Canad. Math. Bull.Vol. **36** (2), 1993 pp. 197–208

A TOPOLOGY FOR THE SOLID SUBSETS OF A TOPOLOGICAL SPACE

ROBERTO LUCCHETTI, ANNA TORRE AND ROGER J.-B. WETS

ABSTRACT. A new topology for the closed subsets of a topological space X which are the closure of their interiors is defined and investigated. Some applications to convergence of regular measures are also given.

1. **Introduction.** In [LSW] a convergence notion τ_B on $c(X)$, the set of the closed subsets of a metric space (X, d), was introduced with the aim of identifying classes of (closed) sets where narrow convergence of probability measures implies their uniform convergence. This issue, which is of great importance in applications, was already considered in [BT, SW].

A generic version of this result would read: narrow convergence to a τ_B-continuous measure implies uniform convergence on every τ_B-compact subclass of sets. The aim of this paper is to define and study a topology τ_r on $c(X)$ which is compatible with τ_B-convergence on the subspace $s(X)$ of the *solid* sets of X, namely the sets that are the closure of their interiors. This subclass of $c(X)$ has a relevant meaning in economics, representing the space of (continuous preferences of) economic agents [Ch, KS]. The topology we introduce can be naturally defined on $c(X)$, but we show that it has poor separation properties there. Instead, it behaves well on $s(X)$.

There is another motivation that led us to the study of this topology. In [Ch], Chichilniski introduced the so called *order topology* on $s(X)$, which is shown to enjoy the following property: a σ*-finite* measure m is continuous at a closed set F which is an m-continuity set (*i.e.* its boundary has null m-measure), if $s(X)$ is endowed with the order topology. Here, stating the topological version of Theorem 4 in [LSW], we show that the same result (and its converse too) holds true for a *finite* measure, if the space $s(X)$ is endowed with the much weaker τ_r-topology. Also, we argue that a *new topology*, intermediate between the order and τ_r topologies, provides the same result if we are given a σ*-finite* or a *locally finite* measure.

The paper is organized as follows: section two gives the notations, assumptions, and definitions of the hyperspace topologies we shall use in the paper. The following section intends to give an idea of the τ_r-topology, by comparing it with other, better known,

---

The first and second authors were partially supported by an MPI grant.
The third author was partially supported by NSF grant DMS 8922396.
Received by the editors July 16, 1991; revised February 14, 1992 and August 14, 1992 .
AMS subject classification: Primary: 54B20.
Key words and phrases: hyperspace topologies, solid sets, narrow convergence of probability measures.

hyperspace topologies. In particular, it is shown that in $c(X)$ the τ_r-topology behaves differently from other topologies, even in the compact case, while on the subset $s(X)$ some useful comparisons can be offered. Section four nails down the basic topological properties of τ_r; for instance, it is shown that, when X is a metrizable locally compact space, then τ_r can be characterized as a *weak* topology. Furthermore, if X is separable too, $\big(s(X), \tau_r\big)$ is metrizable, while the regularity of the hyperspace fails if X is not locally compact. The final section offers some applications.

In closing this introduction, we observe that, when X is locally compact, then the study of the rough topology fits in the theory of the continuous lattices, in the following sense: $c(X)$ is an upper continuous lattice and the upper rough topology is in such a case the so called *Scott topology*, while $o(X)$, the set of the open subsets of X, is a lower continuous lattice and can be provided by the Scott topology, generated by the sets $\{G \in o(X) : K \subset G\}$, where K is a compact set. By means of the weak topology generated by the 1-1 function int: $s(X) \longrightarrow o(X)$ defined as $\mathrm{int}(F) = \mathrm{int}\, F$, then one can see the rough topology as the join of the two Scott topologies on $s(X)$ and also on the set $\{G \in o(X) : \mathrm{int}\,\mathrm{cl}\, G = G\}$ (for more about the Scott topology and continuous lattices, the reader can consult [GH]).

2. **Notations and assumptions.** Let X be a set with at least two points and let τ be a topology on X, which is always supposed to be at least Hausdorff. $c(X)$ indicates the set of the closed subsets of X, which will be called the *hyperspace of* X, when endowed with some topology [Mi]. The closure of a set A is denoted by $\mathrm{cl}\, A$, its complement by A^c, its boundary by ∂A and its interior by $\mathrm{int} A$. Given a closed set F we denote by $S_\varepsilon[F]$ the set $\{x \in X : d(x, F) \leq \varepsilon\}$. $S(x, r)$ will be the open ball centered at x with radius r. The set of the accumulation points of X will be indicated by $A(X)$. We shall consider the following families of subsets of $c(X)$:

$$G^- = \{F \in c(X) \text{ such that } F \cap G \neq \emptyset\}, \text{ and}$$
$$G^+ = \{F \in c(X) \text{ such that } F \subset G\}, \text{ for a subset } G \text{ of } X.$$

