ON $\mathcal{M}$-HARMONIC BLOCH FUNCTIONS AND THEIR CARLESON MEASURES†

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Abstract. On the setting of the unit ball of the complex n-space, some characterizations of $\mathcal{M}$-harmonic Bloch functions are obtained. As an application, Carleson measures are characterized by means of Berezin type integrals of $\mathcal{M}$-harmonic Bloch functions. As one may expect, these results carry over to $\mathcal{M}$-harmonic little Bloch functions and vanishing Carleson measures.

1. Introduction. Let B be the unit ball of the complex n-space C^n with boundary S. For $f \in C^1(B)$, let us define

$$Qf(z) = \sup_{\zeta \in S} \frac{|< \nabla f(z), \bar{\zeta} > + \overline{< \nabla \bar{f}(z), \bar{\zeta} >}|}{\beta(z, \zeta)} \quad (z \in B),$$

where β is the Bergman metric on B and ∇f is the complex gradient of f. Here, the notation $< z, w >$ denotes the usual Hermitian inner product for points $z, w \in C^n$. It is known [4] that Q is invariant under all automorphisms of B in the sense that $Q(f \circ \varphi) = Qf \circ \varphi$ for all $\varphi \in \mathcal{A}$, the group of all automorphisms (i.e. biholomorphic self-maps) of B.

A function $u \in C^2(B)$ is called $\mathcal{M}$-harmonic on B if it is annihilated on B by the invariant Laplacian $\tilde{\Delta}$. See Section 2 for relevant definitions. The *$\mathcal{M}$-harmonic Bloch space $M\mathcal{B}$* is the space of all $\mathcal{M}$-harmonic functions f on B for which

$$\| f \| = \sup_{z \in B} Q f(z) < \infty$$

and the *$\mathcal{M}$-harmonic little Bloch space $M\mathcal{B}_0$* is the subspace of $M\mathcal{B}$, consisting of functions f for which the additional boundary vanishing condition

$$\lim_{|z| \to 1} Q f(z) = 0$$

holds. By the invariance of Q under $\mathcal{A}$ we see that $\| f \circ \varphi \| = \| f \|$, for all $\varphi \in \mathcal{A}$.

If f is holomorphic on B, it is known [10] that f is a Bloch function if and only if $(1 - |z|^2)| \nabla f(z)| = O(1)$ and f is a little Bloch function if and only if $(1 - |z|^2)| \nabla f(z)| = o(1)$. Many other conditions characterizing holomorphic (little) Bloch functions are well known. See, for example, [2], [3], [5], [9], [10], [11] and references therein. In the $\mathcal{M}$-harmonic case, Hahn and Youssfi [4] first studied and characterized $\mathcal{M}$-harmonic Bloch functions in terms of the Berezin transform, invariant Laplacian and BMO type integrals. Recently, Jevitć and Pavlović [6] have shown that many characterizations of holomorphic (little) Bloch functions also characterize $\mathcal{M}$-harmonic ones by giving characterizations in terms of various derivatives.

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In the present paper, we add some other characterizations of $M\mathcal{B}$ and $M\mathcal{B}_0$. Our results imply that recent characterizations of Xiao and Zhong **[12]**, **[13]** for holomorphic (little) Bloch functions (on the disc) continue to hold for $\mathcal{M}$-harmonic ones. To state our result, let V denote the normalized Lebesgue volume measure on B, φ_a be the standard automorphism of B such that $\varphi_a(0) = a$, and write $d(z, w)$ for the Bergman distance between two points $z, w \in B$. For details, see Section 2.

THEOREM A. *Let* $1 \le p < \infty$. *Then, for a function* f $\mathcal{M}$*-harmonic on* B, *the following statements are equivalent.*

(a) $f \in M\mathcal{B}$.

(b) $\sup_{\substack{z,w\in B\\ z\neq w}} \dfrac{|f(z) - f(w)|}{d(z, w)} < \infty$.

(c) $\sup_{a\in B} \int_B |f \circ \varphi_a - f(a)|^p dV < \infty$.

(d) $\sup_{a\in B} \int_B (Qf(z))^p \left(\dfrac{1 - |a|^2}{|1- < z, a > |^2} \right)^{n+1} dV(z) < \infty$.

(e) *There is a constant* $t > 0$ *such that*

$$\sup_{a\in B} \int_B \exp(t|f \circ \varphi_a - f(a)|) dV < \infty.$$

Note that the condition (d) of Theorem A can be rephrased as "the Berezin transform of the measure $(Qf)^p dV$ is bounded". As is well known (see, for example, [**14**, Theorem A]), the Berezin transform of a positive Borel measure μ on B is bounded if and only if μ is a Carleson measure. To be more precise, let $E_r(a) = \varphi_a(rB)$ denote the pseudohyperbolic ball with center $a \in B$ and radius $r \in (0, 1)$. Then, μ is called a *Carleson measure* if

$$\sup_{a\in B} \frac{\mu(E_r(a))}{V(E_r(a))} < \infty$$

for some r. As an application of Theorem A, we prove the following theorem which characterizes Carleson measures by means of their action on Berezin type integrals of $\mathcal{M}$-harmonic Bloch functions.

THEOREM B. *Let* $0 < p < \infty$. *Then, a positive Borel measure* μ *on* B *is a Carleson measure if and only if there is a constant* C *such that*

$$\sup_{a\in B} \int_B |f(z) - f(a)|^p \left(\frac{1 - |a|^2}{|1- < z, a > |^2} \right)^{n+1} d\mu(z) \le C\|f\|^p,$$

for all $f \in M\mathcal{B}$.

