\}$, then $\mathscr{S}$ is a locally finite refinement of $\mathscr{U}$.

(b) $\mathscr{V}$ is uncountable. We shall show that this is impossible. For suppose it is true. Then, it is easy to check, there exists an uncountable subset M of P, and for each $p \in M$ a finite subcollection r_p of Q, such that the family of all $\langle p, A_p\rangle$, with $p \in M$, is point-finite. Bing's proof that F is not collectionwise normal actually proves that such a family cannot be *disjoint:* the proof that it cannot even be *point-finite* is very similar, and we therefore only indicate the necessary modification in Bing's proof. Bing begins by assuming that the collection of all $\langle p, r_p\rangle$ is disjoint, and obtains his contradiction by finally showing that it isn't even point-finite. The only place where Bing actually uses the disjointness of $\{\langle p, r_p\rangle\}_{p \in M}$ is, essentially, to show the existence of an uncountable $W_1' \subset W$ (where W is an uncountable subset of M), and a $q_1 \in Q$, such that $q_1 \in r_p$ for every $p \in W_1'$. To show the existence of q_1 and W_1' even under the weaker assumption that $\{\langle p, r_p\rangle\}_{p \in M}$ is point-finite, we proceed as follows: Let T be a maximal subset of W having the property that $r_p \cap r_{p'} = \phi$ whenever $p \in T$, $p' \in T$, $p \neq p'$; the existence of such a set follows from Zorn's lemma. It is easy to see that

$$\bigcap_{p \in T} \langle p, r_p\rangle \neq \phi,$$

and hence T must be finite. Letting $r = \bigcup_{p \in T} r_p$, we see that r is a finite subcollection of Q. Now for every $q \in r$, let $E_q = \{p \in W | q \in r_p\}$; it follows from the maximality of T that $\bigcup_{q \in r} E_q = W$. Hence E_q must be uncountable for at least one $q \in r$, say for q_1. If we now let $W_1' = E_{q_1}$, then W_1' and q_1 have the required properties.

To obtain a space, satisfying our requirements, which is also perfectly normal (i.e., every closed subset is a G_δ), we need only replace the above space F of Example G of **(1)** by the space F of Example H of **(1)**. The proof goes just as before.

Example 2. We shall construct a normal, non-collectionwise normal space G, every covering of which has a point-finite refinement. (This last property is sometimes called pointwise paracompactness.) This space certainly has all required properties, since if every point-finite covering of G had a locally

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finite refinement, it would follow that G is paracompact, and hence collectionwise normal, which it is not.

To obtain G, we begin with the space F of Bing (**1**, Example G) which was used in Example 1, and then let G be the subspace of F defined by

$$G = F_p \cup \{f \in F \mid f(q) = 0 \text{ except for finitely many } q \in Q\}.$$

Since G is a closed subset of F, G is normal. Bing's proof that F is not collectionwise normal goes through *verbatim* to show that G is not collectionwise normal. All that remains to show is that every covering of G has a point-finite refinement.

Let $\mathscr{U}$ be a covering of G. For each $p \in P$, pick a $U_p \in \mathscr{U}$ such that $f_p \in U_p$, and let $V_p = \{f \in G \mid f(p) = 1\}$. It follows from the definition of G that $\{V_p\}_{p \in P}$ is point-finite. If we now let

$$\mathscr{W} = \Big(\{W_p \cap U_p\}_{p \in P}\Big) \cup \Big(\{\{f\}\}_{f \in G - F_p}\Big),$$

then $\mathscr{W}$ is clearly a point-finite refinement of $\mathscr{U}$. This completes the proof that G has all the required properties.

Just as in Example 1, we can obtain a space, satisfying all our requirements, which is also perfectly normal. In fact, all we need to do is to start with the space F of Example H of (**1**), rather than with the space F of Example G of (**1**). We then let

$$G = F_p \cup \{f \in F \mid f(q) \text{ is even except for finitely many } q \in Q\}.$$

The proof that G does the trick proceeds just as before.

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