These subsets of $c(X)$ allow to define topologies on $c(X)$; for example the *upper Vietoris* topology has as a base the family of sets V^+, where V ranges over the open sets of X, and the *lower Vietoris* topology has as a subbase the family of sets V^-, where V ranges over the open sets of X. The *lower Fell* topology agrees with the lower Vietoris topology and *the upper Fell* topology has as a base the family of sets $(K^c)^+$, where K ranges over the compact sets of X. The corresponding *Vietoris* and *Fell* topologies V and F are defined as the supremum of their lower and upper parts: $V = V^+ \cdot V^-$, and $F = F^+ \cdot F^-$. Thus a base for the Vietoris topology V is given by $G^+ \cap V_1^- \cap \cdots \cap V_n^-$ and for the Fell topology F is given by $(K^c)^+ \cap V_1^- \cap \cdots \cap V_n^-$, where V_i and G range over the open sets of X and K over the compact subsets of X. The paper mainly deals with the so called *rough topology*, which is defined as the supremum of two parts: the upper part, which we shall call the *τ_r^+-topology*, agrees with the upper Fell topology, and the other topology, called the τ_r^--topology, has as a base the sets of the kind:

$$(H)^{r-} = \{F \in c(X) \text{ such that } \mathrm{cl}(F^c) \subset H^c\} = \{F \in c(X) \text{ such that } \mathrm{int}\, F \supset H\},$$

where H ranges over the compact sets of X. Thus a base for the τ_r-topology is given by $(K^c)^+ \cap (H)^{r-}$, where K and H range over the compact subsets of X. Observe that the τ_r^--topology is a lower topology, in the sense that an open neighborhood of a set F automatically contains all the (closed) supersets of F. We shall frequently use in the sequel the following facts, easy to prove: $K_1 \subset K_2$ if and only if $(K_2^c)^+ \subset (K_1^c)^+$ and $H_1 \subset H_2$ if and only if $(H_2)^{r-} \subset (H_1)^{r-}$.

In terms of converging nets, it can be shown that, if X is completely regular, a net $\{F_n\}$, $n \in T$, T a directed set, converges in the τ_r-topology to a set F provided the following conditions hold:

i) for each net $a_k, k \in S$, S a set cofinal to T, such that $a_k \in F_k$ and there is n such that, for all $k > n$, $a_k \in K$, where K is a compact set, and $a = \lim a_k$, then $a \in F$.

ii) for each net $a_k, k \in S$, S cofinal to T, such that $a_k \in (F_k)^c$ and there is n such that, for all $k > n$, $a_k \in K$, where K is a compact set, and $a = \lim a_k$, then $a \in \mathrm{cl}(F^c)$.

This follows from the known equivalence between i) and convergence of nets for the upper Fell topology [KT, Theorem 3.3.10, p. 32], then applying the result to the closure of the complements.

If (X, τ) is locally compact, condition i) of Proposition 4.2 is equivalent to $F \supset \mathrm{Ls}\, F_n$, where $\mathrm{Ls}\, F_n$ denotes the topological superior limit of a net of sets [KT, Definition 3.1.4, p. 24], because every converging net eventually belongs to a compact set. One can also prove that convergence of *nets* in the following sense:

$$(*) \qquad \{F_n\} \text{ converges to } F \text{ if } F \supset \mathrm{Ls}\, F_n \text{ and } \mathrm{cl}(F^c) \supset \mathrm{Ls}\big((F_n)^c\big)$$

is not topological, unless X is (completely regular and) locally compact. Finally the convergence $(*)$ for *sequences* characterizes converging sequences in $\big(c(X), \tau_r\big)$ [LT].

We do not want to exclude, at least in principle, the empty set from our analysis. This is motivated by the fact that we think of the applications of the τ_r-topology mainly in probability, where the impossible event cannot be ignored. However, this creates a problem with the definition of those hyperspace topologies specifically depending from the metric d on X (and not only from the topology engendered by d), as we need to define $d(x, \emptyset)$, for every $x \in X$. Some authors define $d(x, \emptyset) = \infty$ for each $x \in X$ [LL, AF], supposedly with the motivation that for a nonempty set A, $d(x, A)$ is always defined as $\inf\{d(x, a) : a \in A\}$ and that it seems natural that $\inf \emptyset = \infty$. Here we make a different choice, already proposed in [Be2]. We set $d(x, \emptyset) =: \sup\{d(x, y) : y \in X\}$. The reason for this choice is mainly to accommodate some equalities between hyperspaces that were proved with nonempty sets only.

Now we briefly describe some other topologies. The *Choquet-Wijsman* topology CW, on the hyperspace of a metric space (X, d), is defined as the supremum of a lower part, which agrees with the lower Vietoris topology, and an upper part, which has as a local base at a nonempty set A the family $\{F \in c(X) : d(x_i, A) < d(x_i, F) + \varepsilon,\ x_i \in X,\ i = 1, \ldots, k;\ \varepsilon > 0\}$. The same definition holds for the empty set, provided (X, d) is bounded. If (X, d) is unbounded, a local base at $\emptyset$ is given by the family $\{F \in c(X) : \varepsilon < d(x_i, F),\ x_i \in X,\ i = 1, \ldots, k;\ \varepsilon > 0\}$.

https://doi.org/10.4153/CMB-1993-029-5 Published online by Cambridge University Press

Equivalently, a net A_n converges to a set A in the Choquet-Wijsman topology if

$$\lim_{n\to\infty} d(x, A_n) = d(x, A), \text{ for each } x \in X.$$

A net A_n converges to the empty set in the Choquet-Wijsman topology if

$$\lim_{n\to\infty} d(x, A_n) = \sup\{d(x, y) : y \in X\}, \text{ for each } x \in X.$$

It is clear that the upper part of the Choquet-Wijsman topology agrees with the weakest topology making *lower semicontinuous* the family $\{A \longrightarrow d_x(A) =: d(x, A), x \in X\}$, and it is not difficult to see that the lower part agrees with the weakest topology making *upper semicontinuous* the family $\{A \longrightarrow d_x(A) =: d(x, A), x \in X\}$.