The equivalences of Theorem A carry over to $\mathcal{M}$-harmonic little Bloch functions.

THEOREM C. *Let* $1 \le p < \infty$ *and* $0 < r < 1$. *Then, for a function* f *that is* $\mathcal{M}$-*harmonic on* B, *the following statements are equivalent.*

(a) $f \in M\mathcal{B}_0$.

(b) $\lim_{|a|\to 1} \sup_{\substack{z\in E_r(a)\\ z\neq a}} \dfrac{|f(z)-f(a)|}{d(z,a)} = 0.$

(c) $\lim_{|a|\to 1} \int_B |f\circ\varphi_a - f(a)|^p dV = 0.$

(d) $\lim_{|a|\to 1} \int_B (Qf(z))^p \left(\dfrac{1-|a|^2}{|1-<z,a>|^2}\right)^{n+1} dV(z) = 0.$

(e) *There is a constant* $t > 0$ *such that*

$$\lim_{|a|\to 1} \int_B \exp(t|f\circ\varphi_a - f(a)|) dV = 1.$$

Also, the equivalence of Theorem B carries over to vanishing Carleson measures μ on B that satisfy

$$\lim_{|a|\to 1} \frac{\mu(E_r(a))}{V(E_r(a))} = 0,$$

for some r.

THEOREM D. *Let* $0 < p < \infty$. *Then, a positive Borel measure* μ *on* B *is a vanishing Carleson measure if and only if*

$$\lim_{|a|\to 1} \sup_{\substack{f\in MB\\ \|f\|=1}} \int_B |f(z)-f(a)|^p \left(\frac{1-|a|^2}{|1-<z,a>|^2}\right)^{n+1} d\mu(z) = 0.$$

In Section 2, we collect some notations and basic facts needed in the proofs. In Section 3, we prove Theorems A and C. In fact, Theorem A is restated and proved in the form of "quantity equivalence" with weights $(1-|z|^2)^\alpha$. Also, the corresponding weighted version of Theorem C is proved. In Section 4, we first note the Carleson measure characterization of $\mathcal{M}$-harmonic (little) Bloch functions as a consequence of results obtained in the previous section. Then, as an application of results obtained in Section 3, we prove the weighted version of Theorem B in the form of "quantity equivalence". In the course of the proof, we notice that actions of Carleson measures on holomorphic or $\mathcal{M}$-harmonic Bloch functions make no difference in a certain sense (see Theorem 7). Also, we have the corresponding weighted version of Theorem D.

2. Preliminaries. For $z \in B$, the standard automorphism φ_z is given by

$$\varphi_z(w) = \frac{z - P_z w - \sqrt{1-|z|^2}\,Q_z w}{1-<w,z>} \quad (w \in B), \tag{1}$$

where P_z denotes the orthogonal projection of C^n onto the subspace generated by z and $Q_z = I - P_z$. Then $\varphi_z \in \mathcal{A}$, $\varphi_z(0) = z$ and $\varphi_z \circ \varphi_z$ is the identity map on B. Furthermore, the real Jacobian $J_R\varphi_z$ of φ_z is given by

$$J_R\varphi_z(w) = \left(\frac{1 - |z|^2}{|1 - <w, z>|^2}\right)^{n+1} \quad (w \in B) \tag{2}$$

and the identity

$$1 - <\varphi_z(a), \varphi_z(b)> = \frac{(1 - |z|^2)(1 - <a, b>)}{(1 - <a, z>)(1 - <z, b>)} \tag{3}$$

holds for every $a, b \in B$. See [7, Chapter 2] for details.

For $\alpha > -1$, define a measure dV_α on B by $dV_\alpha(z) = \lambda_\alpha(1 - |z|^2)^\alpha dV(z)$, where the constant λ_α is chosen so that $V_\alpha(B) = 1$. For $a \in B$ and $\alpha > -1$, we put

$$k_a^\alpha(z) = \left(\frac{\sqrt{1 - |a|^2}}{1 - <z, a>}\right)^{n+1+\alpha} \quad (z \in B)$$

for notational simplicity. By (2) and (3), we have a useful change-of-variable formula:

$$\int_B h(z) dV_\alpha(z) = \int_B h(\varphi_a(z)) |k_a^\alpha(z)|^2 dV_\alpha(z) \quad (z \in B), \tag{4}$$

for all measurable h on B, whenever the integrals make sense.

For $u \in C^2(B)$, the invariant Laplacian $\tilde{\Delta} u$ is defined by

$$(\tilde{\Delta} u)(z) = \Delta(u \circ \varphi_z)(0) \quad (z \in B),$$

where Δ denotes the ordinary Laplacian. The operator $\tilde{\Delta}$ commutes with automorphisms in the sense that $\tilde{\Delta}(u \circ \varphi) = (\tilde{\Delta} u) \circ \varphi$, for all $\varphi \in \mathcal{A}$. Hence $\mathcal{M}$-harmonic functions are closed under composition with automorphisms. Moreover, by the invariant mean value property [7, Theorem 4.2.4] and a simple application of the integration in polar coordinates, we have the following mean value property for $\mathcal{M}$-harmonic functions f:

$$f(z) = \frac{1}{V_\alpha(rB)} \int_{rB} f \circ \varphi_z \, dV_\alpha \quad (z \in B, 0 < r < 1). \tag{5}$$

Given $z \in B$ and $\zeta \in C^n$, the Bergman metric $\beta(z, \zeta)$, modulo a constant factor, is given by

$$\beta(z, \zeta) = \left(\frac{(1 - |z|^2)|\zeta|^2 + |<z, \zeta>|^2}{(1 - |z|^2)^2}\right)^{1/2}$$

and the corresponding distance $d(z, w)$, called the Bergman distance, has the explicit formula

$$d(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|} \quad (z, w \in B).$$

In particular, for any $0 < p < \infty$ and $\alpha > -1$, the function $d^p(z, 0)$ is integrable with respect to the measure dV_α. We note that

$$\beta(z,\zeta) \le \frac{|\zeta|}{1-|z|^2} \quad (z \in B,\ \zeta \in C^n) \tag{6}$$

and the Bergman distance is invariant under $\mathcal{A}$. See Section 2 of **[8]** for details.

