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Spectral Problems for Non-Linear Sturm-Liouville Equations with Eigenparameter Dependent Boundary Conditions

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Abstract. The nonlinear Sturm-Liouville equation

$$-(py')' + qy = \lambda(1 - f)ry \text{ on } [0, 1]$$

is considered subject to the boundary conditions

$$(a_j\lambda + b_j)y(j) = (c_j\lambda + d_j)(py')(j), \quad j = 0, 1.$$

Here $a_0 = 0 = c_0$ and $p, r > 0$ and q are functions depending on the independent variable x alone, while f depends on x, y and y'. Results are given on existence and location of sets of (λ, y) bifurcating from the linearized eigenvalues, and for which y has prescribed oscillation count, and on completeness of the y in an appropriate sense.

1 Introduction

Linear eigenvalue problems with eigenparameter-dependent boundary conditions have a long history, and we refer to [10], [12], [18] and their reference lists for some of this activity. Typical topics studied have been existence and location of the eigenvalues, oscillation, comparison of the eigenfunctions, their completeness (and more general spectral decompositions), asymptotics, and applications to physics and engineering.

Much of this work concerns problems which may be put in the form

$$ly = \lambda y, \quad \text{on } [0, 1] \tag{1.1}$$

where

$$ly = \frac{1}{r}\big(-(py')' + qy\big)$$

and $r > 0$, $p > 0$, subject to boundary conditions

$$y \in BC_0 = \{u \in C^1([0, 1]; \mathbb{R}) : u(0) \cos\alpha = (pu')(0) \sin\alpha\}, \tag{1.2}$$

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and

$$y \in BC_1^\lambda = \{u \in C^1([0,1];\mathbb{R}) : (a\lambda + b)u(1) = (c\lambda + d)(pu)'(1)\}, \tag{1.3}$$

where $\delta = ad - bc > 0$, $\alpha \in [0, \pi)$ and $c \neq 0$. The conditions $r > 0$ and $\delta > 0$ are sometimes referred to as 'right definiteness'. Without loss of generality we can scale (1.3) so that $\delta = 1$.

The methods of [12] and [18] show that the problem may then be recast in the form

$$AY = \lambda Y \tag{1.4}$$

where A is the linear operator defined in $H = L^2 \oplus \mathbb{C}$ by

$$AY = \begin{pmatrix} ly \\ by(1) - dp(1)y'(1) \end{pmatrix}$$

on

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} y \\ -ay(1) + cp(1)y'(1) \end{pmatrix} : y, py' \in AC, \int_0^1 r|ly|^2 < \infty, \quad y \in BC_0 \right\}.$$

Here and below we write $Y = \binom{y}{\gamma} \in L^2 \oplus \mathbb{C}$. A turns out to be self adjoint and bounded below with compact resolvent, if we equip H with the Hilbert space norm given by

$$\|Y\|^2 = |\gamma|^2 + \int_0^1 r|y|^2.$$

This provides a convenient setting for several of the topics mentioned above.

Appropriate analogues of Sturm's oscillation and comparison theory have been discussed in [2] via modified Prüfer transformation techniques. Indeed (1.3) may be rewritten in the form

$$\cot\theta = \frac{a\lambda + b}{c\lambda + d} \tag{1.5}$$

where the Prüfer angle θ obeys $\cot\theta = py'/y$. Noting that

$$\frac{a\lambda + b}{c\lambda + d} = \frac{a}{c} - \frac{1}{c^2(\lambda + \frac{d}{c})} \tag{1.6}$$

by virtue of $\delta = 1$, we see that (1.5) is close, for large λ, to the λ-independent condition

$$\cot\theta = \frac{a}{c} \tag{1.7}$$

(recall $c \neq 0$). The corresponding Sturm-Liouville problem (1.1), (1.2), (1.7) is called the (right hand) asymptotic problem and indeed it is shown in [2] that the eigenvalues λ_k of (1.1), (1.2), (1.5) are asymptotically very close to those of (1.1), (1.2), (1.7) as $k \to \infty$.

Problems which are nonlinear in λ have also been discussed (*cf.* [9], [17]) but we are aware of no work in this area with nonlinearities in y. For the Sturm-Liouville case (with λ-independent boundary conditions) the subject is well studied (*cf.* [5], [7], *etc.*) but there are some interesting differences, and the standard methods require significant modification, when (1.3) is imposed. A major difficulty is to identify the oscillation count (which turns out not to be constant in general) along the 'bifurcation curves'. We have used a blend of the techniques mentioned earlier for the linear problem with standard (topological and C^1) fixed point methods of bifurcation theory. We remark that we assume (1.2) to be λ-independent mainly to reduce technicalities (*cf.* [2] and [18] for the linear case).

Our aim, then, is to discuss the problem

$$-(py')' + qy = \lambda(1 - f)ry \quad \text{on } [0, 1] \tag{1.8}$$

subject to (1.2) and (1.3). Here f depends on y, y' and the independent variable x, while all other functions depend on x alone. Moreover f is assumed to be continuous, and zero when $y = 0$, so (1.1) is the linearization of (1.8) about $y = 0$. Evidently (1.4) may be replaced by an equation of the form

$$AY = \lambda\big(I - F(Y)\big)Y \tag{1.9}$$

where

$$F(Y)(x) = \begin{bmatrix} f\big(x, y(x), y'(x)\big) & 0 \\ 0 & 0 \end{bmatrix}.$$

It is clear that (1.9) admits the trivial solution $Y = 0$ for all λ. By an eigenpair of (1.9) we mean a pair (λ, Y) with $Y \neq 0$.

In Section 2 we shall combine the above operator approach (using different topologies) with a revised version of the Prüfer approach to provide a setting appropriate for subsequent fixed point theory. Section 3 contains the first existence result, via Schauder's theorem, for eigenpairs of given $\|Y\|$ and given oscillation count for y (Corollary 3.5). This is an analogue of the 'horizontal' approach in [7] where each horizontal line in the $(\lambda, \|Y\|)$ bifurcation diagram contains at least one point on each 'bifurcation curve'. We also bound these 'curves' in vertical strips under various conditions on f. We remark that some of our conditions on f permit (1.8) to have 'indefinite weight', a topic of some interest recently in the linear case, *cf.* [1], [9].

In Section 4 we assume f to be C^1 and we apply a standard result of [8] to give a C^1 bifurcation curve through the point $P_k = (\lambda_k, 0)$ (Theorem 4.1). This curve, which cannot pass through any P_j, $j \neq k$, corresponds to fixed oscillation count near P_k, but that count changes across the line $\lambda = -d/c$. We also adapt the techniques of [4] to give existence of a Riesz basis of (normalized) eigenvectors Y (Theorem 4.4).

