BULL. AUSTRAL. MATH. SOC. 47A05, 47B20
VOL. 33 (1986), 279-291.

A NOTE ON CONTAINMENT OF OPERATORS

RICHARD I. LOEBL

Two new types of containment of operators on Hilbert space, namely partial containment and semi-containment are introduced. We show in Proposition 11 that A is semi-contained in B if and only if the map $p(B) \to p(A)$ for polynomials p extends to be an ultra-weakly continuous completely positive map from all bounded operators on the underlying Hilbert space of B. We show in Theorem 15 that if an isometry is semi-contained in a contraction T, then T has a non-zero invariant subspace on which T is isometric. The semi-equivalence class of the simple unilateral shift is characterized in Theorem 18, and we show that a unilateral positive-weighted shift semi-contains the unilateral shift if and only the weights are eventually 1.

0. The purpose of this note is to make some observations about containment relations for operators on Hilbert space. The basic problem is to derive information about one operator from information about another operator. Most of the containment relations we will mention are well-known but we will introduce two new ones which we feel are interesting and merit further study. All these containment relations are reflexive and transitive, and thus give rise to equivalence relations. Our basic

Received 2 July 1985.

reference for operator theory is [6]; operators are taken to be acting on a separable Hilbert space and are non-zero.

1. We begin with some definitions; A and B denote operators.

DEFINITION 1. A is contained in B if A is the restriction of B to a reducing subspace. We also say A is a suboperator of B in this case.

DEFINITION 2. A is unitarily contained in B if A is unitarily equivalent to a suboperator of B .

DEFINITION 3. A is quasi-contained in B if there is a *-homomorphism ϕ of $\mathfrak{A}(B)$, the von Neumann algebra generated by B , onto $\mathfrak{A}(A)$ such that $\phi(B) = A$. This was introduced by Ernest [4, p. 9] , who said B covers A .

DEFINITION 4. A is weakly-contained in B if there is a *-homomorphism ϕ of $C(B)$, the C^*-algebra generated by B and the identity, onto $C^*(A)$ such that $\phi(B) = A$, cf[4 , Definition 1.46].

Notice that in all four of these situations that if B is normal, then A must also be normal. In particular, if B is the usual bilateral shift, then A cannot be the usual unilateral shift.

DEFINITION 5. A is approximately contained in B if B is the norm limit of operators, each having a suboperator unitarily equivalent to A . Bunce and Deddens, who introduced this notion in [3], say that A is a subspace approximant of B.

It is clear that the first four containments are in decreasing order of restrictiveness. We comment here that [3, Proposition 1] shows that approximate containment implies weak containment. Once again, we have that if B is normal and A is a subspace approximant of B , then A is normal.

Each of these containment relations engenders a corresponding notion of equivalence, namely, equality, unitary equivalence, quasi-equivalence, weak equivalence, and approximate equivalence. We are told that an unpublished work of Wai-Fong Chuan shows that in the special case of irreducible operators, weak equivalence implies approximate equivalence. (In general, approximate equivalence implies weak equivalence, as noted in

https://doi.org/10.1017/S0004972700003154 Published online by Cambridge University Press

the paragraph above.)

2. We now introduce some containments that are probably new. Let K denote a (separable) Hilbert space.

DEFINITION 6. The subspace $S \subseteq H$ is said to be a semi-invariant subspace for the operator $T \in L(H)$ if $P_S T^k P_S = (P_S T P_S)^k$, $k = 0,1,2,\dots$.

This definition is due to Sarason [7], who characterized semi-invariant subspaces as follows.