3. Characterizations of $M\mathcal{B}$ and $M\mathcal{B}_0$. We begin with a simple lemma.

LEMMA 1. *Let $f \in C^1(B)$. Then we have*

$$|f(z) - f(0)| \le \left(\sup_{|w|\le|z|} Qf(w)\right) d(0,z),$$

for all $z \in B$.

Proof. We first note that by (6) we have

$$\begin{aligned}|f(z)-f(0)| &= \left|\int_0^1 \{<\nabla f(tz), \bar z> + \overline{<\nabla \bar f(tz), \bar z>}\}dt\right| \\ &\le \int_0^1 \frac{|<\nabla f(tz), \bar z/|z|> + \overline{<\nabla \bar f(tz), \bar z/|z|>}|}{\beta(tz, z/|z|)} |z|\,\beta(tz, z/|z|)dt \\ &\le \int_0^1 \frac{Qf(tz)|z|}{1-|tz|^2}dt \\ &\le \left(\sup_{|w|\le|z|} Qf(w)\right)\int_0^1 \frac{|z|}{1-|tz|^2}dt,\end{aligned}$$

for all $z \in B$. Since

$$\int_0^1 \frac{|z|}{1-|tz|^2}dt = \frac{1}{2}\log\frac{1+|z|}{1-|z|} = d(0,z),$$

for all $z \in B$, we have the desired result. This completes the proof. □

We are ready to characterize $\mathcal{M}$-harmonic Bloch functions. The equivalence of the quantities in (a) and (c) of the following theorem was proved in **[4**, Theorem 5.4] in the unweighted case of $\alpha = 0$.

THEOREM 2. *Let $1 \le p < \infty$ and $\alpha > -1$. Then the following quantities are equivalent as f runs over all $\mathcal{M}$-harmonic functions on B*:

(a) $\|f\|$,

(b) $\|f\|_b = \sup\limits_{\substack{z,w\in B\\ z\neq w}} \dfrac{|f(z)-f(w)|}{d(z,w)}$,

(c) $\|f\|_{c,p} = \sup\limits_{a\in B}\left(\int_B |f\circ\varphi_a - f(a)|^p dV_\alpha\right)^{1/p}$,

(d) $\|f\|_{d,p} = \sup\limits_{a\in B}\left(\int_B (Qf)^p |k_a^\alpha|^2 dV_\alpha\right)^{1/p}$,

(e) $\|f\|_e = \inf\limits_{t>0}\sup\limits_{a\in B} \log\left(\int_B \exp(t|f\circ\varphi_a - f(a)|)dV_\alpha\right)^{1/t}$.

In the rest of the paper, the same letter C will denote various positive constants which may change from one occurrence to the next. While constants C may depend on variables like n, p, r, α or some others, they will always be independent of functions, points or measures under consideration.

Proof. By Lemma 1,

$$|f(z) - f(0)| \le \|f\|\, d(0, z),$$

for all $z \in B$. Replacing f by $f\circ\varphi_w$ and then z by $\varphi_w(z)$, we get, by the invariance of $\|\;\|$ and d under $\mathcal{A}$,

$$|f(z) - f(w)| \le \|f\circ\varphi_w\|\, d(0, \varphi_w(z)) = \|f\|\, d(z, w),$$

for all $z, w \in B$, and so we have $\|f\|_b \le \|f\|$.

Next, we show that $\|f\|_{c,p} \le C\|f\|_b$. By the invariance of d under $\mathcal{A}$, we see that

$$|f\circ\varphi_a(z) - f(a)| \le \|f\|_b d(z, 0),$$

for all $z, a \in B$. It follows that

$$\int_B |f\circ\varphi_a(z) - f(a)|^p dV_\alpha \le \|f\|_b^p \int_B d^p(z,0)dV_\alpha(z) \le C\|f\|_b^p,$$

for all $a \in B$ and hence $\|f\|_{c,p} \le C\|f\|_b$, as desired.

Next, we show $\|f\|_{d,p} \le C\|f\|_{c,p}$. Assume that $\|f\|_{c,p} < \infty$. Then, by (5), with $r \to 1$ and the change-of-variable formula (4), one can see that

$$f(z) = \int_B f\circ\varphi_z dV_\alpha = \int_B f(w)\left(\frac{1-|z|^2}{|1- <z, w>|^2}\right)^{n+1+\alpha} dV_\alpha(w) \quad (z \in B).$$

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Differentiation under the integral sign yields

$$| < \nabla f(0), \bar{\zeta} > | \leq C \int_B |f| dV_\alpha$$

and

$$| < \nabla \bar{f}(0), \bar{\zeta} > | \leq C \int_B |f| dV_\alpha,$$

for all $\zeta \in S$. It follows from the definition of Q and Jensen's inequality that

$$Qf(0) \leq C \int_B |f| dV_\alpha \leq C \left(\int_B |f|^p dV_\alpha \right)^{1/p}.$$