The *bounded Hausdorff* (or *Attouch-Wets*) metric topology AW, on the hyperspace of a metric space is defined, by means of converging sequences, in the following way: $A = \mathrm{AW} - \lim_{n\to\infty} A_n$ if:

$$\lim_{n\to\infty} \sup\{|d(x, A_n) - d(x, A)| : x \in B, B \text{ a bounded set}\} = 0.$$

In the case $A = \emptyset$, and if (X, d) is unbounded, then $\emptyset = \mathrm{AW} - \lim_{n\to\infty} A_n$ if:

$$\lim_{n\to\infty} \inf\{d(x, A_n) : x \in B, B \text{ a bounded set}\} = \infty.$$

Our analysis will mainly concern the subspace $s(X)$ of $c(X)$ of the solid sets, namely the sets that are the closure of their interior. This subspace has been considered in an economical setting, for instance in [Ch, KS], where the order topology τ_0 has been defined. A base for this topology is given by: $G^+ \cap (H)^{r-}$, where G ranges over the open sets of X and H ranges over the (closed) sets in $s(X)$.

We refer to [En, Ku] as standard texts of general topology. For more about hyperspace topologies the reader can consult [Mi, KT, BL2, BLLN, FLL, LT].

3. **Comparisons.** In order to give some flavor to the τ_r-topology, we shall start our analysis by comparing it with other hyperspace topologies. This will be also useful to derive some of its topological properties. As it was already remarked in [Lu], the τ_r-topology behaves rather differently with respect to other topologies.

EXAMPLE 3.1. Even in the compact case, the τ_r-topology need not agree with the Vietoris topology on $c(X)$. For, consider the following sequences, on the interval $[-1, 1]$: $A_{2n} = \{1\}$, $A_{2n+1} = \{-1\}$, and $F_n = [-1, -\frac{1}{n}] \cup [\frac{1}{n}, 1]$. The first one has a τ_r-limit but not a V-limit; the opposite happens with the second. More generally, observe that all the singletons which do not correspond to isolated points have as the only τ_r^--open neighborhood all of the space $c(X)$, while the element $\{X\}$ is always τ_r^--isolated in the hyperspace, when X is compact. ∎

The previous example shows that the τ_r-topology is not comparable with other hyperspace topologies as well, as the Hausdorff metric, Fell, Choquet-Wijsman, and Attouch-Wets topologies.

The situation changes on $s(X)$.

PROPOSITION 3.2. *On $s(X)$, the τ_r^--topology is finer than the V^- topology. If $A(X)$ is nonempty, then the τ_r^--topology is strictly finer than the V^--topology.*

PROOF. It is finer: given $F \in V_1^- \cap \cdots \cap V_n^-$, observe that there are $x_1, \ldots, x_n$ such that $x_i \in V_i \cap \operatorname{int} F$. Then $F \in (\{x_1\})^{r-} \cap \cdots \cap (\{x_n\})^{r-} \subset V_1^- \cap \cdots \cap V_n^-$. Now, take $x \in A(X)$. We shall show that there is no basic open set of the form $V_1^- \cap \cdots \cap V_n^-$ such that $V_1^- \cap \cdots \cap V_n^- \subset (\{x\})^{r-}$. As $x \in A(X)$, there are points $y_1, \ldots, y_n$ such that $y_i \neq x$ and $y_i \in V_i$ for each i. Take an open set G containing $y_1, \ldots, y_n$ and such that its closure does not contain x. Then $\operatorname{cl} G \in V_1^- \cap \cdots \cap V_n^-$, but $\operatorname{cl} G \notin (\{x\})^{r-}$. ∎

From Proposition 3.2 we easily get the following results concerning the hyperspace $s(X)$: the τ_r-topology is always finer than the Fell topology and it is finer than the Vietoris topology when the space X is compact, as the complement of each open set is a compact set (strictly finer: remember that $\{X\}$ is isolated in $\big(s(X), \tau_r\big)$). When (X, d) is a metric space the τ_r-topology is finer than the Choquet-Wijsman topology if the upper Fell topology is finer than the upper Choquet-Wijsman topology, *i.e.* when each ball in X is either compact or all the space. All the previous relations are strict when $A(X)$ is nonempty.

The following proposition further highlights connections between the τ_r and Choquet-Wijsman topologies. Denote by CW^c the following topology on $s(X)$, described by open neighborhoods of a set F: $\mathcal{B}$ is a CW^c-open neighborhood of F if the set $\mathcal{B}^c =: \{A : \operatorname{cl} A^c \in \mathcal{B}\}$ is an open neighborhood of $\operatorname{cl}(F^c)$ in the Choquet-Wijsman topology.

The following theorem shows that $\mathrm{CW} \cdot \mathrm{CW}^c$ is always finer than τ_r and gives conditions on the metric space (X, d) to ensure that the two topologies agree on $s(X)$.

THEOREM 3.3. *Let (X, d) such that every closed ball is either compact or all the space. Then $\big(s(X), \tau_r\big) = \big(s(X), \mathrm{CW} \cdot \mathrm{CW}^c\big)$.*