LEMMA. [7]: *S is a semi-invariant subspace for T if and only if $S = S_1 \ominus S_2$ where $S_2 \subseteq S_1$ are invariant subspaces for T. Note that since $S_1 \ominus S_2 = S_2^c \ominus S_1^c$, S will be semi-invariant for T^* also.*

For $B \in L(H)$, we let $B^{(\infty)}$ denote the operator $B \oplus B \oplus B\dots$ countably many times, acting on the Hilbert space $H^{(\infty)} = H \oplus H \oplus H \dots$ countably many times; or equivalently, $B^{(\infty)} = B \otimes I$, I the identity on a fixed separable infinite-dimensional Hilbert space H_1 . Notice that even if B is irreducible, $B^{(\infty)}$ is very far from being irreducible. [Note that the *-commutant of $B \otimes I$ contains $I \otimes L(H_1)$] . In fact $B^{(2)}$ already has $H \oplus 0$ as a non-trivial reducing subspace.

DEFINITION 7. A is partially contained in B if A is unitarily equivalent to $B^{(\infty)}$ restricted to an invariant subspace; that is, $A \cong P_M B^{(\infty)} P_M$, where M is an invariant subspace for $B^{(\infty)}$.

DEFINITION 8. For operators A and B , we say A is semi-contained in B if A is unitarily equivalent to $B^{(\infty)}$ restricted to a semi-invariant subspace, that is, $A \cong P_S B^{(\infty)} P_S$, where S is a semi-invariant subspace for $B^{(\infty)}$. It is clear that partial-containment implies semi-containment.

We recall that A is a part of B if A is unitarily equivalent to B restricted to an invariant subspace [6]; and A is a suboperator of B if A is unitarily equivalent to B restricted to a reducing

https://doi.org/10.1017/S0004972700003154 Published online by Cambridge University Press

subspace [4]. Thus if A is a part of B, or A is a suboperator of B, then A is partially-contained in B; the converse fails, as we shall see from the example below. For example, if S is the usual unilateral shift and B is the unitary bilateral shift, then S is partially-contained in B. Hence, A partly contained in B and B normal does not imply A normal; the case of suboperators shows that even if A is normal and actually contained in B, then B need not be normal.

Notice also that if A is contained in B in any of the seven ways listed above, then $||A|| \leq ||B||$.

Here is an example of partial-containment, see [2].

EXAMPLE: In M_n, suppose $||T|| \leq 1$ and $T^n = 0$. Let $C_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ & & \ddots & & \\ & & & \ddots & 0 \\ \vdots & & & \ddots & \\ & & & & 1 \\ & & & & \ddots \\ 0 & \cdots & & \cdots & 0 \end{bmatrix}$

Then T is partially contained in C_n.

Proof. Let $D = \sqrt{1-T^*T}$, and for $z \in \mathbb{C}^n$, let $Vz = (Dz, DTz, \ldots, DT^{n-1}z)$. Then $\langle Vz, Vz \rangle = \langle (1-T^*T)z, z\rangle + \langle T^*(1-T^*T)Tz, z\rangle + \ldots + \langle (T^*)^{n-1}(1-T^*T)T^{n-1}z, z\rangle = \langle [1-(T^*)^n T^n]z, z\rangle = \langle z, z\rangle$. Thus V is isometric, hence unitary. Let

$$\tilde{C}_n = \begin{bmatrix} 0 & I & \cdot & \cdot & 0 \\ \cdot & \cdot\cdot & & & \\ & & \cdot & & \\ \cdot & \cdot & & & 0 \\ & & & \cdot & \\ \cdot & & \cdot & & I \\ & & & \cdot & \\ 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} = I_n \otimes C_n ,$$

so $\tilde{C}_n$ is unitarily equivalent to $C_n \otimes I_n = C_n^{(n)}$. An easy calculation

https://doi.org/10.1017/S0004972700003154 Published online by Cambridge University Press

shown that $Vt = \tilde{C}_n V$, and so $T = V^* \tilde{C}_n V$. Then T is unitarily equivalent to a part of $C_n^{(n)}$, which is a suboperator of $C_n^{(\infty)}$. By the comments after Definition 8, it now follows that T is partially contained in C_n .