Apply the above inequalities to $f \circ \varphi_z - f(z)$ to obtain

$$Qf(z) \leq C \left(\int_B |f \circ \varphi_z - f(z)|^p dV_\alpha \right)^{1/p}, \tag{7}$$

for all $z \in B$. Note that k_a^α has norm 1 in $L^2(dV_\alpha)$, for all $a \in B$, by (4). It follows from (7) that

$$\begin{aligned} \int_B (Qf)^p |k_a^\alpha|^2 dV_\alpha &\leq C \int_B \int_B |f \circ \varphi_z - f(z)|^p |k_a^\alpha(z)|^2 dV_\alpha dV_\alpha(z) \\ &\leq C \left(\sup_{z \in B} \int_B |f \circ \varphi_z - f(z)|^p dV_\alpha \right) \int_B |k_a^\alpha|^2 dV_\alpha \\ &= C \sup_{z \in B} \int_B |f \circ \varphi_z - f(z)|^p dV_\alpha, \end{aligned}$$

for all $a \in B$, and so we have $\|f\|_{d,p} \leq C \|f\|_{c,p}$.

Next, we show $\|f\| \leq C \|f\|_{d,p}$. Fix $r \in (0,1)$. By (5), we have, for each $t \in (-1, 1)$ and $\zeta \in S$,

$$f(t\zeta) = \frac{1}{V_\alpha(rB)} \int_{rB} f \circ \varphi_{t\zeta} \, dV_\alpha.$$

Fixing ζ, w and denoting the j-th component of $\varphi_{t\zeta}(w)$ by $\varphi_j(t)$, one can see that

$$\varphi_j'(0) = \zeta_j - < w, \zeta > w_j \text{ and } \bar{\varphi}_j'(0) = \overline{\zeta_j - < w, \zeta > w_j},$$

for each j. Thus,

$$\frac{d}{dt} f \circ \varphi_{t\zeta}(w)|_{t=0} = < \nabla f(w), \overline{\zeta - < w, \zeta > w} > + \overline{< \nabla \bar{f}(w), \overline{\zeta - < w, \zeta > w} >},$$

for each $w \in B$ and $\zeta \in S$. It follows that

$$\begin{aligned}
&< \nabla f(0), \bar{\zeta} > + \overline{< \nabla \bar{f}(0), \bar{\zeta} >} \\
&= \frac{d}{dt} f(t\zeta)|_{t=0} \\
&= \frac{1}{V_\alpha(rB)} \int_{rB} \frac{d}{dt} f \circ \varphi_{t\zeta}(w)|_{t=0}\, dV_\alpha(w) \\
&= \frac{1}{V_\alpha(rB)} \int_{rB} < \nabla f(w), \overline{\zeta - < w, \zeta > w} > + \overline{< \nabla \bar{f}(w), \overline{\zeta < w, \zeta > w} >} dV_\alpha(w).
\end{aligned}$$

Hence by (6), one obtains

$$Qf(0) \le C \int_{rB} \frac{Qf(w)}{1 - |w|^2} dV_\alpha(w) \le C \int_B Qf dV_\alpha.$$

Now replace f by $f \circ \varphi_a$. Then use Jensen's inequality and the change-of-variable formula (4) to see that

$$Qf(a) \le C\left(\int_B (Qf(\varphi_a))^p dV_\alpha\right)^{1/p} = C\left(\int_B (Qf)^p |k_a^\alpha|^2 dV_\alpha\right)^{1/p}, \tag{8}$$

for all $a \in B$, so that we get $\|f\| \le C\|f\|_{d,p}$.

Consequently, $\|f\|, \|f\|_b, \|f\|_{c,p}$ and $\|f\|_{d,p}$ are all equivalent for each p with $1 \le p < \infty$. Since $\|f\|$ is independent of p and equivalent to $\|f\|_{c,p}$, for each p in $[1,\infty)$, it is equivalent, in particular, to $\|f\|_{c,1}$. Thus, in order to finish the proof, it is sufficient to prove the inequalities $\|f\|_{c,1} \le \|f\|_e \le C\|f\|$.

By Lemma 1, we get as before

$$|f \circ \varphi_a(z) - f(a)| \le \|f\| d(z, 0) = \frac{\|f\|}{2} \log \frac{1 + |z|}{1 - |z|}, \tag{9}$$

for all $z, a \in B$. Assume $0 < \|f\| < \infty$. Then, by taking $t = (\alpha + 1)/\|f\|$, one can see from (9) that

$$\|f\|_e \le \frac{\|f\|}{\alpha + 1} \sup_{a \in B} \left(\log \int_B \exp(\frac{\alpha + 1}{\|f\|} |f \circ \varphi_a - f(a)|)\, dV_\alpha \right)$$

$$\le \frac{\|f\|}{\alpha + 1} \log \int_B \left(\frac{1 + |z|}{1 - |z|}\right)^{\frac{\alpha+1}{2}} dV_\alpha(z).$$

Since the last integral above is finite, we have $\|f\|_e \le C\|f\|$.

Finally, the inequality $\|f\|_{c,1} \le \|f\|_e$ is an easy consequence of Jensen's inequality. The proof is complete. □

As a result corresponding to Theorem 2, we characterize the $\mathcal{M}$-harmonic little Bloch space. In the following theorem, the equivalences of (a), (b) and (e) were proved for holomorphic functions on the disk in [**13**, Theorem 2.1] and the equivalence of (a) and (c) is given in [**4**, Theorem 5.6] in the unweighted case of $\alpha = 0$.

THEOREM 3. *Let* $1 \le p < \infty, \alpha > -1$ *and* $0 < r < 1$. *Then the following statements are equivalent for a function f that is* $\mathcal{M}$*-harmonic on B.*

(a) $f \in MB_0$.