In the final section, we give various conditions for the bifurcation 'curves' (or more precisely connected sets of eigenpairs) to intersect given 'vertical' lines. For example, assuming that $f \in C^1$, $|f(x, y, y')| > 1$ for large $|y| + |y'|$ and that the asymptotic problem (see above) is left semidefinite we show that the 'curve' through P_k intersects each line $\lambda = h$ for $h \geq \lambda_k$, except for $k = 0$: the curve through P_0 always lies to the left of $\lambda = -d/c$. If the semidefiniteness condition is dropped, $k = 1$ can be execeptional also (Theorem 5.6).

2 Preliminaries

We begin with smoothness conditions on the coefficients. We assume $r, q \in C^0([0,1];\mathbb{R})$ and $p \in C^1([0,1];\mathbb{R})$. Let $f \in C^0([0,1] \times \mathbb{R}^2;\mathbb{R})$ with $f(x,0,0) = 0$ for all $x \in [0,1]$.

Define

$$|u|_1 = \max_{x\in[0,1]} |u(x)| + \max_{x\in[0,1]} |u'(x)| \tag{2.1}$$

and

$$\left\| \begin{pmatrix} u \\ \gamma \end{pmatrix} \right\|_1 = |u|_1 + |\gamma| \tag{2.2}$$

for all $u \in C^1([0,1];\mathbb{R})$ and $\gamma \in \mathbb{R}$. We denote by $V_j = \binom{v_j}{\gamma_j}$ the eigenvector of A (1.4) corresponding to the eigenvalue, λ_j with $v_j(x) > 0$ for small $x > 0$ and normalized by $\|V_j\|_1 = 1$. All (real) eigenpairs of (1.4) are thus given by (λ_j, sV_j), $s \in \mathbb{R} \setminus \{0\}$, $j = 0, 1, 2, \ldots$.

We define D to be the Banach space

$$D = \left\{ \begin{pmatrix} u \\ -au(1) + cp(1)u'(1) \end{pmatrix} : u \in BC_0 \right\}$$

with norm given by (2.2). Let S be the subset of D given by

$$S = \left\{ \begin{pmatrix} u \\ \gamma \end{pmatrix} \in D : |u(x)| + |u'(x)| > 0 \, \forall x \in [0,1] \right\},$$

with metric inherited from D. The elements of S are precisely those $\binom{u}{\gamma} \in D$ for which u has only (finitely many) simple zeros in $[0,1]$. Since $p \in C^1([0,1];\mathbb{R})$, it follows that if $\binom{u}{\gamma} \in \mathcal{D}(A)$ then $u' \in AC([0,1];\mathbb{R})$. Thus $u \in C^1([0,1];\mathbb{R})$ and $\mathcal{D}(A) \subseteq D$.

In order to discuss our analogues of (1.5), let β be the continuous function defined by

$$\cot \beta(\lambda) = \frac{a\lambda + b}{c\lambda + d}, \quad \beta\left(-\frac{d}{c}\right) = 0. \tag{2.3}$$

It follows from (1.6) that β is a strictly decreasing function on $\mathbb{R}$. For each $\binom{u}{\gamma} \in S$ we define $\theta\big(\binom{u}{\gamma}, \cdot\big)$ to be the continuous function on $[0,1]$ satisfying

$$\cot \theta \left(\begin{pmatrix} u \\ \gamma \end{pmatrix}, x \right) = \frac{p(x)u'(x)}{u(x)}, \quad \theta \left(\begin{pmatrix} u \\ \gamma \end{pmatrix}, 0 \right) = \alpha. \tag{2.4}$$

It is apparent that $\theta \colon S \times [0,1] \to \mathbb{R}$ is continuous. From (2.3), (2.4) and [2] we obtain that

$$\theta(sV_j, 1) - \beta(\lambda_j) = j\pi, \quad s \in \mathbb{R} \setminus \{0\}, \quad j = 0, 1, 2, \ldots. \tag{2.5}$$

Subsets of S with fixed oscillation count can now be defined by

$$S^{\sigma}_{k,\lambda} = \left\{ \begin{pmatrix} u \\ \gamma \end{pmatrix} \in S : \theta\left(\begin{pmatrix} u \\ \gamma \end{pmatrix}, 1 \right) - \beta(\lambda) = k\pi, \sigma u(x) > 0 \text{ for small } x > 0 \right\},$$

with $S_{k,\lambda} = S^{+}_{k,\lambda} \cup S^{-}_{k,\lambda}$, and

$$E_\lambda = \left\{ \begin{pmatrix} u \\ \gamma \end{pmatrix} \in D : u \in BC^{\lambda}_{1} \right\}.$$

These subsets all inherit the topology of S, *i.e.*, of D. We also use the notation $B_\rho = \{U \in D : \|U\|_1 \leq \rho\}$, $B^{o}_{\rho} = \{U \in D : \|U\|_1 < \rho\}$, $\partial B_\rho = \{U \in D : \|U\|_1 = \rho\}$. Here $\sigma \in \{+, -\}$, $\lambda \in \mathbb{R}$, $k = 0, 1, 2, \ldots$ and $\rho > 0$.

Lemma 2.1

(a) Let $U \in S$. If $\theta(U, x) = n\pi$ where $n \in \mathbb{Z}$, then $\frac{\partial\theta}{\partial x}(U, x) = \frac{1}{p(x)} > 0$. If $\theta(U, x) \geq n\pi$ where $n \in \mathbb{Z}$, then $\theta(U, y) > n\pi$ for all $y > x$.

(b) The $S^{\sigma}_{k,\lambda}$ are pairwise disjoint open subsets of E_λ.

(c) If $\alpha \in [0, \pi)$ and $\lambda \geq -\frac{d}{c}$ then $S^{\sigma}_{0,\lambda}$ is empty.

(d) Let $\binom{u}{\gamma} \in S^{\sigma}_{k,\lambda}$. If $\lambda < -\frac{d}{c}$, then u has precisely k zeros in $(0, 1)$. If $\lambda \geq -\frac{d}{c}$, then u has precisely $k - 1$ zeros in $(0, 1)$.

Proof (a) Let $U \in S$. Directly from the definition of θ we have that if $\theta(U, x) = n\pi$ where $n \in \mathbb{Z}$, then $\frac{\partial\theta}{\partial x}(U, x) = \frac{1}{p(x)} > 0$. The continuity of $\theta(U, x)$ with respect to x and the assumption that $p(x) > 0$ yield $\theta(U, y) > n\pi$ for all $y > x$.

(b) The sets $S^{\sigma}_{k,\lambda}$ are pairwise disjoint by their definition. Let $U = \binom{u}{\gamma} \in S^{\sigma}_{k,\lambda}$. Then $\theta(U, 1) - \beta(\lambda) = k\pi$, which implies that $u \in BC^{\lambda}_{1}$. Hence $S^{\sigma}_{k,\lambda} \subset E_\lambda$. That $S^{\sigma}_{k,\lambda}$ is open in E_λ follows from

$$\{u \in BC_0 \cap BC^{\lambda}_{1} : \sigma u(x) > 0 \,\forall x \text{ near } 0, |u(x)| + |u'(x)| > 0 \,\forall x, u \text{ has } m \text{ zeros in } (0, 1)\}$$

being open in

$$\{u \in BC_0 \cap BC^{\lambda}_{1} : |u(x)| + |u'(x)| > 0 \,\forall x\},$$

which can be proved by straightforward classical analysis.