Actually more is true; the above proof shows that T is quasi-contained in C_n . Note that since C_n is irreducible, A is unitarily contained in C_n if and only if $A = 0$ or A is unitarily equivalent to C_n . On the other hand, the interesting operator $x = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is (partially) quasi-contained in C_3 , (at least after we scale by $||X|| = (1 + \sqrt{5})/2)$, but is not unitarily contained in C_3 .

As an elementary remark, we note that $A^{(n)}$ is partially and semi-contained in $A^{(m)}$ for $1 \leq n,m \leq \infty$.

LEMMA 9. *If* A *is partially (semi)-contained in* B *, then* A^* *is partially (semi)-contained in* B^*.

Proof. We have $A = U^*P\, B^{(\infty)}PU$ for some unitary U and appropriate projection P . Taking adjoints and using the fact that $[B^{(\infty)}]^* = [B^*]^{(\infty)}$, we are done.

Note that for the first five containments, the conclusion of Lemma 9 also holds.

PROPOSITION 10. A *is semi-contained in* B *if and only if there is an isometry* V *such that* $A^n = V^*[B^{(\infty)}]^n V$ *for all* $n \geq 0$.

Proof. If A is semi-contained in B , we have $A = U^*P(B^{(\infty)})PU$, so $A^n = U^*[P(B^{(\infty)})P]^n U$; but by Definition 5, $[P(B^{(\infty)})P]^n = P[B^{(\infty)}]^n P$, and $[B^{(\infty)}]^n = [B^n]^{(\infty)}$. Hence $A^n = U^*P[B^n]^{(\infty)}PU = (PU)^*[B^n]^{(\infty)}(PU)$. Letting $V = PU$, we see that V is the desired isometry.

Conversely, let P be the projection on VH , that is, $P = VV^*$.

https://doi.org/10.1017/S0004972700003154 Published online by Cambridge University Press

Then there is a unitary U from H to the range of P with $V = PU$. Thus, $A = V^* B^{(\infty)} V = (PU)^* B^{(\infty)} (PU) = U^*(PB^{(\infty)}P)U$. But then $U^* P[B^{(\infty)}]^n PU = V^*[B^n]^{(\infty)} V = A^n = U^*(PB^{(\infty)}P)^n U$. Since U is unitary, $P[B^{(\infty)}]^n P = (P B^{(\infty)} P)^n$, and so the range of P is a semi-invariant subspace.

Hence, if A is semi-contained in B, then A^k is semi-contained in B^k for $k \geq 0$. Since an invariant subspace is perforce semi-invariant, the first paragraph of the proof of Proposition 10 shows that if A is partially contained in B, then A^k is partially contained in B^k. The same result is true for the other five containments, with elementary proofs.

We recall that the linear map ϕ between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be completely positive if for all $n \geq 0$, the map $\phi_n = \phi \otimes id_n : \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n$ is positive, where M_n denotes the $n \times n$ complex matrices. The fundamental theorem of Stinespring asserts that ϕ is completely positive if and only if $\phi(A) = W^* \pi(A) W$, where π is a $*$-representation of $\mathcal{A}$ [8].

PROPOSITION 11. *A is semi-contained in B if and only if the map $p(B) \to p(A)$ for polynomials p extends to be an ultra-weakly continuous positive map ϕ from $L(H_B) \to L(H_A)$ with, in particular, $\phi(B^n) = A^n$ for $n \geq 0$.*

Proof. Suppose there exists such a ϕ; by Stinespring's theorem $\phi = V^* \pi V$, for some V and *-representation π of $L(H_B)$. Then $\phi(1) = 1$ implies that V is an isometry; this fact and the ultra-weak continuity of ϕ imply that $\pi(T) = T^{(n)}$, $1 \leq n$. By separability $n \leq \infty$, so $\phi(T) = V^* T^{(n)} V$. If $n < \infty$, we can write $\phi(T) = V_1^* T^{(\infty)} V_1$ where V_1 is still isometric. Hence $\phi(T) = V^* T^{(\infty)} V$ for some isometry V, and hence $A^n = \phi(B^n) = V^*(B^n)^{(\infty)} V$ for $n \geq 0$, so A is semi-contained in B by Proposition 10.