(b) $\lim_{|a| \to 1} \sup_{\substack{z \in E_r(a) \\ z \neq a}} \frac{|f(z) - f(a)|}{d(z, a)} = 0$.

(c) $\lim_{|a| \to 1} \int_B |f \circ \varphi_a - f(a)|^p dV_\alpha = 0$.

(d) $\lim_{|a| \to 1} \int_B (Qf)^p |k_a^\alpha|^2 dV_\alpha = 0$.

(e) *There exists a constant* $t > 0$ *such that*

$$\lim_{|a| \to 1} \int_B \exp(t|f \circ \varphi_a - f(a)|)\, dV_\alpha = 1.$$

Before proceeding to the proof, we note that

$$1 - |w|^2 \approx 1 - |a|^2 \quad (w \in E_r(a)), \tag{10}$$

for each fixed $r \in (0, 1)$. This follows from (3). Here and elsewhere, the notation $A(w) \approx B(a)$ means that two quantities have ratio bounded and bounded away from 0 by constants independent of the points w, a under consideration.

Proof. We first prove the equivalence of (a) and (b). We shall assume (a) holds and prove (b). By Lemma 1, we have

$$|f(z) - f(0)| \le \left(\sup_{|w| < r} Qf(w) \right) d(0, z),$$

for $|z| < r$. Replacing f by $f \circ \varphi_a$ and, using the invariance of Q under $\mathcal{A}$, one obtains

$$|f \circ \varphi_a(z) - f(a)| \le \left(\sup_{|w| < r} Qf(\varphi_a(w)) \right) d(0, z),$$

for $|z| < r$. It follows from the invariance of d under $\mathcal{A}$ that

$$\sup_{\substack{z\in E_r(a)\\ z\neq a}} \frac{|f(z)-f(a)|}{d(z,a)} = \sup_{0<|z|<r} \frac{|f\circ\varphi_a(z)-f(a)|}{d(\varphi_a(z),a)}$$
$$= \sup_{0<|z|<r} \frac{|f\circ\varphi_a(z)-f(a)|}{d(z,0)}$$
$$\le \sup_{|w|<r} Qf(\varphi_a(w))$$
$$= \sup_{w\in E_r(a)} Qf(w),$$

for all $a \in B$. Now, letting $|a| \to 1$, we obtain (b) by (10).

Assume (b) holds. Using (5), one can easily see as before that

$$Qf(0) \le C\int_{rB} |f|dV.$$

Replace f by $f\circ\varphi_a - f(a)$ and then use the change-of-variable formula (4) to see that

$$\begin{aligned} Qf(a) &\le C\int_{rB} |f\circ\varphi_a - f(a)|dV \\ &= C\int_{E_r(a)} |f(z)-f(a)|\frac{(1-|a|^2)^{n+1}}{|1-<z,a>|^{2n+2}}dV(z) \\ &\le C\left(\sup\frac{|f(z)-f(a)|}{d(z,a)}\right)\int_{E_r(a)} d(z,a)\frac{(1-|a|^2)^{n+1}}{|1-<z,a>|^{2n+2}}dV(z) \\ &= C\left(\sup\frac{|f(z)-f(a)|}{d(z,a)}\right)\int_{rB} d(z,0)dV(z) \\ &\le C\left(\sup\frac{|f(z)-f(a)|}{d(z,a)}\right), \end{aligned}$$

for each $a \in B$, where sup is taken over all $z \in E_r(a)$ with $z \neq a$. Letting $|a| \to 1$, we have proved (a).

We assume (a) holds and prove (c). Let $a \in B$. Then, by (9) and the invariance of d under $\mathcal{A}$, one obtains

$$\begin{aligned} &\int_B |f\circ\varphi_a - f(a)|^p dV_\alpha \\ &= \int_{rB} |f\circ\varphi_a - f(a)|^p dV_\alpha + \int_{B\setminus rB} |f\circ\varphi_a - f(a)|^p dV_\alpha \\ &\le \left(\sup_{0<|z|<r}\frac{|f\circ\varphi_a(z)-f(a)|}{d(z,0)}\right)^p\int_{rB} d^p(z,0)\,dV_\alpha(z) + \|f\|^p\int_{B\setminus rB} d^p(z,0)dV_\alpha(z) \\ &\le C\left(\sup_{\substack{z\in E_r(a)\\ z\neq a}}\frac{|f(z)-f(a)|}{d(z,a)}\right)^p + \|f\|^p\int_{B\setminus rB} d^p(z,0)\,dV_\alpha(z). \end{aligned}$$

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Having seen that (a) and (b) are equivalent, one can see that the first term of the expression above tends to 0 as $|a| \to 1$, for each r. Consequently, first taking the limit as $|a| \to 1$ and then as $r \to 1$, we obtain (c).

Assume (c) and show (d). Note that $f \in M\mathcal{B}$ by Theorem 2. By (10), we have

$$\lim_{|a|\to 1} \sup_{z\in E_t(a)} \int_B |f\circ\varphi_z - f(z)|^p dV_\alpha = 0, \tag{11}$$

for each $t \in (0, 1)$. Now, by the change-of-variable formula (4) and (7), we have

$$\begin{aligned}\int_B (Qf)^p |k_a^\alpha|^2 dV_\alpha &= \int_{tB} (Qf)^p(\varphi_a) dV_\alpha + \int_{B\setminus tB} (Qf)^p(\varphi_a)\, dV_\alpha \\ &\le \sup_{z\in E_t(a)} (Qf)^p(z) + \|f\|^p V_\alpha(B\setminus tB) \\ &\le C\left(\sup_{z\in E_t(a)} \int_B |f\circ\varphi_z - f(z)|^p dV_\alpha\right) + \|f\|^p V_\alpha(B\setminus tB).\end{aligned}$$

Consequently, first taking the limit as $|a| \to 1$ and then as $t \to 1$, we obtain (d) by (11).

