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## LETTER TO THE EDITOR

Dear Editor,

*A note on the ageing character of the run length of Markov-type quality-control schemes*

In this letter, we refer to a paper of Li and Shaked (1997) concerning a discrete-time Markov chain $\{X_n,\ n \geq 0\}$, with state space $\mathbb{N}_+ = \{0, 1, 2, \dots\}$ and initial state $k$. They showed that the first-passage time of $\{X_n,\ n \geq 0\}$ to surpass a given threshold $x$ or for the maximal increment of this process to exceed a fixed critical value $y$, denoted $T_k(x, y)$, has increasing failure rate as long as

(a) the transition matrix $\boldsymbol{P}$ is stochastically monotone convex and

(b) the matrix with the left partial sums of $\boldsymbol{P}$ is totally positive of order 2.

We show that if the assumption (a) is replaced by an assumption of spatial homogeneity (with a reflecting boundary in state 0) of $\boldsymbol{P}$, then $T_k(x, y)$ still has increasing failure rate when $k = 0$. This result is of special interest in statistical process control since this sort of first-passage time naturally arises when dealing with Markov-type quality-control schemes, such as the combined upper one-sided CUSUM–Shewhart scheme, and stochastically monotone convex transition matrices are fairly unusual in a quality-control setting.

### 1. Preliminaries

Let $\{X_n,\ n \geq 0\}$ be a discrete-time Markov chain with infinite state space $\mathbb{N}_+$, initial state $k$, and space homogeneous transition probability matrix $\boldsymbol{P} = [p_{ij}]_{i,j\in\mathbb{N}_+}$ (with a reflecting boundary in state 0), with entries given by

$$p_{ij} = \begin{cases} F_Y(-i), & i \in \mathbb{N}_+,\ j = 0, \\ P_Y(j - i), & i \in \mathbb{N}_+,\ j \in \mathbb{N}, \end{cases} \tag{1}$$

where $\mathbb{N} = \{1, 2, \dots\}$ and $F_Y(\cdot)$ and $P_Y(\cdot)$ are, respectively, the distribution and probability functions of an integer-valued random variable $Y$ with support on a set of consecutive integers including the positive values $y$ and $y + 1$.

In this setting, let $T_k(x, y)$ be the first-passage time of the Markov chain $\{X_n,\ n \geq 0\}$ to surpass a given threshold $x$ or for the maximal increment of this process to exceed a fixed critical value $y$, i.e.

$$T_k(x, y) = \min\{n \in \mathbb{N}_+ : X_n > x \text{ or } X_n - X_{n-1} > y \mid X_0 = k\}.$$

According to Li and Shaked (1997), for $y = 0, 1, \dots, x$, $T_k(x, y)$ has the same distribution as the first-passage time

$$\tilde{T}_k(x) = \min\{n \in \mathbb{N}_+ : \tilde{X}_n > x \mid \tilde{X}_0 = k\},$$

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where $\{\tilde{X}_n,\ n \geq 0\}$ is a discrete-time Markov chain with finite state space $\{0, 1, 2, \dots, x+1\}$, initial state $k$ and transition probability matrix $\tilde{\boldsymbol{P}} = [\tilde{p}_{ij}]$, resulting from a simple modification of $\boldsymbol{P}$, given by

$$\begin{bmatrix}
p_{0\,0} & p_{0\,1} & \cdots & p_{0\,y} & 0 & \cdots & 0 & 1-\sum_{l=0}^{y} p_{0\,l} \\
p_{1\,0} & p_{1\,1} & \cdots & \cdots & p_{1\,y+1} & \ddots & \vdots & 1-\sum_{l=0}^{y+1} p_{1\,l} \\
\vdots & \vdots & & & & \ddots & 0 & \vdots \\
p_{(x-y)\,0} & p_{(x-y)\,1} & \cdots & & & \cdots & p_{(x-y)\,x} & 1-\sum_{l=0}^{x} p_{(x-y)\,l} \\
p_{(x-y+1)\,0} & p_{(x-y+1)\,1} & \cdots & & & \cdots & p_{(x-y+1)\,x} & 1-\sum_{l=0}^{x} p_{(x-y+1)\,l} \\
\vdots & \vdots & & & & & \vdots & \vdots \\
p_{x\,0} & p_{x\,1} & \cdots & & & \cdots & p_{x\,x} & 1-\sum_{l=0}^{x} p_{x\,l} \\
0 & 0 & \cdots & & & \cdots & 0 & 1
\end{bmatrix}$$

Before we proceed, we recall a few preparatory notions required to prove the main result. They are adapted from Shaked and Shanthikumar (1994, pp. 4, 25) and Kijima (1997, Section 3.2). The survival function associated with a distribution function $F$ is denoted by $\bar{F} = 1 - F$.