(c) Let $\alpha \in [0, \pi)$ and $\lambda \geq -\frac{d}{c}$. Suppose that there exists $U \in S^{\sigma}_{0,\lambda}$. Then, by (a), $\theta(U, 1) > 0$. Since β is a decreasing function and $\beta(-\frac{d}{c}) = 0$, it follows that $\beta(\lambda) \leq 0$. Together with the definition of $S^{\sigma}_{0,\lambda}$ we have

$$0 < \theta(U, 1) = \beta(\lambda) \leq 0,$$

a contradiction.

(d) Let $U = \binom{u}{\gamma} \in S^{\sigma}_{k,\lambda}$. Then $\theta(U, 1) = \beta(\lambda) + k\pi$. For $\lambda < -\frac{d}{c}$ we have $\pi > \beta(\lambda) > 0$ which gives

$$k\pi < \theta(U, 1) < (1 + k)\pi \tag{2.6}$$

and for $\lambda \geq -\frac{d}{c}$ we have $-\pi < \beta(\lambda) \leq 0$ which gives

$$(k-1)\pi < \theta(U,1) \leq k\pi. \tag{2.7}$$

With the aid of part (a) of the Lemma, (2.6) and (2.7) complete the proof of (d). ■

In order to obtain our eigentuples, we shall need the following constructions based on those above:

$$T_k^\sigma = \bigcup_{\lambda\in\mathbb{R}} (\{\lambda\} \times S_{k,\lambda}^\sigma),$$

$$T_k = T_k^+ \cup T_k^-,$$

$$E = \bigcup_{\lambda\in\mathbb{R}} (\{\lambda\} \times E_\lambda).$$

We give the above sets the metric induced by the norm

$$\|(\lambda, V)\| = |\lambda| + \|V\|_1.$$

Lemma 2.2

(a) E is a complete metric space.
(b) The T_k^σ are disjoint open subsets of E.
(c) If (λ, U) is an eigenpair of (1.9) then $(\lambda, U) \in \cup_k T_k$.
(d) If (λ, U) is a solution of (1.9) and $(\lambda, U) \in \partial T_k$, then $U = 0$.
(e) For each $\rho > 0$, $(\lambda_k, \rho\sigma V_k)$ is the eigenpair of (1.4) from $T_k^\sigma \cap \partial B_\rho$.

Proof (a) That E is a complete metric space follows directly from E being a closed subset of $\mathbb{R} \times \big(C^1([0,1]) \times \mathbb{C}\big)$.

(b) That the T_k^σ are disjoint subsets of E is obvious. To prove that T_k^σ is open in E, let $(\lambda, U) \in T_k^\sigma$ where $U = \binom{u}{\gamma}$. As $\sigma u(x)$ is positive for $x > 0$ small and as u has only simple zeros in $[0,1]$, there exists $\epsilon_1 > 0$ such that if $\|v - u\|_1 < \epsilon_1$, $v \in C^1([0,1]) \cap BC_0$, then $\sigma v(x)$ is positive for $x > 0$ small and v has only simple zeros in $[0,1]$. Let $\|V - U\|_1 < \epsilon_1$, $V \in D$, then $\theta(V,1)$ exists and, by the continuity of θ, there exists $\epsilon_2, \epsilon_1 > \epsilon_2 > 0$, such that if $\|V - U\|_1 < \epsilon_2$, $V \in D$, then $|\theta(V,1) - \theta(U,1)| < \frac{\pi}{4}$. By the continuity of β, there exists $\epsilon, \epsilon_2 > \epsilon > 0$, for which $|\mu - \lambda| < \epsilon$ implies $|\beta(\mu) - \beta(\lambda)| < \frac{\pi}{4}$. Thus if $\|(\mu, V) - (\lambda, U)\| < \epsilon$, $(\mu, V) \in E$, then

$$\big|[\theta(V,1) - \beta(\mu)] - [\theta(U,1) - \beta(\lambda)]\big| < \frac{\pi}{2}. \tag{2.8}$$

But for $\|(\mu, V) - (\lambda, U)\| < \epsilon$, $(\mu, V) \in E$, we have

$$\theta(V,1) = \beta(\mu) + n\pi \tag{2.9}$$

for some $n \in \mathbb{Z}$ (as $(\mu, V) \in E$ and $\theta(V,1)$ exists), and by definition of U

$$\theta(U,1) = \beta(\lambda) + k\pi. \tag{2.10}$$

The combination of (2.8), (2.9) and (2.10) gives $|n-k| < \frac{1}{2}$. Since n and k are integers this implies that $n = k$. Thus, by (2.9), $(\mu, V) \in T_k^\sigma$, and T_k^σ is and open set.

(c) Let $(\lambda, U), U = \binom{u}{\gamma}$, be an eigenpair of (1.9). Since $\mathcal{D}(A) \subseteq D$ we have $(\lambda, U) \in E$. To complete the proof of (c) we only need to prove that the zeros of u in $[0, 1]$ are simple. If u has a non-simple zero then, as u is a solution to the homogeneous linear second order equation $-(pu')' + hu = 0$ where $h = q - \lambda r\big(1 - f(x, u, u')\big)$, u is identically zero. This contradicts (λ, U) being an eigenpair.

(d) Let $(\lambda, U) \in \partial T_k^\sigma$ be a solution of (1.9). Then, by part (b), $(\lambda, U) \notin \bigcup_k T_k^\sigma$, and hence, by (c), $U = 0$.

(e) That $(\lambda_k, \rho\sigma V_k) \in T_k^\sigma \cap \partial B_\rho$ is obvious. From [2], the definition of T_k^σ and the eigenspaces of (1.4) being 1-dimensional it follows that (1.4) has precisely one eigenpair $(\lambda, V) \in T_k^\sigma \cap \partial B_\rho$. ∎

3 Horizontal Theory

We start with the following result which bounds sets of eigentuples (λ, V) with fixed oscillation count inside certain vertical strips in the $(\lambda, \|V\|_1)$ bifurcation diagram.

Theorem 3.1 *Let $(\lambda, V) \in T_k$ be an eigenpair of (1.9) with $\|V\|_1 = \rho > 0$.*

(a) Let

$$M_\rho = \max\{1 - f(x, \xi, \eta) : x \in [0, 1], |\xi| + |\eta| \le \rho\}.$$

If $\lambda \ge 0$ then $\lambda \ge \frac{\lambda_k}{M_\rho}$, while if $\lambda \le 0$ then $\lambda \le \frac{\lambda_k}{M_\rho}$.