The implication (d) $\Rightarrow$ (a) is a consequence of (8).

Consequently, (a), (b), (c) and (d) are all equivalent. Since (a) is independent of p and equivalent to (c), for each p in $[1,\infty)$, it is equivalent to (c) when $p = 1$. Thus, in order to finish the proof, it is sufficient to show, (a) $\Rightarrow$ (e) $\Rightarrow$ (c) when $p = 1$.

We assume (a) holds and prove (e). By Lemma 1 with $f \circ \varphi_a$ in place of f, we have

$$|f\circ\varphi_a(z) - f(a)| \le \left(\sup_{w\in E_{|z|}(a)} Qf(w)\right) d(0, z) \quad (z \in B). \tag{12}$$

Since $f \in M\mathcal{B}_0$ by assumption, it follows from (10) that $|f\circ\varphi_a(z) - f(a)| \to 0$ as $|a| \to 1$, for each fixed $z \in B$. Choose $t > 0$ such that $t\|f\| < 2(\alpha + 1)$. Then, by (12), one can see that

$$\exp(t|f\circ\varphi_a(z) - f(a)|) \le \left(\frac{1+|z|}{1-|z|}\right)^{\frac{t\|f\|}{2}},$$

for all $z, a \in B$. Since the right side of the above expression is integrable with respect to the measure dV_α, (e) is a consequence of the Lebesgue dominated convergence theorem.

Finally, the implication (e) $\Rightarrow$ (c) with $p = 1$ easily follows from Jensen's inequality. The proof is complete. □

4. Carleson measures. Fix $\alpha > -1$, $r \in (0, 1)$ and let μ be a positive Borel measure on B. We say that μ is an *α-weighted Carleson measure* if

$$\sup_{a\in B} \frac{\mu(E_r(a))}{V_\alpha(E_r(a))} < \infty.$$

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If, in addition, μ satisfies the condition

$$\lim_{|a|\to 1} \frac{\mu(E_r(a))}{V_\alpha(E_r(a))} = 0,$$

we say that μ is an *α-weighted vanishing Carleson measure*. It turns out that the notion of (vanishing) Carleson measures is independent of the choice of r. In fact, it is known (see for example, [**14**, Theorems A and B]) that μ is an α-weighted Carleson measure if an only if its α-weighted Berezin transform is bounded; that is

$$\sup_{a\in B} \int_B |k_a^\alpha|^2 d\mu < \infty.$$

Similarly, μ is an α-weighted vanishing Carleson measure if and only if

$$\lim_{|a|\to 1} \int_B |k_a^\alpha|^2 d\mu = 0. \tag{13}$$

Hence the following corollary is an immediate consequence of Theorems 2 and 3.

COROLLARY 4. *Let $1 \le p < \infty, \alpha > -1$, and assume that f is $\mathcal{M}$-harmonic on B.*
(a) *$f \in M\mathcal{B}$ if and only if $(Qf)^p dV_\alpha$ is an α-weighted Carleson measure.*
(b) *$f \in M\mathcal{B}_0$ if and only if $(Qf)^p dV_\alpha$ is an α-weighted vanishing Carleson measure.*

It is also well known that, given $0 < p < \infty$, μ is an α-weighted Carleson measure if and only if

$$\int_B |f|^p d\mu \le C \int_B |f|^p dV_\alpha,$$

for all holomorphic functions f in $L^p(dV_\alpha)$. In [**12**], Xiao observed that α-weighted Carleson measures on the disc can be characterized by a similar integral condition, where L^p-integrals are replaced by Berezin type integrals of holomorphic Bloch functions. Here, we prove in Theorem 7 below that α-weighted Carleson measures are also characterized by the same Berezin type integral condition for $\mathcal{M}$-harmonic Bloch functions. We first need a submean value type inequality for $\mathcal{M}$-harmonic functions.

PROPOSITION 5. *Let $0 < p < \infty, 0 < t < s < 1$ and $\alpha > -1$. Then, there exists a constant C such that*

$$\sup_{z\in E_t(a)} |f(z)|^p \le \frac{C}{V_\alpha(E_s(a))} \int_{E_s(a)} |f|^p dV_\alpha,$$

for all $a \in B$ and f an $\mathcal{M}$-harmonic function on B.

Before proceeding to the proof, we first note that, for a given r, we have

$$V_\alpha(E_r(a)) \approx (1 - |a|^2)^{n+1+\alpha} \quad (a \in B). \tag{14}$$

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Proof. Fix a point $a \in B$ and an $\mathcal{M}$-harmonic f. Let $z \in E_t(a)$ and $r = s - t$. Note that $E_r(z) \subset E_s(a)$ and hence $1 - |w|^2 \approx 1 - |a|^2$, for all $w \in E_r(z)$, by (10). By Proposition 10.1 of **[8]** and (14), we have

$$|f(z)|^p \leq C \int_{E_r(z)} \frac{|f(w)|^p}{(1-|w|^2)^{n+1+\alpha}} dV_\alpha(w)$$
$$\leq \frac{C}{(1-|a|^2)^{n+1+\alpha}} \int_{E_s(a)} |f|^p dV_\alpha$$
$$\leq \frac{C}{V_\alpha(E_s(a))} \int_{E_s(a)} |f|^p dV_\alpha,$$

which completes the proof. □

Before turning to Theorem 7, we need a simple lemma.