Conversely, Proposition 10 says that $A^n = V^*(B^n)^{(\infty)} V$; then the map

https://doi.org/10.1017/S0004972700003154 Published online by Cambridge University Press

$\phi(T) = V^* T^{(\infty)} V$ provides the desired extension.

COROLLARY 12. *Semi-containment is transitive.*

Proof. Suppose A is semi-contained in B and B is semi-contained in C. Then by Proposition 5, there are completely positive maps ϕ, Ψ with $\phi(B^n) = A^n$, $\Psi(C^n) = B^n$ for $n \geq 0$. Then $\tau = \phi \circ \Psi$ is completely positive and $\tau(C^n) = A^n$ for $n \geq 0$. By Proposition 10, A is semi-contained in C.

COROLLARY 13. *If A is semi-contained in B, then $A^{(\infty)}$ is semi-contained in B.*

Proof. By an earlier remark, $A^{(\infty)}$ is semi-contained in A. The result then follows from Corollary 12.

LEMMA 14. *Partial-containment is transitive.*

Proof. Suppose A is partially contained in B, and B is partially contained in C, then $V^*AV = QB^{(\infty)}Q$, and $U^*BU = PC^{(\infty)}P$, where P,Q are projections onto invariant subspaces and U,V are unitaries.

Then $(U^*BU)^{(\infty)} = U^*BU \otimes I_\infty = (U^* \otimes I_\infty)(B \otimes I_\infty)(U \otimes I_\infty) = \tilde{U}\, B^{(\infty)}\tilde{U}$, where $\tilde{U}$ is a unitary. Further, $(PC^{(\infty)}P)^{(\infty)} = PC^{(\infty)}P \otimes I_\infty =$ $(P \otimes I_\infty)(C^{(\infty)} \otimes I_\infty)(P \otimes I_\infty)$; however, $C^{(\infty)} \otimes I_\infty$ is unitarily equivalent to $C^{(\infty)}$, and therefore we can say that $(P \otimes I_\infty)(C^{(\infty)} \otimes I_\infty)(P \otimes I_\infty) \cong$ $\tilde{P}\, C^{(\infty)}\, \tilde{P}$, where $\tilde{P}$ is also a projection.

Then we can write $\tilde{U}^*(V^*AV)\tilde{U} = \tilde{U}^*(Q\, B^{(\infty)}Q)\tilde{U} = (\tilde{U}^*Q\tilde{U})(\tilde{U}^*B^{(\infty)}\tilde{U})(\tilde{U}^*Q\tilde{U}) =$ $\tilde{Q}\,\tilde{P}\, C^{(\infty)}\, \tilde{P}\,\tilde{Q}$. Remembering that P and Q were projections onto invariant subspaces, we see that A is now unitarily equivalent to $C^{(\infty)}$ restricted to an invariant subspace.

COROLLARY. *If A is partially-contained in B, then $A^{(\infty)}$ is partially contained in B.*

https://doi.org/10.1017/S0004972700003154 Published online by Cambridge University Press

Proof. $A^{(\infty)}$ is partially contained in A.

We will now establish the main result of this paper. Although every invariant subspace for T yields a (semi-)invariant subspace for $T^{(\infty)}$, there is no a priori connection between a semi-invariant subspace of $T^{(\infty)}$ and an invariant subspace of T. In this light, the following result is somewhat surprising.