**Definition 1.** Let $Y$ and $Z$ be two nonnegative integer random variables. Then $Y$ is said to be stochastically smaller than $Z$ in the

(i) usual sense (written $Y \leq_{\text{st}} Z$) if

$$\bar{F}_Y(m) = 1 - F_Y(m) \leq \bar{F}_Z(m), \qquad m \in \mathbb{N}_+;$$

(ii) reversed hazard rate sense (written $Y \leq_{\text{rh}} Z$) if

$$\bar{\lambda}_Y(m) := \frac{P_Y(m)}{F_Y(m)} \leq \frac{P_Z(m)}{F_Z(m)} =: \bar{\lambda}_Z(m), \qquad m \in \mathbb{N}_+.$$

The variable $Y$ is said to have

(iii) increasing hazard rate (written $Y \in \text{IHR}$) if

$$\lambda_X(m) := \frac{P_Y(m)}{\bar{F}_Y(m-1)}$$

is increasing in $m \in \mathbb{N}_+$.

**Definition 2.** The Markov chain $\{X_n,\ n \in \mathbb{N}_+\}$ is said to be stochastically monotone in the ‘st’ sense (written $\boldsymbol{P} \in \mathcal{M}_{\text{st}}$) if

$$(X_{n+1} \mid X_n = i) \leq_{\text{st}} (X_{n+1} \mid X_n = m), \qquad i \leq m,\ n \in \mathbb{N}_+,$$

and is said to be stochastically monotone in the 'rh' sense (written $\boldsymbol{P} \in \mathcal{M}_{\mathrm{rh}}$) if

$$(X_{n+1} \mid X_n = i) \leq_{\mathrm{rh}} (X_{n+1} \mid X_n = m), \qquad i \leq m,\ n \in \mathbb{N}_+.$$

In addition, $\{X_n,\ n \in \mathbb{N}_+\}$ is said to be stochastically monotone convex (written $\boldsymbol{P} \in \mathcal{M}_{\mathrm{c}}$) if

$$\sum_{j=k}^{\infty} p_{ij} \leq \sum_{j=k+1}^{\infty} p_{(i+1)\,j}, \qquad i, k \in \mathbb{N}_+.$$

Recall that the stochastic monotonicity in the convex sense implies that $p_{ij} = 0$ for all $i > j$ (that is, the associated Markov chain has increasing paths). Moreover, note that if $\boldsymbol{P}$ has entries given by (1), then $\boldsymbol{P} \in \mathcal{M}_{\mathrm{c}}$ if and only if the random variable $Y$ is nonnegative.

## 2. Some special features of $\tilde{\boldsymbol{P}}$ and the ageing of $T_k(x, y)$

The transition matrix $\tilde{\boldsymbol{P}}$ has a few special features that play a vital role in establishing the ageing character of $T_k(x, y)$. We recall that $\boldsymbol{P}$ is totally positive of order 2 (written $\boldsymbol{P} \in \mathrm{TP}_2$) if all the $2 \times 2$ minors of $\boldsymbol{P}$ are nonnegative, that is, if

$$p_{ij} \times p_{i'j'} \geq p_{i'j} \times p_{ij'}, \qquad i \leq i',\ j \leq j'.$$

**Theorem 1.** *If $\boldsymbol{P}$ is spatially homogeneous with a reflecting boundary in state* 0 *(as in (1)) and if, moreover,* $\boldsymbol{P} \in \mathrm{TP}_2$*, then* $\tilde{\boldsymbol{P}} \in \mathcal{M}_{\mathrm{st}}$ *and* $\tilde{\boldsymbol{P}} \in \mathcal{M}_{\mathrm{rh}}$ *for* $y = 0, 1, \dots, x$.

*Proof.* A direct substitution yields that, for $j = 0, 1, \dots, x$,

$$\begin{aligned}\sum_{l=0}^{j} \tilde{p}_{il} &= \sum_{l=0}^{\min\{j, i+y\}} p_{il} \\ &= F_Y(\min\{j - i, y\})\end{aligned}$$

is a decreasing function of $i$, regardless of the distribution of $Y$ under the conditions mentioned in Section 1. In the light of this result we can conclude that $\tilde{\boldsymbol{P}} \in \mathcal{M}_{\mathrm{st}}$.

To prove that $\tilde{\boldsymbol{P}} \in \mathcal{M}_{\mathrm{rh}}$, it suffices to show that the matrix with the left partial sums of $\tilde{\boldsymbol{P}}$,

$$\begin{bmatrix}
p_{00} & \cdots & \sum_{l=0}^{y-1} p_{0l} & \sum_{l=0}^{y} p_{0l} & \cdots & \cdots & \sum_{l=0}^{y} p_{0l} & 1 \\
p_{10} & \cdots & \cdots & \sum_{l=0}^{y} p_{1l} & \sum_{l=0}^{y+1} p_{1l} & \cdots & \sum_{l=0}^{y+1} p_{1l} & 1 \\
\vdots & & & & \ddots & \ddots & \vdots & \vdots \\
p_{(x-y)\,0} & \cdots & & & \cdots & \sum_{l=0}^{x-1} p_{(x-y)\,l} & \sum_{l=0}^{x} p_{(x-y)\,l} & 1 \\
p_{(x-y+1)\,0} & \cdots & & & \cdots & \sum_{l=0}^{x-1} p_{(x-y+1)\,l} & \sum_{l=0}^{x} p_{(x-y+1)\,l} & 1 \\
\vdots & & & & & \vdots & \vdots & \vdots \\
p_{x\,0} & \cdots & & & \cdots & \sum_{l=0}^{x-1} p_{x\,l} & \sum_{l=0}^{x} p_{x\,l} & 1 \\
0 & \cdots & & & \cdots & 0 & 0 & 1
\end{bmatrix},$$