PROOF. We study separately later the case of the empty set, because the results we refer to in the first part of the proof apply to the subspace of the (closed) *nonempty* subsets of X. At first, observe that $\tau_r \leq \mathrm{CW} \cdot \mathrm{CW}^c$ (without assumptions on X). For, $\tau_r^+ \leq \mathrm{CW}^+$ [FLL, Proposition 2.3] and, by symmetry, $\tau_r^- \leq (\mathrm{CW}^c)^+$, hence $\tau_r = \tau_r^+ \cdot \tau_r^- \leq \mathrm{CW}^+$. $(\mathrm{CW}^c)^+ \leq \mathrm{CW} \cdot \mathrm{CW}^c$. From the assumptions on the space X, $\tau_r^+ \geq \mathrm{CW}^+$ [Be1, Theorem 2.3], hence $\tau_r \geq \mathrm{CW}$, from Proposition 3.2. Now, using the homeomorphic character of the function c: $\big(s(X), \tau_r\big) \longrightarrow \big(s(X), \tau_r\big)$ defined by $c(F) = \operatorname{cl}(F^c)$, we conclude that $\tau_r \geq \mathrm{CW} \cdot \mathrm{CW}^c$. Next, we prove the theorem for the open neighborhoods of the empty set. If (X, d) is compact, then $\emptyset$ is isolated in both topologies. This is evident for τ_r^+. Let us show it for CW^+. Set $\operatorname{diam} X =: 5a > 0$. There are $x_1, \ldots, x_n$ such that $\bigcup_{i=1}^n S(x_i, a) = X$. We want to show that the CW^+-open set $N =: \{F \in s(X) : d(x_i, F) > 2a$ for each $i\} = \{\emptyset\}$. $\emptyset \in N$, because $d(x_i, \emptyset) > 2a$ for each i, due to the triangle inequality. Suppose $F \in N$ and F nonempty. Take $y \in F$. Then $d(x_i, y) > 2a$, *i.e.* $y \notin \bigcup_{i=1}^n S(x_i, a)$, a contradiction. We provide the rest of the proof in the case X bounded (but not compact), the case X unbounded being similar (and more standard). To begin with, we show that, given a basic τ_r^+-open neighborhood of $\emptyset$ of the form $(K^c)^+$, there is a basic CW^+-open neighborhood N of $\emptyset$, such that $N \subset (K^c)^+$ (with the only assumption X being metric). Take any compact set K. There is $x \in X$ such that $d(x, K) = 3a > 0$. Then there are $k_1, \ldots, k_n$

such that $K \subset \bigcup_{i=1}^{n} S(k_i, a)$. Consider the following basic CW$^+$-open neighborhood N of $\emptyset$: $N = \{F \in s(X) : d(k_i, F) > 2a\}$. Suppose $F \in N$. Then $F \cap S(k_i; a) = \emptyset$ for each i, and hence $F \in (K^c)^+$. Finally, suppose (X, d) is bounded, noncompact and such that all the closed proper balls are compact. Take any basic CW$^+$-open neighborhood N of $\emptyset$. N is of the form $N =: \{F \in s(X) : d(x_i, F) > \alpha_i - 2\varepsilon\}$, where we set $\alpha_i = \sup\{d(x_i, y) : y \in X\}$. Since $\operatorname{cl} S(x_i, \alpha_i - \varepsilon)$ is not all of the space, it is compact. Set $K = \bigcup_{i=1}^{n} S(x_i, \alpha_i - \varepsilon)$, a compact set. It is not difficult to show that $(K^c)^+ \subset N$ and the proof is complete. ∎

Observe that the function c, defined in the proof of the previous proposition, is not usually continuous on all of $c(X)$: for the sequence $[-\frac{1}{n}, \frac{1}{n}] \subset R$ τ_r-converges in $c(X)$, while the sequence of the closures of the complements does not. ∎

Now, let (X, d) be a normed linear space: we want to compare the AW and τ_r-topologies, on the subspace $\operatorname{conv}(X)$ of $c(X)$ of the convex sets of X.

PROPOSITION 3.4. *On* $\operatorname{conv}(X)$, $\mathrm{AW} \geq \tau_r$.

PROOF. It is clear that $\mathrm{AW}^+ \geq \tau_r^+$, for instance by comparing both with CW$^+$. It remains to show that $\mathrm{AW}^- \geq \tau_r^-$. Take $A \in (H)^{r-}$. Then there is $a > 0$ such that for each $h \in H$, $S(h, 2a) \subset \operatorname{int} A$. Take $h_1, \ldots, h_k$ such that $\bigcup_{l=1}^{n} S(h_i, a) \supset H$. As it is shown in [BL1], $\big(S(h_i, a)\big)^{r-}$ is an AW$^-$ neighborhood $\mathcal{B}_i$ of A. Then $A \in \mathcal{B} =: \bigcap_{l=1}^{k} \mathcal{B}_i \subset (H)^{r-}$. ∎

REMARK 3.5. It can be useful to observe that the former result *does not apply*, in general, to the case X a closed convex subset of a linear space. This happens because, usually, the rough topology *does not pass* to subspaces. This is not surprising, because of the complement operation appearing in the definition of τ_r. Thus, for instance, if I is a compact interval of the real line, then it is isolated in $\big(s(I), \tau_r\big)$, but it is not in the topology $s(I)$ inherits *as a subspace* of $\big(s(R), \tau_r\big)$. ∎

Finally, we compare the τ_r-topology with the order topology τ_0.