(b) Suppose that $f(x, \xi, \eta) < 1$ for all $x \in [0, 1]$, $|\xi| + |\eta| \le \rho$ and let

$$\epsilon_\rho = \min\{1 - f(x, \xi, \eta) : x \in [0, 1], |\xi| + |\eta| \le \rho\}.$$

If $\lambda \ge 0$ then $\lambda \le \frac{\lambda_k}{\epsilon_\rho}$, while if $\lambda \le 0$ then $\lambda \ge \frac{\lambda_k}{\epsilon_\rho}$.

Proof (a) Let $\lambda \ge 0$ and suppose that $\lambda < \frac{\lambda_k}{M_\rho}$. Then

$$\lambda M_\rho < \lambda_k. \tag{3.1}$$

From the definition of M_ρ and the assumption that $\lambda \ge 0$ we obtain $\lambda(1 - f) < \lambda_k$, from which the Comparison Theorem, [6, Theorem 8.1.2], enables us to conclude that

$$\theta(V, 1) \le \theta(V_k, 1). \tag{3.2}$$

From the overall assumption that $f(x, 0, 0) = 0$ we are ensured that $M_\rho \ge 1$, which combined with (3.1) and the assumption that $\lambda \ge 0$ gives $\lambda < \lambda_k$. As $\beta(\cdot)$ is a strictly decreasing function we may thus conclude that

$$\beta(\lambda) > \beta(\lambda_k). \tag{3.3}$$

Combining (3.2) and (3.3) we obtain the contradiction

$$k\pi = \theta(V, 1) - \beta(\lambda) < \theta(V_k, 1) - \beta(\lambda_k) = k\pi.$$

Hence $\lambda \geq \frac{\lambda_k}{M_\rho}$.

The proof for the case of $\lambda \leq 0$ is analogous.

(b) From the overall assumption that $f(x,0,0) = 0$ and from the additional assumption that $f(x,\xi,\eta) < 1$ for all $x \in [0,1]$, $|\xi| + |\eta| \leq \rho$, we have that $0 < \epsilon_\rho \leq 1$.

Let $\lambda \geq 0$ and suppose that

$$\lambda > \frac{\lambda_k}{\epsilon_\rho}. \tag{3.4}$$

From the definition of ϵ_ρ and the assumption that $\lambda \geq 0$ we obtain $\lambda(1-f) > \lambda_k$, from which the comparison theorem enables us to conclude that

$$\theta(V,1) \geq \theta(V_k,1). \tag{3.5}$$

Since $\epsilon_\rho \leq 1$, (3.4) and the assumption that $\lambda \geq 0$ yield $\lambda > \lambda_k$. As β is a strictly decreasing function we may thus conclude that

$$\beta(\lambda) < \beta(\lambda_k). \tag{3.6}$$

Combining (3.5) and (3.6) we obtain the contradiction

$$k\pi = \theta(V,1) - \beta(\lambda) > \theta(V_k,1) - \beta(\lambda_k) = k\pi.$$

Hence $\lambda \leq \frac{\lambda_k}{\epsilon_\rho}$.

The proof for the case $\lambda \leq 0$ is analogous. ∎

As a direct consequence of the above theorem we have

Corollary 3.2 *Let $(\lambda, V) \in T_k$ be an eigenpair of (1.9).*

(a) Suppose that there exists M such that $M \geq 1 - f$ on $[0,1] \times \mathbb{R}^2$. If $\lambda \geq 0$ then $\lambda \geq \frac{\lambda_k}{M}$, while if $\lambda \leq 0$ then $\lambda \leq \frac{\lambda_k}{M}$.

(b) Suppose that there exists $\epsilon > 0$ such that $\epsilon \leq 1 - f$ on $[0,1] \times \mathbb{R}^2$. If $\lambda \geq 0$ then $\lambda \leq \frac{\lambda_k}{\epsilon}$, while if $\lambda \leq 0$ then $\lambda \geq \frac{\lambda_k}{\epsilon}$.

In particular, if $f \geq 0$ then we can choose $M = 1$ and the k-th bifurcation 'curve' is bounded on the left by the line $\lambda = \lambda_k$. While considering such bounds it should also be noted that Lemma 2.1(c) shows that the 0-th bifurcation 'curve' lies strictly to the left of the line $\lambda = -d/c$.

In order to apply Schauder's theorem, we shall need certain continuity and compactness properties, and the following is a basic ingredient.

Lemma 3.3 *Let λ be in the resolvent of A. Then $(A-\lambda)^{-1} : C^0([0,1]) \oplus \mathbb{C} \to \mathcal{D}(A)$ is a continuous and compact map, where $C^0([0,1]) \oplus \mathbb{C}$ has norm given by*

$$\left\| \begin{pmatrix} u \\ \gamma \end{pmatrix} \right\|_0 = |\gamma| + \max_{x \in [0,1]} |u(x)|. \tag{3.7}$$

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Proof From [3],

$$(A-\lambda)^{-1}\begin{pmatrix} w \\ \gamma \end{pmatrix} = \begin{pmatrix} g \\ \tau \end{pmatrix},$$

where

$$g(x) = f_+(x)\int_x^1 f_- wr\,dy + f_-(x)\int_0^x f_+ wr\,dy - \gamma f_+(x),$$
$$\tau = -ag(1) + cp(1)g'(1),$$

in which $f_+, f_- \in C^2([0,1];\mathbb{R})$ are suitably chosen solutions of $(l-\lambda)u = 0$.

By Kantorovich's compactness theorem, [11, Theorem 9.5.8], the following maps on $C^0([0,1])$ are compact:

$$M_0\colon w \to f_+(x)\int_x^1 f_- wr\,dy + f_-(x)\int_0^x f_+ wr\,dy,$$
$$M_1\colon w \to f_+'(x)\int_x^1 f_- wr\,dy + f_-'(x)\int_0^x f_+ wr\,dy.$$

Hence the map

$$M\colon \begin{pmatrix} w \\ \gamma \end{pmatrix} \to \begin{pmatrix} f_+(x)\int_x^1 f_- wr\,dy + f_-(x)\int_0^x f_+ wr\,dy \\ \gamma \end{pmatrix}$$

from $C^0([0,1]) \times \mathbb{C}$ to $C^1([0,1]) \times \mathbb{C}$ is compact. Since the composition of a continuous and a compact map is compact, the Lemma follows. ■

Theorem 3.4 *Let $\epsilon > 0$, $K > 0$, and $f(x,\xi,\eta) \le 1-\epsilon$ for all $x \in [0,1]$ and $|\xi| + |\eta| \le K$. For each $U \in (C^1 \oplus \mathbb{C}) \cap B_K$ let $R_k(U) = \big(\mu_k(U), U_k(U)\big)$ denote the eigenpair of*

$$AV = \lambda\big(I - F(U)\big)V, \quad V \in \mathcal{D}(A) \tag{3.8}$$

from T_k^+ and having $\|U_k(U)\|_1 = 1$. Then the mappings

$$\mu_k\colon (C^1 \oplus \mathbb{C}) \cap B_K \to \mathbb{R} \tag{3.9}$$
$$U_k\colon (C^1 \oplus \mathbb{C}) \cap B_K \to \partial B_1 \cap S_{k,\lambda}^+ \tag{3.10}$$
$$R_k\colon (C^1 \oplus \mathbb{C}) \cap B_K \to \mathbb{R} \times \partial B_1 \cap T_k^+ \tag{3.11}$$

are continuous and compact for each $k = 0, 1, 2, \ldots$.