THEOREM 15. *Suppose* $T \in L(H)$ *is a contraction and* W *is an isometry. If* W *is semi-contained in* T, *then there is a non-zero subspace* M *of* H, *invariant under* T, *such that* $T|_M$ *is isometric.*

Proof. Let K denote the space on which W acts. By Proposition 10, $W^k = V^* (T^{(\infty)})^k V$ for $k \geq 0$, where $V: K \to H^{(\infty)}$ is isometric. We can write $V = \oplus V_i$, where each $V_i: K \to H$.

Since V and W are isometric, we also have that for $k \geq 0$ and $z \in K$, $\Sigma||V_i W^k z||^2 = ||VW^k z||^2 = ||W^k z||^2 = ||z||^2$. Hence $||z||^2 = \Sigma||V_i W^k z||^2 = ||W^k z||^2 = ||V^*(\infty \cdot T)^k Vz||^2 \leq ||(\infty \cdot T)^k Vz||^2 = \Sigma||T^k V_i z||^2 \leq ||T||^{2k}\Sigma||V_i z||^2 = ||T||^{2k}||Vz||^2 = ||T||^{2k}||z||^2$. Since T is a contraction, one can conclude that $||T^k V_i z|| = ||V_i z||$ for all $k \geq 0$, $z \in K$, and all i. Furthermore, for $k = 1$, we state that $||Vz|| = ||z|| = ||Wz|| = ||VWz|| = ||T^{(\infty)} Vz||$, so $T^{(\infty)}$ is isometric on the range of V. Further, since $I = V^* V = \Sigma V_i^* V_i$, there is an i_0 with $V_{i_0} \neq 0$, and so $||T^k V_{i_0} z|| = ||V_{i_0} z||$ for all $k \geq 0$, $z \in H$. Let $M = \{x \in H: ||T^k x|| = ||x|| \text{ for all } k \geq 0\}$. Then $M \supseteq V_{i_0} K \supset \{0\}$, M is invariant under T, and moreover $T|_M$ is isometric.

Notice that if instead of requiring that T be a contraction, it had been required that T^{n_0} were a contraction, we could have produced a non-zero subspace M, invariant under T^{n_0}, so that $T^{n_0}|_M$ would be isometric.

More importantly, notice that a contraction semi-contains an isometry if and only if the contraction has an isometric part. We now study an important case in which an isometry can semi-contain a contraction. S will again denote the simple unilateral shift. The following result generalizes the earlier example of semi-containment.

PROPOSITION 16. *T is semi-contained in S if and only if $||T|| \leq 1$ and $(T^*)^m$ converges to 0 in the strong operator topology.*

Proof. If T is semi-contained in S, then $||T|| \leq ||S|| = 1$ and T^* is semi-contained in S^*, so by Proposition 10, $(T^*)^n = V^*[(S^*)^n]^{(\infty)}V$ for $n \geq 0$ and appropriate isometry V. But the map $X \to V^*X^{(\infty)}V$ is strongly continuous, and $(S^*)^n \to 0$ strongly, hence $(T^*)^n \to 0$ strongly.

The converse is a well-known result [6]. For completeness we include the proof, which shows that T^* is a part of $(S^*)^{(n)}$ for some n, $1 \leq n \leq \infty$, so that T^* is semi-contained in $(S^*)^n$, which is semi-contained in S^*, hence T^* is semi-contained in S^* and T is semi-contained in S.

Let $B = \sqrt{1 - TT^*}$, let S be the closure of the range of B. Let $K = S \oplus S \oplus S \oplus \ldots$ countably many times. For $z \in H$, we have $B(T^*)^n z \in S$ for $n \geq 0$, and moreover $\sum_{j=0}^{k} || B(T^*)^j z||^2 = \sum_{j=0}^{k} ((1-TT^*)(T^*)^j z,$

$$(T^*)^j z) = \sum_{j=0}^{k} [||(T^*)^j z||^2 - ||(T^*)^{j+1} z||^2] = ||z||^2 - ||(T^*)^{k+1} z||^2 .$$