LEMMA 6. *For every a, b and w in B, we have*

$$\frac{1 - |\varphi_a(b)|^2}{1 - <\varphi_a(w), \varphi_a(b)>} = 1 - <\varphi_b(w), \varphi_b(a)> .$$

Proof. A direct calculation by (3) completes the proof. □

In the following the notation $\mathcal{B}$ denotes the holomorphic Bloch space.

THEOREM 7. *Let* $0 < p < \infty, 0 < r < 1$ *and* $\alpha > -1$. *Then the following quantities are equivalent as* μ *runs over all positive Borel measures on* B.

(a) $\|\mu\|_{a,p} = \sup_{a \in B} \sup_{\substack{f \in M\mathcal{B} \\ \|f\|=1}} \int_B |f - f(a)|^p |k_a^\alpha|^2 d\mu.$

(b) $\|\mu\|_{b,p} = \sup_{a \in B} \sup_{\substack{f \in \mathcal{B} \\ \|f\|=1}} \int_B |f - f(a)|^p |k_a^\alpha|^2 d\mu.$

(c) $\|\mu\|_{c,r} = \sup_{a \in B} \dfrac{\mu(E_r(a))}{V_\alpha(E_r(a))}.$

Proof. The inequality $\|\mu\|_{b,p} \leq \|\mu\|_{a,p}$ is clear because $\mathcal{B} \subset M\mathcal{B}$.

Next, we show that $\|\mu\|_{c,r} \leq C\|\mu\|_{b,p}$. Let $t = (1 + r)/2$. Corresponding to each $a = |a|\zeta$ in $B, \zeta \in S$, let $b = -t\zeta$ and put

$$f_a(z) = \frac{1}{1 - <z, a_0>}, a_0 = \varphi_a(b) \quad (z \in B).$$

Note that $a_0 \neq 0$. Since f_a is holomorphic, we have from **[10]** that

$$\|f_a\| \approx \sup_{z \in B} (1 - |z|^2) |\nabla f_a(z)|$$

and therefore one can see from (3) that

$$\|f_a\| \approx \sup_{z\in B} \frac{|a_0|(1-|z|^2)}{|1-<z,a_0>|^2} = \sup_{z\in B} \frac{|a_0|(1-|\varphi_{a_0}(z)|^2)}{1-|a_0|^2} = \frac{|a_0|}{(1-|a_0|^2)}.$$

Also, by (3), one can easily verify that

$$1-|a_0|^2 \approx 1-|a|^2 \approx |1-<z,a_0>| \quad (z\in E_r(a)).$$

Thus, it follows from (14) that

$$\begin{aligned} \|\mu\|_{b,p} &\geq \frac{1}{\|f_a\|^p}\int_{E_r(a)} |f_a(z)-f_a(a_0)|^p |k^\alpha_{a_0}(z)|^2 d\mu(z) \\ &\geq \frac{C}{V_\alpha(E_r(a))}\int_{E_r(a)} \left(\frac{1}{|a_0|}\left|1-\frac{1-|a_0|^2}{1-<z,a_0>}\right|\right)^p d\mu(z). \end{aligned} \tag{15}$$

On the other hand, using the explicit formula (1) of the standard automorphism and simple manipulations, one can easily see that

$$\varphi_b(a) = -\left(\frac{t+|a|}{1+t|a|}\right)\zeta$$

and hence that

$$\frac{1}{|\varphi_b(a)|}|<\varphi_b(w),\varphi_b(a)>| = |<\varphi_b(w),\zeta>| = \left|\frac{t+<w,\zeta>}{1+t<w,\zeta>}\right|,$$

for all $w\in B$. Note from (3) that $|\varphi_z(w)| = |\varphi_w(z)|$, for all $z, w\in B$. Hence, it follows from Lemma 6 that

$$\begin{aligned} \inf_{z\in E_r(a)} \frac{1}{|a_0|}\left|1-\frac{1-|a_0|^2}{1-<z,a_0>}\right| &= \inf_{|w|<r} \frac{1}{|\varphi_a(b)|}\left|1-\frac{1-|\varphi_a(b)|^2}{1-<\varphi_a(w),\varphi_a(b)>}\right| \\ &= \inf_{|w|<r} \frac{1}{|\varphi_b(a)|}|<\varphi_b(w),\varphi_b(a)>| \\ &= \inf_{|w|<r}\left|\frac{t+<w,\zeta>}{1+t<w,\zeta>}\right| \\ &\geq \frac{1-r}{4}. \end{aligned}$$

Combining the above with (15), we have

$$\sup_{a\in B} \frac{\mu(E_r(a))}{V_\alpha(E_r(a))} \leq C\|\mu\|_{b,p},$$

as desired.

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Finally, we show that $\|\mu\|_{a,p} \le C\|\mu\|_{c,r}$. Using the same method of Axler [1, Lemma 3.5], we can choose a sequence $\{w_i\}$ of points in B and a positive integer M such that $\bigcup_{i=1}^{\infty} E_r(w_i) = B$ and each $z \in B$ is in at most M of the sets $E_{(1+r)/2}(w_i)$. Let $a \in B$ and $f \in \mathcal{MB}$ with $\|f\| = 1$. Note that

$$1 - |\varphi_a(z)|^2 \approx 1 - |\varphi_a(w)|^2, \quad 1 - |z|^2 \approx 1 - |w|^2,$$

for $z \in E_l(w)$ and $a \in B$ by (10). It follows from (3) that, for each fixed $l \in (0,1)$, $|k_a^\alpha(z)| \approx |k_a^\alpha(w)|$, for $z \in E_l(w)$ and $a \in B$. Thus we obtain from Proposition 5, with $t = r$ and $s = (1+r)/2$, that