Proof By Theorem 3.1, $\mu_k(U) \in [-\frac{|\lambda_k|}{\epsilon}, \frac{|\lambda_k|}{\epsilon}]$ for all $(C^1 \oplus \mathbb{C}) \cap B_K$. Hence the map $\mu_k(\cdot)$ is compact.

Let μ be in the resolvent of A, then

$$U_k(U) = (A-\mu)^{-1}\big[\mu_k(U)\big(I - F(U)\big) - \mu\big]U_k(U).$$

The map $U \to \big[\mu_k(U)\big(I - F(U)\big) - \mu\big]U_k(U)$ taking $(C^1 \oplus \mathbb{C}) \cap B_K \to (C^0 \oplus \mathbb{C})$ is a bounded map. By Lemma 3.3, $(A-\mu)^{-1}$ is a compact mapping. Thus U_k is a compact map and consequently R_k is a compact mapping.

It remains to prove that R_k is continuous. Let $W_j \to W$ in $C^1 \oplus \mathbb{C}$. We show that $R_k(W_j) \to R_k(W)$. As $\{W_j\}$ is convergent, it is bounded, and, by the compactness of R_k, we may move to a subsequence of $\{W_j\}$ for which $\{R_k(W_j)\}$ is convergent, say to (μ', U). Then, by the continuity of $(A-\mu)^{-1}$ we have $U = (A-\mu)^{-1}\big[\mu'\big(I - F(W)\big) - \mu\big]U$. But $\|U\|_1 = 1$ and, by Lemma 2.2, $U \in T_k^+$. Thus $(\mu', U) = R_k(W)$. So every subsequence of $\{R_k(W_j)\}$ contains a subsequence convergent to $R_k(W)$, and consequently $\{R_k(W_j)\}$ converges to $R_k(W)$. ■

Our first existence result is a consequence of the above.

Corollary 3.5 *Suppose $\rho > 0$ and there exists $\epsilon_\rho > 0$ such that $f(x, \xi, \eta) \le 1 - \epsilon_\rho$ for all $x \in [0,1]$ and $|\xi| + |\eta| \le \rho$. Then, for each $k = 0, 1, 2, \ldots$, there exists an eigenpair $(\lambda^\sigma_{k,\rho}, V^\sigma_{k,\rho}) \in T^\sigma_k$ with $\|V^\sigma_{k,\rho}\|_1 = \rho$.*

Proof Let μ_k and U_k be as defined in Theorem 3.4. Let $P^\sigma_{k,\rho}(\lambda, V) = \big(\mu_k(V), \sigma\rho U_k(V)\big)$. Then, from Theorem 3.4, $P^\sigma_{k,\rho}\colon \mathbb{R} \times [(C^1 \oplus \mathbb{C}) \cap B_\rho] \to (\mathbb{R} \times \partial B_\rho) \cap T^\sigma_k$, is compact and continuous. Applying Schauder's Fixed Point Theorem to $P^\sigma_{k,\rho}$, we see that there exists a pair $(\lambda^\sigma_{k,\rho}, V^\sigma_{k,\rho}) \in (\mathbb{R} \times \partial B_\rho) \cap T^\sigma_k$ such that $P^\sigma_{k,\rho}(\lambda^\sigma_{k,\rho}, V^\sigma_{k,\rho}) = (\lambda^\sigma_{k,\rho}, V^\sigma_{k,\rho})$. Thus $AU^\sigma_{k,\rho} = \mu^\sigma_{k,\rho}\big(I - F(U^\sigma_{k,\rho})\big)U^\sigma_{k,\rho}$. ■

Remark In Corollary 3.5, if we assume in addition f to be C^1, then, using Theorem 4.3, for each $k = 0, 1, 2, \ldots$, there exists a connected set of eigenpairs $(\lambda_{k,\rho}, V_{k,\rho}) \in T_k$ with $\|V_{k,\rho}\|_1 = \rho$ and $\lim_{\rho \to 0}(\lambda_{k,\rho}, \frac{V_{k,\rho}}{\rho}) = (\lambda_k, V_k)$.

4 Local Theory

In this section we dispense with the bounds on f, but we shall impose the extra smoothness condition $f \in C^1([0,1] \times \mathbb{R}^2; \mathbb{R})$. Let μ' be in the resolvent of A and

$$\Phi(\lambda, U) = U - (A - \mu')^{-1}\big[\lambda\big(I - F(U)\big) - \mu'\big]U, \tag{4.1}$$

for $\lambda \in \mathbb{R}$ and $U \in C^1 \oplus \mathbb{C}$. Then (1.9) is equivalent to $\Phi(\lambda, U) = 0$. Our second existence theorem is for C^1 local bifurcation curves.

Theorem 4.1 *Let Z_k denote a complement of $\operatorname{span}(V_k)$ in $C^1 \oplus \mathbb{C}$. Then, for each $k = 0, 1, 2, \ldots$, there exists a neighbourhood $\Omega_k = (\lambda_k - \delta, \lambda_k + \delta) \times W_k$ of $(\lambda_k, 0)$ in $\mathbb{R} \times (C^1 \oplus \mathbb{C})$, an interval $(-\delta_k, \delta_k)$, $\delta_k > 0$, and functions $U_k \in C\big((-\delta_k, \delta_k); Z_k\big)$ and $\mu_k \in C\big((-\delta_k, \delta_k); \mathbb{R}\big)$ such that:*

(a) $\Big(\mu_k(t), t\big(V_k + U_k(t)\big)\Big)$ *is an eigenpair of (1.9) from $T_k^{\operatorname{sgn}(t)}$ for all $t \in (-\delta_k, \delta_k) \setminus \{0\}$;*

(b) $\mu_k(0) = \lambda_k$ *and* $U_k(0) = 0$*;*

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(c) If $(\mu, U) \in \Omega \setminus (\mathbb{R} \times \{0\})$ *is an eigenpair of (1.9), then*

$$(\mu, U) = \big(\mu_k(t), t[V_k + U_k(t)]\big), \tag{4.2}$$

for some unique $t \in (-\delta_k, \delta_k)$.