Since $||(T^*)^{k+1} z|| \to 0$ by hypothesis, we have that for $z \in H$, the mapping V defined by $Vz = (Bz, BT^*z, B(T^*)^2 z, \ldots)$ is an isometry from H to K. If $\tilde{S}$ denotes the unilateral shift on K, that is, $\tilde{S}(f_0, f_1, f_2, \ldots) = (0, f_0, f_1 \ldots)$, then $VT^*z = \tilde{S}^*Vz$ for all $z \in H$.

https://doi.org/10.1017/S0004972700003154 Published online by Cambridge University Press

This last equation also shows that $\tilde{S}^*(VH) \subseteq VH$, so that VH is invariant under $\tilde{S}^*$, hence $\tilde{T}^*$ is a part of $\tilde{S}^*$. However $\tilde{S}^*$ is unitarily equivalent to $(S^*)^{(n)}$, where n is the dimension of S, and the proof is complete.

In particular, Proposition 16 implies that S dominates every contractive unilateral weighted shift and also every nilpotent contraction. Since S is irreducible [6], hence has no proper suboperators, and since every part of S is isometric, we see that suboperators and parts of S do not constitute all the operators dominated by S. Also, it follows that S^* does not dominate S^*; hence S^* does not dominate S.

Remark. Theorem 15 suggests the conjecture that W is unitarily equivalent to $T|_M$. We previously noted that $A^{(\infty)}$ is always semi-contained in A; so consider the case of $A = S$, the simple unilateral shift. The proof of Theorem 15 gives $M = H$ in this case, and it is evident that $S^{(\infty)}$ is not unitarily equivalent to S itself, nor to any part of S.

DEFINITION 17. We say that A is semi-equivalent to B if both A semi-contains B and B semi-contains A.

It is clear that semi-equivalence is an equivalence relation.

THEOREM 18. *Let S be the simple unilateral shift. Then T is semi-equivalent to S if and only if $||T|| \leq 1$, $(T^*)^n \to 0$ strongly, and there is an infinite dimensional subspace M, invariant under T, such that $T|_M$ is a completely non-unitary isometry.*

Proof. By Proposition 16, T is semi-contained in S if and only if $||T|| \leq 1$ and $(T^*)^n \to 0$ strongly; so the question is whether also S is semi-contained in T.

If S is semi-contained in T, then by Theorem 15, there is a non-zero subspace M, invariant under T, such that $T|_M$ is isometric. But the condition $(T^*)^n \to 0$ strongly forces M to be infinite-

https://doi.org/10.1017/S0004972700003154 Published online by Cambridge University Press

dimensional for otherwise $TM = M$ and $T|_M$ is unitary. Since M is infinite-dimensional, $T|_M$ contains a completely non-unitary isometry.

Conversely, suppose there is such an M ; then $T|_M$ is unitarily equivalent to a multiple of S [6] . Hence, for suitable n , we have $S^{(n)}$ is a part of T , so that T semi-contains $S^{(n)}$ which in turn semi-contains S .

COROLLARY 19. *If* $||T|| \leq 1$ *and* $(T^*)^n \to 0$ *strongly, then* S *is semi-contained in* T *if and only if* T *contains a copy of* S .

COROLLARY 20. $S^{(m)}$ *is semi-equivalent to* S^n , 1 _ m,n _ .
(Notice that $S^{(m)}$ *is unitarily equivalent to* S^n *if and only if* $m = n$.)

COROLLARY 21. *Let* T *be a unilateral weighted shift with weights* $\{w_n\}$, $0 \leq w_n \leq 1$. *Then* T *is semi-contained in* S *always, and* S *is semi-contained in* T *if and only if* $T \geq S$ *if and only if* $w_n = 1$ *for* $n \geq N$.

Proof. This is an easy calculation based on the fact that T , hence all powers of T , are isometric on an infinite-dimensional subspace.