PROPOSITION 3.6. *Let X be a regular space. Then $\big(s(X), \tau_0\big)$ is finer than $\big(s(X), \tau_r\big)$. They coincide if and only if X is compact.*

PROOF. The comparison of the two upper parts is straightforward. As far as the lower parts are concerned, take $A \in (H)^{r-}$, where H is a closed set. Then there is an open set N in X such that $H \subset N$ and $\operatorname{cl} N \cap (\operatorname{int} A)^c = \emptyset$. Then $A \in (\operatorname{cl} N)^{r-} \subset (H)^{r-}$. If X is not compact, there is a closed proper non compact set F_1. Take $x \notin F_1$ and N open such that $\operatorname{cl} N \cap F_1 = \emptyset$. Then $\operatorname{cl} N \in \big((F_1)^c\big)^+$. Suppose there is K such that $\operatorname{cl} N \in (K^c)^+ \subset \big((F_1)^c\big)^+$. Then $F_1 \subset K$, which easily gives a contradiction. ∎

4. **Topological results.** We shall briefly analyze here some of the topological properties of the τ_r-topology on $c(X)$; then we shall focus on the subset $s(X)$. The following properties of $\big(c(X), \tau_r^+\big)$ are easy to prove: $\big(c(X), \tau_r^+\big)$ is T_0, for if a point x belongs to F_1 and not to F_2, then F_2 belongs to $(\{x\}^c)^+$, while F_1 does not. $\big(c(X), \tau_r^+\big)$ is not T_1, like all the upper (and lower) topologies. If there are two distinct points x and y which are not isolated, then $\big(c(X), \tau_r^-\big)$ is not T_0, as it is not possible to separate any pair of sets with

empty interiors. Finally, if there are two distinct points x and y which are not isolated, $\big(c(X), \tau_r\big)$ is T_0 but not T_1.

We start with the analysis of the embedding of X in the hyperspace, to show *admissibility* of the rough topology. Recall that Michael [Mi] calls *admissible* those hyperspace topologies such that $i\colon (X, \tau) \longrightarrow \big(c(X), \tau_r\big)$, defined as $i(x) = \{x\}$ is a homeomorphism (onto $i(X)$).

PROPOSITION 4.1. *$i\big(A(X)\big)$ is compact in $\big(c(X), \tau_r\big)$.*

PROOF. It is easy to see that $(K^c)^+ \cap i(X) = \big\{\{x\} : x \in K^c\big\}$. As far as the lower part is concerned, observe that $(H)^{r-} \cap i(X) = i(X)$ if $H = \emptyset$, $(H)^{r-} \cap i(X) = \emptyset$ if H contains more than one point and moreover, if $H = \{x\}$, then $(H)^{r-} \cap i(X) = \{x\}$ if and only if x is an isolated point: otherwise it is again empty. Thus, given a basic open covering $\mathcal{V}$ of $i\big(A(X)\big)$, the only case to consider is when $\mathcal{V}$ is of the form:

$$\mathcal{V} = \big\{(K_j^c)^+ \cap i\big(A(X)\big), j \in J\big\}.$$

Then $\bigcup_{j\in J}(K_j^c)^+ \cap i\big(A(X)\big) = i\big(A(X)\big)$ and hence $\bigcap_{j\in J} K_j \cap A(X) = \emptyset$. Then $\bigcap_{j\in J}(K_1 \cap K_j) \cap A(X) = \emptyset$. Then there is a finite family $j_1, \ldots, j_n$ such that $\bigcap_{i=1}^{n}\big(K_1 \cap K_{j_i} \cap A(X)\big) = \emptyset$, due to the compactness of K_1 and closedness of $A(X)$. As a result,

$$i\big(A(X)\big) \subset (K_1^c)^+ \cup (K_{j_1}^c)^+ \cup \cdots \cup (K_{j_n}^c)^+. \qquad \blacksquare$$

Proposition 4.1 is the first step to see under which conditions on the space X the rough topology is admissible. In the next theorem, we shall see that the continuity of i holds with no assumption on the topology on X; instead, contunuity of its inverse requires strong conditions.

THEOREM 4.2. *The τ_r-topology is admissible in the sense of Michael if and only if X is either compact or discrete.*

PROOF. The continuity of i easily follows from the characterization of the sets $(K^c)^+ \cap i(X)$ and $(H)^{r-} \cap i(X)$ given in the proof of Proposition 4.1. Then observe that i maps isolated points in X into isolated points in $\big(i(X), \tau_r\big)$, because, if x is isolated, then $i(x) = \{x\} = (\{x\})^{r-} \cap i(X)$. Thus, admissibility follows in both cases. As far as the only if part is concerned, suppose X is not discrete, *i.e.* $A(X)$ is nonempty. Take $x \in A(X)$. Take any open set A containing x and such that $A^c \neq \emptyset$. From the continuity of i^{-1} it follows that some open basic neighborhood $\mathcal{V}$ of $\{x\}$ must exist such that $i^{-1}(\mathcal{V}) \subset A$. As $x \in A(X)$, $\mathcal{V}$ must be of the form $(K^c)^+$, as already observed. Thus, there exists a compact set K such that $K^c \subset A$, *i.e.* $A^c \subset K$. Summarizing, we have seen that each closed set F which does not contain x is actually compact. It follows that X must be compact. $\blacksquare$

We now switch our attention to $\big(s(X), \tau_r\big)$, where more regularity properties hold, as we shall see. We shall assume that X is at least a regular space. We first remark that $s(X)$ is a *dense*, (*usually not closed*) subset of $c(X)$. Density follows from the following fact: take a basic open set $(K^c)^+ \cap (H)^{r-}$. Take $F \in (K^c)^+ \cap (H)^{r-}$ and consider an open set A such that $A \supset F$ and $\operatorname{cl} A \cap K = \emptyset$. Then $\operatorname{cl} A \in s(X) \cap (K^c)^+ \cap (H)^{r-}$.