Proof The theorem is a direct consequence of [8] applied to Φ, with the exception of the statement that $\Big(\mu_k(t), t\big(V_k + U_k(t)\big)\Big) \in T_k^{\mathrm{sgn}(t)}$ for all $t \in (-\delta_k, \delta_k) \setminus \{0\}$ in part (a). As $(\lambda_k, V_k) \in T_k^+$, $U_k(0) = 0$, $\lambda_k = \mu_k(0)$ and since U_k and μ_k are continuous it follows from T_k^+ being an open set that for $|t|$ small $\big(\lambda_k, V_k + U_k(t)\big) \in T_k^+$. Thus by the connectedness and openness of T_k^+ along with the fact that $t\big(V_k + U_k(t)\big) \neq 0$ if $t \neq 0$, Lemma 2.2 enables us to conclude that $\Big(\mu_k(t), t\big(V_k + U_k(t)\big)\Big) \in T_k^{\mathrm{sgn}(t)}$ for all $t \in (-\delta_k, \delta_k) \setminus \{0\}$. ■

Extension of the above curves may be carried out subject to certain conditions at secondary bifurcation points. We omit details, but we shall give a nonexistence result, to the effect that no extended curve can join two distinct points of the form $(\lambda_j, 0)$.

Theorem 4.2 *Under the hypotheses of Theorem 4.1, let K denote the closure in* $\mathbb{R} \times (C^1 \oplus \mathbb{C})$ *of* $\{(\lambda, V) \in E : \Phi(\lambda, V) = 0, V \neq 0\}$ *and* K_k *denote the connected component of K containing* $(\lambda_k, 0)$*. Then every element of* K_k *is a solution of (1.9),* $(\mu, 0) \notin K_k$ *for all* $\mu \neq \lambda_k$ *and* $K_k \subseteq T_k \cup \{(\lambda_k, 0)\}$.

Proof Suppose that there exists $\mu \neq \lambda_k$ such that $(\mu, 0) \in K_k$. Then there is some component J of $K_k \setminus (\mathbb{R} \times \{0\})$ having limit points $(\lambda_k, 0)$ and $(\mu, 0)$. We also note that, as Φ is continuous, $\{(\lambda, V) \in E : \Phi(\lambda, V) = 0\}$ is a closed set and so if $(\lambda, V) \in K_k$ then $\Phi(\lambda, U) = 0$. Thus all $(\lambda, V) \in J$ are eigenpairs of (1.9). By the connectedness of J, Lemma 2.2 and Theorem 4.1 it follows that $J \subset T_k^\sigma$ for either $\sigma = +$ or $\sigma = -$. Since $(\mu, 0) \in \bar{J}$, there exists a sequence $\{(\mu_j, U_j)\}_j \subset J$ such that $(\mu_j, U_j) \to (\mu, 0)$. From $\Phi(\mu_j, U_j) = 0$ it follows that

$$0 = \frac{U_j}{\|U_j\|_1} - \mu_j A^{-1}[I - F(U_j)]\frac{U_j}{\|U_j\|_1}. \tag{4.3}$$

But the sequence

$$[I - F(U_j)]\frac{U_j}{\|U_j\|_1} \tag{4.4}$$

is bounded and A^{-1} is a compact operator and so (4.4) has a convergent subsequence. Passing to this subsequence we have

$$A^{-1}[I - F(U_j)]\frac{U_j}{\|U_j\|_1} \to U$$

for some U. Thus $\frac{U_j}{\|U_j\|_1} \to \mu U$ and $\|\mu U\|_1 = 1$. From this, along with (4.3), we may conclude that $\mu \neq 0$, $U \in \mathcal{D}(A)$, $U \neq 0$ and $0 = U - \mu A^{-1} U$. Thus $\mu = \lambda_n$ for some $n \neq k$ and by Theorem 4.1 and openness of T_n, $(\mu_j, U_j) \in T_n \cap T_k = \phi$ for large j. ■

Corollary 4.3 *Under the hypotheses of Theorem 4.1, let K_k denote the connected set of eigenpairs of (1.9) containing $(\lambda_k, 0)$ as a limit point. Then K_k is an unbounded subset of E, $K_k \subseteq T_k \cup \{(\lambda_k, 0)\}$ and the only limit point of K_k not in K_k is the point $(\lambda_k, 0)$.*

Proof From [5] we see that K_k is either unbounded or $(\mu, 0) \in K_k$ for some $\mu \neq \lambda_k$. Theorem 4.1 and Lemma 2.2 show that if $(\mu, 0) \notin K_k$ for all $\mu \neq \lambda_k$, then $K_k \subset T_k \cup \{(\lambda_k, 0)\}$. But Theorem 4.2 allows us to conclude $(\mu, 0) \notin K_k$ for all $\mu \neq \lambda_k$. The remaining contentions come directly from Theorem 4.2. ■

We conclude this section with a completeness result of a similar nature to [4].

Theorem 4.4 *For $\sigma = +, -$, there is a sequence of eigenpairs of (1.9), $(l_j^\sigma, U_j^\sigma) \in T_j^\sigma$, $j = 0, 1, 2, \ldots$, such that the normalized eigenvectors $\{\frac{U_j^\sigma}{\|U_j^\sigma\|}\}_{j=0,1,2,\ldots}$ form a Riesz basis for $L^2 \oplus \mathbb{C}$.*

Proof We prove the result for $\sigma = +$. The proof for $\sigma = -$ is similar, differing only in the detail that one chooses $t_k \in (-\delta_k, 0)$.

Let $\Sigma(t, k) = \bigl(\mu_k(t), t[V_k + U_k(t)]\bigr)$, $t \in (-\delta_k, \delta_k)$, where $\mu_k(t)$, $U_k(t)$ and δ_k are as in Theorem 4.1. Then, for each $k = 0, 1, 2, \ldots$, $\Sigma(t, k) \in T_k^{\operatorname{sgn} t} \cup \{(\lambda_k, 0)\}$, for $t \in (-\delta_k, \delta_k)$, is a continuous curve of solutions to (1.9) and $\Sigma(t, k) = (\lambda_k, 0)$ if and only if $t = 0$.

Let $a_k = \|V_k\|$, then, by [14], $\{\frac{V_k}{a_k}\}_{k=0,1,2,\ldots}$ is an orthonormal basis for $L^2 \oplus \mathbb{C}$. Let $t_k \in (0, \delta_k)$ be such that $\frac{\|U_k(t_k)\|_1}{a_k} \leq 2^{-k-2}$. Let $l_j^+ = \mu(t_k)$ and

$$U_k^+ = t_k[U_k(t_k) + V_k].$$

Then

$$\left\| \frac{U_k^+}{a_k t_k} - \frac{V_k}{a_k} \right\| = \frac{\|U_k(t_k)\|}{a_k} \leq \frac{\|U_k(t_k)\|_1}{a_k} \leq 2^{-k-2},$$

and

$$\bigl|\|U_k^+\| - t_k a_k\bigr| \leq t_k a_k 2^{-k-2},$$

giving

$$\sum \left\| \frac{U_k^+}{\|U_k^+\|} - \frac{V_k}{a_k} \right\| < 1.$$

Hence, by [14, Theorem 2.20 and Corollary 2.22], $\{\frac{U_k^+}{\|U_k^+\|}\}$ is a Riesz basis of $L^2 \oplus \mathbb{C}$ under the norm $\| \cdot \|$. ■

5 Vertical Theory

In place of the vertical strips used in Section 3, we aim in this section to bound the bifurcation curves between the λ axis and the graph of a function m which we construct next.