We remark that T is similar to S if and only if $\Sigma(1-w_n) < \infty$ [6]. Thus T semi-equivalent to S implies T similar to S , but not conversely. Since T is semi-equivalent to S if and only if T^* is semi-equivalent to S^* , it follows that S^* is the model for certain backward shifts. Further, we have that for any two irreducible operators R and T that R is quasi-equivalent in the sense of Ernest if and only if R is unitarily equivalent to T [3, Theorem 1.34]. Thus an irreducible weighted shift quasi-equivalent to S in the sense of Ernest is semi-equivalent, but not conversely.

We now make some simple ovservations about semi-containment.

1. If A is semi-contained in $T = T^*$, then A is self-adjoint; this follows from Proposition 10.

2. If A is semi-contained in N and N is normal, then A need not be normal; the case of the unilateral and bilateral shifts is an example.

3. If A is semi-contained in B, then $\mathrm{Re}\, A$ is semi-contained in $\mathrm{Re}\, B$ and $\mathrm{Im}\, A$ is semi-contained in $\mathrm{Im}\, B$. The converse need not hold.

If we temporarily ignore separability of the spaces, then the following hold.

4. If A is semi-contained in B and B is separably acting, then A is also separably acting. For $B^{(\infty)}$ is separably acting, hence $B^{(\infty)}$ restricted to any subspace is separably acting.

5. If A is semi-contained in B and B is irreducible, then A is separably acting; for B irreducible implies that B is separably acting.

6. For all A, A is semi-contained in 0 implies $A = 0$. However, it is not true that $A \neq 0$ implies 0 is semi-contained in A, for 1 does not semi-contain 0. If A does not have dense range, then the infinite-dimensional 0 is semi-contained by A. Let P denote the projection onto the closure of the range of A so $A = PA$. Then the range of $1 - P$ is a semi-invariant subspace for A, with $(1-P)A(1-P) = 0$. Thus we can produce an infinite dimensional semi-invariant subspace for $A^{(\infty)}$ on which $A^{(\infty)} = 0$.

We conclude with some questions. If T_α and T_β are weighted shifts given by sequences $\alpha = \{\alpha_i\}_1^\infty$, $0 \leq \alpha_i \leq 1$ and $\beta = \{\beta_i\}_1^\infty$, $0 \leq \beta_i \leq 1$, what conditions on α and β imply that T_α is semi-contained in T_β and T_β is semi-equivalent to T_β? Further, what conditions on α and β imply that T_α is a part of T_β? It is our belief that the methods presented in this paper may help in the study of weighted shifts, see Corollary 21.

Also, Ernest has defined a notion of weak equivalence [4]. What is the relationship, say for irreducible operators, between Ernext's weak equivalence and our semi-equivalence? Aspects of Ernest's work are also treated in [5].

References

[1] W. B. Arveson, "Subalgebras of C^*-algebras", *Acta Math.* 123(1969), 141-224.

[2] W. B. Arveson, "Subalgebras of C^*-algebras II", *Acta Math.* 128 (1972), 271-308.

[3] J. W. Bunce and J. Deddens, "Subspace approximants and GCR operators", *Indiana U. Math. J.* 24 (1974), 341-349.

[4] J. Ernest, *Charting the operator terrain*, (Memoirs Amer. Math. Soc. #171, 1976).

[5] D. Hadwin, "An operator-valued Spectrum", *Indiana U. Math. J.* 26 (1977), 329-340.

[6] P. R. Halmos, *A Hilbert Space Problem Book*, (Van Nostrand, Princeton, 1967).

[7] D. E. Sarason, "On spectral sets having connected complement", *Acta Sci. Math. (Szeged)* 26(1965), 289-299.

[8] W. F. Stinespring, "Positive functions on C^*-algebras", *Proc. Amer. Math. Soc.* 6(1955), 211-216.

Department of Mathematics
Wayne State University
Detroit
Michigan 48202
U.S.A.