PROPOSITION 4.3. *$\big(s(X), \tau_r\big)$ is T_2.*

PROOF. The proof relies on the following fact, which is straightforward to prove: if A and F are two distinct sets in $s(X)$, then there is a point x belonging to $\operatorname{int}(A \setminus F) \cup \operatorname{int}(F \setminus A)$. Suppose $x \in \operatorname{int}(A \setminus F)$. Then observe that $F \in (\{x\}^c)^+$ and $A \in (\{x\})^{r-}$. ∎

PROPOSITION 4.4. *Let (X, τ) be metrizable and suppose $\big(s(X), \tau_r\big)$ is first countable. Then (X, τ) is a σ-compact space, hence it is separable.*

PROOF. We shall show that $\{X\}$ does not have a countable fundamental system of open neighborhoods, unless (X, τ) is σ-compact. A fundamental system of open neighborhoods of X must be of the form $\{(H_i)^{r-} : i \in N\}$, where H_i are compact, and hence separable subsets of X. Suppose X is not σ-compact. Then there is $p \in X$ such that for no $i, p \in H_i$. Then $X \in (\{p\})^{r-}$, but for no i, $(\{p\})^{r-} \supset (H_i)^{r-}$. ∎

Observe that the result concerning separability holds true for all topologies having the lower Vietoris topology as their lower parts.

The following theorem shows that we must impose some conditions on the space X in order to get more separability properties on the hyperspace.

THEOREM 4.5. *Let (X, d) be a metric space. If $\big(s(X), \tau_r\big)$ is regular, then X is locally compact.*

PROOF. Suppose x is a point without compact neighborhoods. Let $\mathcal{A} = \{A \in s(X) : x \in A\}$ and $B \in s(X)$ such that $x \notin B$.

We claim that it is not possible to separate B from $\mathcal{A}$. Let $(K^c)^+ \cap (H)^{r-}$ be a basic open neighborhood of B. Observe that $\mathcal{A} \cap (H)^{r-} \neq \emptyset$. We claim that there is $A \in s(X)$ such that $A \cap K \subset \{x\}$. This is obvious if $x \notin K$. If $x \in K$, consider a sequence $x_n \in S(x, \frac{1}{n}) \cap K^c$, such that $x_n \neq x_m$ for $n \neq m$. Take a_n such that $d(x_n, K) \geq 2a_n$ and $\operatorname{cl} S(x_n, a_n) \cap \operatorname{cl} S(x_m, a_m) = \emptyset$, for $n \neq m$. If for all large n, say $n > n_0$, $\operatorname{cl} S(x_n, a_n) = \{x_n\}$, then $A = \bigcup_{n=n_0}^{\infty} \operatorname{cl} S(x_n, a_n) \cup \{x\}$ is such that $A \in s(X)$ and $x \notin \operatorname{int}(A)$, because A is compact. Otherwise suppose that for a subsequence n_k, there is $y_{n_k} \neq x_{n_k}$ such that $y_{n_k} \in \operatorname{cl} S(x_{n_k}, a_{n_k})$. Set $2b_k = d(x_{n_k}, y_{n_k})$ and call $A' = \bigcup_{k=1}^{\infty} \operatorname{cl} S(x_{n_k}, b_k) \cup \{x\}$. Then $A' \in s(X)$ and $x \notin \operatorname{int}(A')$. Consider the set $C = A \cup B$ in the first case and the set $C = A' \cup B$ in the second case. C cannot have an open neighborhood disjoint from $(K^c)^+ \cap (H)^{r-}$ because, given any basic open neighborhood N of C, for sufficiently small ε, $C \cap S(x, \varepsilon)^c \in N \cap (K^c)^+ \cap (H)^{r-}$. ∎

On the other hand local compactness allows us to get much stronger results.

THEOREM 4.6. *Let (X, d') be a locally compact metrizable space. Then $\big(s(X), \tau_r\big)$ is completely regular.*

PROOF. Local compactness allows claiming the existence of a distance d, equivalent to d', such that all the proper closed balls in (X, d) are compact [Be2, Theorem 2]. Obviously, the rough topology in the hyperspace is not affected by switching from d' to d in the space X. We shall prove the claim by showing that the rough topology is a weak topology. From Theorem 3.3, we know that $\big(s(X), \tau_r\big)$ agrees with $\Big(s\big((X, d), \mathrm{CW} \cdot \mathrm{CW}^c\big)\Big)$.

The claim now easily follows from the fact that the Choquet-Wijsman topology is the weakest one, on $s(X)$, making continuous the family of functionals, defined on $s(X)$, $\{A \longrightarrow d_x(A), x \in X\}$. Then the τ_r-topology is the weakest topology on $s(X)$ making continuous the following family of functionals:

$$A \longrightarrow d_x(A) \text{ and } A \longrightarrow d_x(A^c), \text{ for each } x \in X. \qquad \blacksquare$$

The importance of describing hyperspace topologies as weak topologies, as is done for the τ_r-topology in the proof of Theorem 4.6, is highlighted in [BL2].

THEOREM 4.7. *Let X be a metrizable space. Then the following are equivalent:*

i) X is separable and locally compact.

ii) $\big(s(X), \tau_r\big)$ is metrizable.

iii) $\big(s(X), \tau_r\big)$ is second countable.