Theorem 5.1 *Suppose $q \geq 0$ on $[0, 1]$, $\alpha \in [0, \frac{\pi}{2}]$, f is bounded on $[0, 1] \times \mathbb{R}^2$ and that there exists $K > 0$ such that if $|\xi| + |\eta| \geq K$ then $f(x, \xi, \eta) > 1$ for all $x \in [0, 1]$. There exists a continuous, positive increasing function, $m(\lambda)$, $\lambda \geq 0$, such that if $(\lambda, V) \in T_k$, $\lambda \geq 0$, is an eigenpair of (1.9) and $\|V\|_1 \geq m(\lambda)$ then $k = 0$ if $\lambda < -\frac{d}{c}$ and $k = 1$ if $\lambda > -\frac{d}{c}$. If $\lambda = -\frac{d}{c}$ or $\frac{a}{c} \leq 0$ and $\lambda > -\frac{d}{c}$ there are no eigenpairs (λ, V) of (1.9) with $\|V\|_1 \geq m(\lambda)$.*

Proof Let $k_1 \geq |r(x)(1-f)(x,\xi,\eta)|$ for all $(x,\xi,\eta) \in [0,1] \times \mathbb{R}^2$. Let $V = \binom{v}{\gamma}$ and

$$k(v) = \min_{x\in[0,1]} [|v(x)| + |v'(x)|].$$

As (λ, V) is an eigenpair, $k(v) \neq 0$. Let $x_0 \in [0,1]$ be such that $k(v) = |v(x_0)| + |v'(x_0)|$. Finally, let $Q = \max_{x\in[0,1]} q(x)$, $P_1 = \max_{x\in[0,1]} p(x)$ and $P_2 = \max_{x\in[0,1]} \frac{1}{p(x)}$.

Since (λ, V) is an eigenpair of (1.9), we have

$$-(pv')' = [r\lambda(1-f) - q]v.$$

Integrating the above we obtain

$$v'(x) - \frac{p(x_0)v'(x_0)}{p(x)} = -\frac{1}{p(x)} \int_{x_0}^{x} [r\lambda(1-f) - q]v \, d\tau, \tag{5.1}$$

and thus

$$|v'(x)| \leq P_1P_2|v'(x_0)| + P_2[\lambda k_1 + Q] \operatorname{sgn}(x - x_0) \int_{x_0}^{x} |v| \, d\tau. \tag{5.2}$$

Integration again from x_0 to x gives

$$|v(x)| \leq |v(x_0)| + |x - x_0|P_1P_2|v'(x_0)| + P_2[\lambda k_1 + Q] \int_{x_0}^{x} (x - \tau)|v| \, d\tau. \tag{5.3}$$

Considering separately the two cases of $x \geq x_0$ and $x \leq x_0$, we apply Gronwall's Lemma, [13], to (5.3) to give

$$|v(x)| \leq [|v(x_0)| + P_1P_2|v'(x_0)|]e^{P_2[\lambda k_1 + Q]},$$

and thus

$$|v(x)| \leq k(v)[1 + P_1P_2]e^{P_2[\lambda k_1 + Q]}. \tag{5.4}$$

Using (5.4) in conjunction with (5.2) we obtain

$$|v'(x)| \leq k(v)\big[P_1P_2 + P_2[\lambda k_1 + Q][1 + P_1P_2]e^{P_2[\lambda k_1 + Q]}\big]. \tag{5.5}$$

The definition of the domain of A together with (5.4) and (5.5) allows us to conclude that

$$\|V\|_1 \leq 2k(v)(1 + |b| + |d|)\big[P_1P_2 + P_2[\lambda k_1 + Q][1 + P_1P_2]e^{P_2[\lambda k_1 + Q]}\big]. \tag{5.6}$$

Let

$$m(\lambda) = 2K(1 + |b| + |d|)\big[P_1P_2 + P_2[\lambda k_1 + Q][1 + P_1P_2]e^{P_2[\lambda k_1 + Q]}\big].$$

Suppose that $\|V\|_1 \geq m(\lambda)$. Then from (5.6)

$$K \leq k(v) = \min_{x\in[0,1]} [|v(x)| + |v'(x)|],$$

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and consequently

$$f\big(x, v(x), v'(x)\big) > 1 \ \forall x \in [0,1].$$

Let

$$s(x) = r(x)\big[1 - f\big(x, v(x), v'(x)\big)\big].$$

Then $s(x) < 0$ for all $x \in [0,1]$, and u is a solution of the linear boundary value problem

$$-(pv')' = [\lambda s(x) - q(x)]v, \ v \in BC_0 \cap BC_1^{\lambda},$$

where $\lambda s(x) - q(x) \le 0$. But $\theta(V,x)$ obeys the differential equation

$$\frac{d\theta(V,x)}{dx} = \frac{1}{p(x)} \cos^2\theta(V,x) + [\lambda s(x) - q(x)] \sin^2\theta(V,x),$$

where $\theta(V,0) = \alpha \in [0, \frac{\pi}{2}]$. Thus $\theta(V,1) \in (0, \frac{\pi}{2}]$. For $\lambda \le -\frac{d}{c}$ this implies $\theta(V,1) - \beta(\lambda) = 0$ and $(\lambda, V) \in T_0$. But Lemma 2.1 shows that S_λ^σ is empty for all $\lambda \ge -\frac{d}{c}$. If $\lambda > -\frac{d}{c}$ this implies $\theta(V,1) - \beta(\lambda) = \pi$, which is possible only if $\cot^{-1}\frac{a}{c} - \pi < -\pi/2$. In the case $\cot^{-1}\frac{a}{c} \ge \pi/2$ no such eigenpair is possible. ∎

The following gives some variants of the above theorem proved using similar techniques.