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is $\mathrm{TP}_2$ (see, e.g. Definition 3.11 of Kijima (1997, p. 129)). Thus, taking into consideration the fact that $\boldsymbol{P} \in \mathrm{TP}_2$, in order to prove that $\tilde{\boldsymbol{P}} \in \mathcal{M}_{\mathrm{rh}}$, we only need to prove that

$$\begin{bmatrix} \sum_{l=0}^{y+h} p_{h\,l} & 1 \\ \sum_{l=0}^{y+h+1} p_{(h+1)\,l} & 1 \end{bmatrix} \in \mathrm{TP}_2, \qquad h = 0, 1, \ldots, x - y - 1, \tag{2}$$

and

$$\begin{bmatrix} \sum_{l=0}^{x} p_{h\,l} & 1 \\ \sum_{l=0}^{x} p_{(h+1)\,l} & 1 \end{bmatrix} \in \mathrm{TP}_2, \qquad h = x - y, x - y + 1, \ldots, x - 1 \tag{3}$$

(see the proof of Theorem 2.8 of Li and Shaked (1997)). But (2) follows from the fact that

$$\sum_{l=0}^{y+h} p_{h\,l} = \sum_{l=0}^{y+h+1} p_{(h+1)\,l} = F_Y(y),$$

due to the spatial homogeneity of the Markov chain ruled by $\boldsymbol{P}$, while (3) is a consequence of the stochastic monotonicity of $\boldsymbol{P}$ in the usual sense.

**Corollary 1.** *If $\boldsymbol{P}$ is spatially homogeneous with a reflecting boundary in state 0 (as in (1)) and if, moreover, $\boldsymbol{P} \in \mathrm{TP}_2$, then $T_0(x, y) \in \mathrm{IHR}$.*

*Proof.* The result follows from Theorem 1 and Durham *et al.* (1990).

Note that, if the random variable $Y$ in (1) is nonnegative, then Corollary 1 also follows from Theorem 2.8 or Example 3.2 of Li and Shaked (1997).

## 3. An application

Considerable benefit is to be gained by combining Shewhart (see Shewhart (1931)) and CUSUM (see Page (1954)) schemes since we can take advantage of two well-known facts: the Shewhart schemes are favoured when a large shift has occurred and the CUSUM schemes allow a fast detection of small and moderate shifts (see Yashchin (1985)).

Morover, if we privilege the detection of upward shifts, we should use a combined upper one-sided CUSUM–Shewhart scheme whose CUSUM constituent has the following summary statistic:

$$X_n = \begin{cases} k, & n = 0, \\ \max\{0, X_{n-1} + (Z_n - \xi)\}, & n \in \mathbb{N}, \end{cases}$$

where $k$ represents the initial value given to the summary statistic (if $k$ is larger than 0, then a head start has been given to the control scheme, as proposed by Lucas and Crosier (1982)), the nonnegative integer random variable $Z_n$ denotes the summary statistic of the upper one-sided Shewhart scheme, and $\xi$, usually called the reference value, is a positive integer constant.

The combined upper one-sided CUSUM–Shewhart scheme triggers a signal at time $n$ if

$$X_n > x \quad \text{or} \quad Z_n > y + \xi,$$

where $y \in \{0, 1, \ldots, x-1\}$ turns out to be the critical value for the increment in the summary statistic $X_n$ that should also be taken as an indication of a shift in the process parameter, even if the summary statistic $X_n$ does not exceed the upper control limit $x$.

Therefore, the run length (i.e. the number of sampling periods before the control scheme triggers a signal) has the same distribution as the first-passage time $T_k(x, y)$. Moreover, $\{X_n,\ n \in \mathbb{N}_+\}$ is governed by a spatially homogeneous transition matrix $\boldsymbol{P}$ with a reflecting boundary in state 0, as described by (1).

If $Z$ has a decreasing likelihood ratio $\mathrm{P}(Z = m)/\mathrm{P}(Z = m-1)$ (see Kijima (1997, Section 3.2)), then $\boldsymbol{P} \in \mathrm{TP}_2$. As a result, we can conclude from Corollary 1 that the run length of a combined upper one-sided CUSUM–Shewhart scheme without head start has increasing hazard rate function. That is, the alarm rate of this control scheme increases as we proceed with the sampling procedure, in contrast to the nonmonotonous alarm-rate function of the run length of combined upper one-sided CUSUM–Shewhart schemes with head starts and the constant alarm-rate function of the geometric run length of any Shewhart scheme.

This result casts interesting light on the ageing behaviour of the most popular performance measure in the online quality-control setting, where stochastically monotone convex transition matrices are fairly unusual.

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Yours sincerely,

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