PROOF. At first, let us show that condition i) implies second countability. Without loss of generality, we can suppose that X has proper closed balls which are compact. Consider the following family of compact sets: $\big\{A_{ir} =: \operatorname{cl} S(x_i, r)$ such that $\{x_i : i \in N\}$ is a dense family in X, r is a rational number, and the ball is proper$\big\}$. Now consider the family of open sets

$$\mathcal{A} = \Big\{\big((A_{ir})^c\big)^+ \cap (A_{js})^{r-}; i, j \in N; r, s \in Q\Big\}.$$

It is not difficult to see that $\mathcal{A}$ is a (countable) subbase for the τ_r-topology. Thus i) implies iii). Moreover i) and Theorem 4.6 allow us to conclude that $\big(s(X), \tau_r\big)$ is metrizable, *i.e.* ii). From Proposition 4.4 and Theorem 4.5 we see that metrizability of $\big(s(X), \tau_r\big)$ implies i). To conclude the proof, let us show that second countability of $\big(s(X), \tau_r\big)$ implies that X is separable and locally compact. Again, separability follows from Proposition 4.4. Now, suppose there is a point $x \in X$ without compact neighborhoods. If there is a countable base for $\big(s(X), \tau_r\big)$, then from the Lindelöf Theorem there must be a countable base of the form $\{(K_n^c)^+ \cap (H_n)^{r-}, n \in N\}$, for compact sets K_n and H_n, and, without loss of generality, $K_n \cap H_n = \emptyset$. As $\operatorname{cl} S(x, \frac{1}{n})$ is not compact, there is $x_n \in \operatorname{cl} S(x, \frac{1}{n}) \cap (H_n)^c$. Call $H = \{x_n, x, n \in N\}$. Consider, for each n, $2a_n =: \min\{d(K_n, H_n), d(x_n, H_n)\} > 0$, where, for given sets A and B, $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Consider $C_n = S_{a_n}[H_n]$. Then $C_n \in (K_n^c)^+ \cap (H_n)^{r-}$, but $C_n \notin H^{r-}$ and the contradiction shows that X must be locally compact.

5. Applications.

THEOREM 5.1. *Let (X, τ) be a Polish space, $\mathcal{B}(X)$ the family of the borel sets of X and m: $\big(s(X), \tau_r\big) \longrightarrow [0, \infty)$ a positive Borel measure. Then the following are equivalent:*

i) m is continuous at F.

ii) m is continuous at $\operatorname{cl} F^c$.

iii) F is a m-continuity set, i.e. $m(\partial F) = 0$.

PROOF. Equivalence between ii) and iii) is an easy corollary of the equivalence between i) and iii). Hence we prove this equivalence. Suppose there is F such that $m(\partial F) =$

$a > 0$. Take a basic open set N of the form: $N = \{A \in s(X) : A \cap K = \emptyset\} \cap \{A \in s(X) : \operatorname{int} A \supset H\} = (K^c)^+ \cap (H)^{r-}$, where F misses the compact set K and $\operatorname{int} F$ contains the compact set H. Pick an open set G such that $G \supset H$ and $\operatorname{cl} G \cap \partial F = \emptyset$. Consider the set $A = \operatorname{cl}(\operatorname{int} F \cap G)$. Then it is not difficult to see that $A \in (K^c)^+ \cap (H)^{r-}$ and that $m(A) \leq m(F) - a$, showing that m is not continuous at F. Conversely, let us observe that m is an inner regular measure, because every Polish space is a Radon space [Sc]. Then, for a given Borel set A, we have $m(A) = \sup\{m(K), K \text{ is compact and } A \supset K\}$. Now, take F such that $m(\partial F) = 0$, and let $a > 0$. There are compact sets H and K such that $\operatorname{int} F \supset H$ and $F^c \supset K$ and such that $m(H) > m(\operatorname{int} F) - a$ and $m(K) > m(F^c) - a$. It is easy to show that, for $A \in (K^c)^+ \cap (H)^{r-}$, we have $|m(A) - m(F)| < a$. ■

REMARK 5.2. Observe that the proof of Theorem 5.1 shows the equivalence between i) and iii) for sets in $c(X)$ and not only in $s(X)$. Also, it is easy to see that both imply ii) for all the sets in $c(X)$. Instead, the relation ii) implies i) usually works only for solid sets.

REMARK 5.3. Theorem 5.1 is true for every normal (for the only if part) and Radon (for the if part) space X: the proof is the same. More interestingly, one can wonder whether the claim holds for σ-finite or locally finite measures too. Paralleling the proof of the theorem, one can see that m is continuous at a point F where $m(F) = \infty$, if and only if $m(\operatorname{int} F) = \infty$, without further conditions on $m(\partial F)$. If $m(F) < \infty$ and $m(X) = \infty$, then m is *never* continuous at F, because every τ_r-neighborhood of F contains sets of arbitrarily big measure. One can get continuity in such a case by considering the topology which is the supremum of the lower τ_r-topology and the upper Vietoris topology, and a measure which is inner and outer regular at the same time. Observe that this last one is finer than the rough topology and coarser than the order topology.

THEOREM 5.4. *Suppose we have a probability measure P on the borel sets of a locally compact space (X, d), and a family $\mathcal{A}$ such that $\mathcal{A}$ is a compact subset of $\big(s(X), \tau_r\big)$. Suppose moreover P: $\big(c(X), \tau_r\big) \longrightarrow [0, 1]$ is continuous at every $A \in \mathcal{A}$. Then*

$$\lim_{\varepsilon \to 0} \sup\{P(S_\varepsilon[\partial A]) : A \in \mathcal{A}\} = 0.$$

PROOF. By virtue of Theorem 5.1 and Remark 5.2, we can suppose that $\mathcal{A}$ is a compact subset of $\big(c(X), \tau_r\big)$. As a first step, of independent interest, let us show that the function ∂: $\big(c(X), \tau_r\big) \longrightarrow \big(c(X), \tau_r\big)$ such that $\partial(A) = \partial A$, is continuous. A neighborhood of ∂A is of the kind $(K^c)^+$, and $K = K_1 \cup K_2$, where K_1 and K_2 are compact sets such that $K_2 \subset \operatorname{int} A$ and $A \cap K_1 = \emptyset$. Then $\partial\big((K_1^c)^+ \cap (K_2)^{r-}\big) \subset (K^c)^+$, as it is easy to see.