Proposition 5.2 *Suppose $q \ge 0$ on $[0,1]$, $\alpha \in [0,\pi)$, f is bounded on $[0,1] \times \mathbb{R}^2$ and that there exists $K > 0$ such that if $|\xi| + |\eta| \ge K$ then $f(x,\xi,\eta) > 1$ for all $x \in [0,1]$. Then there exists a continuous, positive increasing function, $m(\lambda)$, $\lambda > 0$, such that if $(\lambda, V) \in T_k$ is an eigenpair of (1.9) and $\|V\|_1 \ge m(\lambda)$ then $k \in \{0,1\}$ if $0 < \lambda < -\frac{d}{c}$ and $k \in \{1,2\}$ if $\lambda \ge -\frac{d}{c}$. If in addition we assume that $\frac{a}{c} \le 0$ then for $\lambda \ge -\frac{d}{c}$ we have $k = 1$.*

Lemma 5.3 *Suppose $q \ge 0$ on $[0,1]$, $\alpha \in [0, \frac{\pi}{2}]$ and $\lambda_k < 0$. Then $k = 0$ if $\lambda_k < -\frac{d}{c}$ while $k = 1$ if $\lambda_k > -\frac{d}{c}$. If $\lambda < 0$ and $\lambda = -\frac{d}{c}$ then λ is not an eigenvalue of (1.4). If in addition we assume that $\frac{a}{c} \le 0$ then there is no eigenvalue, $0 > \lambda > -\frac{d}{c}$, of (1.4).*

Proof Suppose $\lambda_k < 0$, then

$$-(pv_k')' = [r\lambda_k - q]v_k.$$

But $\theta(V_k, x)$ obeys the differential equation

$$\frac{d\theta(V_k,x)}{dx} = \frac{1}{p(x)} \cos^2\theta(V_k,x) + [\lambda_k r(x) - q(x)] \sin^2\theta(V_k,x), \tag{5.7}$$

where $\theta(V_k, 0) = \alpha \in [0, \frac{\pi}{2}]$. From (5.7) and the assumption that $\lambda_k < 0$ we have

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$$\theta(V_k, 1) \in \left(0, \frac{\pi}{2}\right). \tag{5.8}$$

For $\lambda_k \le -\frac{d}{c}$ (5.8) implies $\theta(V_k, 1) - \beta(\lambda_k) = 0$ and hence that $k = 0$, but by Lemma 2.1 $\lambda_0 \neq -\frac{d}{c}$. For $\lambda_k > -\frac{d}{c}$ (5.8) implies that

$$\theta(V_k, 1) - \beta(\lambda_k) = \pi, \tag{5.9}$$

and consequently $k = 1$. But (5.9) is only possible if $\cot^{-1}\frac{a}{c} - \pi < -\pi/2$. In the case of $\cot^{-1}\frac{a}{c} \ge \pi/2$ (5.9) is not possible. ■

We are now in a position to prove the following structure theorem for the bifurcation curve though $(\lambda_k, 0)$.

Theorem 5.4 *Let $f \in C^1([0,1] \times \mathbb{R}^2; \mathbb{R})$ be bounded, $M = \sup\{1 - f(x, \xi, \eta) : x \in [0,1], \xi, \eta \in \mathbb{R}\}$, and assume that there exists $K > 0$ such that if $|\xi| + |\eta| \ge K$ then $f(x, \xi, \eta) > 1$ for all $x \in [0,1]$. Suppose $q \ge 0$ on $[0,1]$, $\frac{a}{c} \le 0$ and $\alpha \in [0, \frac{\pi}{2}]$. Then:*

(a) for each $k = 0, 1, 2, \ldots$ there is an unbounded connected set of eigenpairs of (1.9) from T_k having $(\lambda_k, 0)$ in its closure, denoted by say K_k;
(b) $\lambda_k \ge 0$ for all $k \ge 1$;
(c) there exists a positive, increasing function $m(\lambda)$ such that if $(\lambda, V) \in K_k$, $k \ge 1$, then $\|V\|_1 < m(\lambda)$ and $\lambda \ge \frac{\lambda_k}{M}$;
(d) for each $k \ge 1$ and $\lambda > \lambda_k$ there exists V such that $(\lambda, V) \in K_k$;
(e) $\lambda_0 < -\frac{d}{c}$;
(f) if $\lambda_0 \ge 0$ then for each $\rho > 0$ there exist λ and V such that $(\lambda, V) \in K_0$ and $\|V\|_1 = \rho$.

Proof (a) and (b) follow from Corollary 4.3 and Lemma 5.3 respectively.

(c) From Theorem 3.1 (a), we observe that if $\lambda_k \ge 0$ then $\lambda \ge 0$ for all $(\lambda, V) \in K_k$. Thus, by (b), if $(\lambda, V) \in K_k$, $k \ge 1$, then $\lambda \ge 0$. By Theorem 5.1 we can then assert the existence of the function m such that if $(\lambda, V) \in K_k$, $k \ge 1$, then $\|V\|_1 < m(\lambda)$. Corollary 3.2 gives that $\lambda \ge \lambda_k/M$.

(d) From Corollary 4.3 K_k, $k \ge 1$, is an unbounded connected set. By part (c) K_k is bounded above by the continuous function $m(\lambda)$ and to the left by the line $\lambda = \lambda_k/M$. Hence the result follows.

(e) This follows from Lemma 2.1 (c) and the existence of λ_0.

(f) If $\lambda_0 \ge 0$ and $(\lambda, V) \in K_0$ then by (e) and Corollary 4.3, K_0 is an unbounded connected set bounded on the left by $\lambda = 0$ and on the right by $\lambda = -d/c$. ■

An analogous but somewhat weaker result can be formulated for $\frac{a}{c} > 0$.

Theorem 5.5 *Let f, M and K be as in Theorem 5.4. Suppose $q \geq 0$ on $[0,1]$, $\frac{a}{c} > 0$ and $\alpha \in [0, \frac{\pi}{2}]$. Then*

(i) *(a) and (e) of Theorem 5.4 hold.*
(ii) *(b), (c) and (d) of Theorem 5.4 are true for $k \geq 2$.*
(iii) *if $-\frac{d}{c} \geq 0$ then λ_0 is the only possibly negative eigenvalue and K_1 is an unbounded set lying in the $\lambda \geq 0$ half of the bifurcation plane. If $\lambda_0 < 0$ then K_0 is unbounded and is in the $\lambda < 0$ half of the bifurcation plane. If $\lambda_0 \geq 0$ then K_0 lies between the lines $\lambda = 0$ and $\lambda = -\frac{d}{c}$, and for each $\rho > 0$ there exists $(\lambda, V) \in K_0$ having $\|V\|_1 = \rho$.*
(iv) *if $-\frac{d}{c} < 0$ then $\lambda_0 < 0$ and λ_1 is the only other possibly negative eigenvalue. K_0 is an unbounded set lying in the $\lambda < 0$ half of the bifurcation plane. If $0 > \lambda_1$ then $\lambda_1 > -\frac{d}{c}$ and K_1 is an unbounded set lying in the $\lambda < 0$ half of the bifurcation plane. If $0 \geq \lambda_1$ then K_1 is an unbounded set lying in the $\lambda \geq 0$ half of the bifurcation plane.*

Proof The proof of (i) is as in Theorem 5.4 while the proof of (ii) follows the same reasoning as presented in the proof of Theorem 5.4(b), (c) and (d) but with $k \geq 1$ replaced by $k \geq 2$. Finally, (iii) and (iv) are a consequence of Corollary 3.2, Corollary 4.3 and Lemma 5.3. ■

Results can be obtained for the cases of $\alpha \in (\frac{\pi}{2}, \pi)$ and q taking on negative values, but these results are weaker than those obtained in Theorems 5.4 and 5.5 and require substantial reworking of the Lemmas and Theorems on which Theorems 5.4 and 5.5 are based.

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