$$\begin{aligned}
\int_B |f - f(a)|^p |k_a^\alpha|^2 d\mu &\le \sum_{i=1}^{\infty} \int_{E_r(w_i)} |f - f(a)|^p |k_a^\alpha|^2 d\mu \\
&\le C \sum_{i=1}^{\infty} \left(\sup_{z \in E_r(w_i)} |f(z) - f(a)|^p \right) |k_a^\alpha(w_i)|^2 \mu(E_r(w_i)) \\
&\le C \sum_{i=1}^{\infty} \frac{\mu(E_r(w_i)) |k_a^\alpha(w_i)|^2}{V_\alpha(E_s(w_i))} \int_{E_s(w_i)} |f - f(a)|^p dV_\alpha \\
&\le C \sum_{i=1}^{\infty} \frac{\mu(E_r(w_i))}{V_\alpha(E_r(w_i))} \int_{E_s(w_i)} |f - f(a)|^p |k_a^\alpha|^2 dV_\alpha \\
&\le C\|\mu\|_{c,r} \sum_{i=1}^{\infty} \int_{E_s(w_i)} |f - f(a)|^p |k_a^\alpha|^2 dV_\alpha \\
&\le CM\|\mu\|_{c,r} \int_B |f \circ \varphi_a - f(a)|^p dV_\alpha.
\end{aligned}$$

Thus, for $1 \le p < \infty$, the desired inequality follows from Theorem 2. For $0 < p < 1$, an application of Jensen's inequality shows that the last integral of the expression above is less than or equal to

$$\sup_{a \in B} \left(\int_B |f \circ \varphi_a - f(a)| dV_\alpha \right)^p \approx \|f\|^p = 1,$$

by Theorem 2 again. The proof is complete. □

Also, a slight modification of the above proof gives a corresponding result for α-weighted vanishing Carleson measures as follows.

THEOREM 8. *Let $0 < p < \infty$ and $\alpha > -1$. Then the following statements are equivalent for a positive Borel measure μ on B.*

(a) $\displaystyle \lim_{|a| \to 1} \sup_{\substack{f \in \mathcal{MB} \\ \|f\|=1}} \int_B |f - f(a)|^p |k_a^\alpha|^2 d\mu = 0.$

(b) $\displaystyle \lim_{|a| \to 1} \sup_{\substack{f \in \mathcal{B} \\ \|f\|=1}} \int_B |f - f(a)|^p |k_a^\alpha|^2 d\mu = 0.$

(c) *μ is an α-weighted vanishing Carleson measure.*

Proof. A trivial modification of the proof of Theorem 7 yields the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c). Now, we assume (c) holds and prove (a). Let $\{w_i\}$ be the sequence chosen in the proof of Theorem 7. Note that $|w_i| \to 1$ as $i \to \infty$. Since $\mu(E_r(a))/V_\alpha(E_r(a))$ tends to 0 as $|a| \to 1$, by assumption, for any $\epsilon > 0$ there is a positive integer N such that

$$\frac{\mu(E_r(w_i))}{V_\alpha(E_r(w_i))} < \epsilon \quad (i > N). \tag{16}$$

Let $a \in B$ and $f \in M\mathcal{B}$, $\|f\| = 1$. By an argument similar to the proof of Theorem 7, one can see by Hölder's inequality that

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{E_r(w_i)} |f - f(a)|^p |k_a^\alpha|^2 d\mu \\
&\quad \le \sum_{i=1}^{N} \left(\int_{E_r(w_i)} |k_a^\alpha|^2 d\mu \right)^{1/2} \left(\int_{E_r(w_i)} |f - f(a)|^{2p} |k_a^\alpha|^2 d\mu \right)^{1/2} \\
&\quad \le C \left(\int_B |k_a^\alpha|^2 d\mu \right)^{1/2} \left(\int_B |f \circ \varphi_a - f(a)|^{2p} dV_\alpha \right)^{1/2} \sum_{i=1}^{N} \left(\frac{\mu(E_r(w_i))}{V_\alpha(E_r(w_i))} \right)^{1/2} \\
&\quad \le C \left(\int_B |k_a^\alpha|^2 d\mu \right)^{1/2} \sum_{i=1}^{N} \left(\frac{\mu(E_r(w_i))}{V_\alpha(E_r(w_i))} \right)^{1/2}
\end{aligned}$$

and from (16), if we set $2s = 1 + r$, then

$$\begin{aligned}
&\sum_{i=N+1}^{\infty} \int_{E_r(w_i)} |f - f(a)|^p |k_a^\alpha|^2 d\mu \\
&\quad \le C \sum_{i=N+1}^{\infty} \frac{\mu(E_r(w_i))}{V_\alpha(E_r(w_i))} \int_{E_s(w_i)} |f - f(a)|^p |k_a^\alpha|^2 dV_\alpha \\
&\quad \le CM\epsilon \int_B |f \circ \varphi_a - f(a)|^p dV_\alpha \\
&\quad \le CM\epsilon.
\end{aligned}$$

Consequently,

$$\int_B |f - f(a)|^p |k_a^\alpha|^2 d\mu \le C \left(\int_B |k_a^\alpha|^2 d\mu \right)^{1/2} \sum_{i=1}^{N} \left(\frac{\mu(E_r(w_i))}{V_\alpha(E_r(w_i))} \right)^{1/2} + CM\epsilon,$$

for each $a \in B$. Now, since $\epsilon > 0$ is arbitrary, letting $|a| \to 1$, we get (a) by (13). The proof is complete □

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