Now, let us consider the map t: $[0, 1] \times \big(c(X), \tau_r\big) \longrightarrow \big(c(X), \tau_r\big)$, defined by $t(\varepsilon, A) = S_\varepsilon[\partial A]$, (where $S_0[\partial A] = \partial A$). We want to show that is is continuous at $(0, A)$, for $A \in \mathcal{A}$. Take a compact set K such that $\partial A \cap K = \emptyset$. Then, for some $s > 0$, $S_s[K]$ is compact and $\partial A \in (S_s[K])^{c+}$. From the first step, there is a neighborhood $\mathcal{V}$ of A such that $\partial B \in (S_s[K])^{c+}$, for each $B \in \mathcal{V}$. It follows that $S_\varepsilon[\partial B] \cap K = \emptyset$, for each $\varepsilon < s$, showing the continuity of t at $(0, A)$. Finally, continuity of $P \circ t$, compactness of $\mathcal{A}$ and the Berge maximum theorem allow us to conclude the proof.

COROLLARY 5.5. *Let (X, d) be locally compact, let $\mathcal{A}$ and P be as in Theorem 5.4, and P_ν a sequence of probability measures converging narrowly to P. Then*

$$\lim_{\nu\to\infty} \sup\{|P_\nu(A) - P(A)| : A \in \mathcal{A}\} = 0,$$

i.e. the convergence of P_ν to P is uniform on $\mathcal{A}$.

PROOF. This is a consequence of Theorem 5.3 and of Theorem 1 of [BT].

To conclude, let us observe that the same conclusion of Corollary 5.5 also holds when $\mathcal{A}^c$ has the same properties required for $\mathcal{A}$ in Theorem 5.4, where $\mathcal{A}^c := \{\mathrm{cl}(A^c)$ such that $A \in \mathcal{A}\}$. Moreover, Corollary 5.5 holds without requiring local compactness of (X, d) if we ask for *sequential compactness* (rather than compactness) of $\mathcal{A}$ (see also [LSW]).

ACKNOWLEDGEMENT. We are grateful to C. Costantini and to the referee for a careful reading of the paper that led to many improvements to the first version of the paper. Moreover, the referee pointed out the connections of our topology with the Scott topology mentioned at the end of the introduction, and the role of the isolated points in Propositions 3.2 and 4.1, helping us to avoid a possible mistake.

REFERENCES

[AF] J-P. Aubin and H. Frankowska, *Set valued analysis*, Birkauser, Boston, 1990.

[ALW] H. Attouch, R. Lucchetti and R. J.-B. Wets, *The topology of the ρ-Hausdorff distance*, Annali Mat. Pura e Appl., Serie IV, **160**(1992), 303–320.

[Be1] G. Beer, *Metric spaces with nice closed balls and distance functions for closed sets*, Bull Australian Math. Soc., (1987), 81–96.

[Be2] ———, *An embedding theorem for the Fell topology*, Michigan Math. J. **35**(1988), 3–9.

[BLLN] G. Beer, A. Lechicki, S. Levi and S. Naimpally, *Distance functionals the suprema of hyperspace topologies*, Annali Mat. Pura e Appl., Serie IV, **162**(1992), 367–381.

[BL1] G. Beer and R. Lucchetti, *Convex optimization and the epi-distance topology*, Trans. Amer. Math. Soc. **327**(1991), 795–814.

[BL2] ———, *Weak topologies for the closed subsets of a metrizable space*, Trans. Amer. Math. Soc., to appear.

[BT] P. Billingsley and F. Topsoe, *Uniformity in weak convergence*, Z. für Wahrscheinlichkeitstheorie und verwandte Gebiete **7**(1967), 1–16.

[Ch] G. Chichilniski, *Spaces of economic agents*, J. of Economic Theory **15**(1977), 160–173.

[En] R. Engelking, *General Topology*, Polish Scientific Publishers, Warsaw, 1977.

[FLL] S. Francaviglia, A. Lechicki and S. Levi, *Quasi-uniformization of hyperspaces and convergence of nets of semicontinuous multifunctions*, J. Math. Anal. Appl. **112**(1985), 347–370.

[GH] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *A compendium of continuous lattices*, Springer Verlag, New York, 1980.

[KS] A. Khan and Y. Sun, *On complete regularity of spaces of economic agents endowed with the order topology*, Archiv der Mathematic **54**(1990), 389–396.

[KT] E. Klein and A. Thompson, *Theory of correspondences*, Wiley, New York, 1984.

[Ku] K. Kuratowski, *Topology*, Academic Press, New York, 1966.

[Lu] R. Lucchetti, Ph.D. Dissertation, University of California at Davis, 1989.

[LSW] R. Lucchetti, G. Salinetti and R. J-B. Wets, *Uniform convergence of probability measures: topological criteria*, to appear.

[LT] R. Lucchetti and A. Torre, *Hyperspace topologies*, in preparation.

[Mi] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71**(1951), 152–182.

[Sc] L. Schwartz, *Radon spaces*, Oxford University Press, 1977.

[SW] G. Salinetti and R. J.-B. Wets, *A Glivenko-Cantelli type theorem: an application of the convergence theory of stochastic suprema*, Annals of Oper. Res., (1991), to appear.

Department of Mathematics
via Saldini 50
20133 Milano
Italy

Department of Mathematics
Strada Nuova 65
Pavia
Italy

Department of Mathematics
University of California
Davis, California 95616
U.S.A.