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On a class of diophantine inequalities

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Dedicated to B. Segre, on his 70th birthday, 16 February 1973.

As a special case of more general results, it is proved in this note that, if α is any real number and δ any positive number, then there exists a positive integer X such that the inequality

$$|X(\tfrac{3}{2})^h - Y_h - \alpha| < \delta$$

has infinitely many solutions in positive integers h and Y_h.

The method depends on the study of infinite sequences of real linear forms in a fixed number of variables. It has relations to that used by Kronecker in the proof of his classical theorem and can be generalised.

1.

For real α put

$$\|\alpha\| = \min_{y=0,\pm1,\pm2,\ldots} |\alpha - y| ,$$

so that $\|\alpha\|$ denotes the distance of α from the nearest integer and hence that

$$0 \le \|\alpha\| \le \tfrac{1}{2} .$$

By H_0 we understand a fixed strictly increasing infinite sequence of positive integers h (H_0 usually will be the set of all positive

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integers), and H denotes some infinite subsequence of H_0, not necessarily always the same.

2.

Let r be a fixed and n a variable positive integer; let further S_n be the set of all r-vectors $\mathrm{x} = (x_1, \ldots, x_r)$ with integral components satisfying

$$1 \le \max(|x_1|, \ldots, |x_r|) \le n .$$

Thus S_n is a finite set, and all vectors in S_n are distinct from the zero vector

$$\mathrm{o} = (0, \ldots, 0) .$$

Next consider an infinite sequence of r-vectors

$$\mathrm{a}_h = (a_{h1}, \ldots, a_{hr}) \quad (h \in H_0)$$

with real components and the associated linear forms

$$L_h(\mathrm{x}) = a_{h1}x_1 + \ldots + a_{hr}x_r \quad (h \in H_0)$$

in x. Then put

$$M_h(n) = \min_{\mathrm{x} \in S_n} \|L_h(\mathrm{x})\| \quad (h \in H_0)$$

and

$$M(n) = \limsup_{\substack{h \to \infty \\ h \in H_0}} M_h(n) .$$

It is obvious that

$$0 \le M_h(n) \le \tfrac{1}{2} \quad (h \in H_0)$$

and hence that also

$$0 \le M(n) \le \tfrac{1}{2} .$$

3.

For $n \ge 3$ these upper bounds for $M_h(n)$ and $M(n)$ can be improved.

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For this purpose, denote by y a further integral variable. The system of $r + 1$ linear forms

$$n^{-1}x_1, \ldots, n^{-1}x_r, \quad n^r(a_{h1}x_1 + \ldots + a_{hr}x_r - y) \quad (h \in H_0)$$

in $x_1, \ldots, x_r$, y has the determinant -1. Hence, by Minkowski's Theorem on linear forms, there exist integers

$$x_{h1}, \ldots, x_{hr}, y_h$$

not all zero, which in general will depend on h, such that simultaneously

$$\max(|x_{h1}|, \ldots, |x_{hr}|) \le n, \quad |a_{h1}x_{h1} + \ldots + a_{hr}x_{hr} - y_h| < n^{-r} \quad (h \in H_0).$$

Here at least one of the first r *integers*

$$x_{h1}, \ldots, x_{hr}$$

does not vanish. For otherwise $y_h \neq 0$, whence

$$1 \le |y_h| < n^{-r} \le 1,$$

which is impossible.

The vector

$$\mathbf{x}_h = (x_{h1}, \ldots, x_{hr})$$

therefore lies in S_n and in addition satisfies the inequality

$$\|L_h(\mathbf{x}_h)\| < n^{-r} \quad (h \in H_0)$$

From this it follows immediately that

(1) $$0 \le M_h(n) < n^{-r} \quad (h \in H_0)$$

and hence also that

(2) $$0 \le M(n) \le n^{-r}.$$

On the other hand, since obviously $S_n \subset S_{n+1}$, it is clear that

$$M_h(1) \ge M_h(2) \ge M_h(3) \ge \ldots \ge 0 \quad (h \in H_0),$$

from which it is easily deduced that also

$$M(1) \geq M(2) \geq M(3) \geq \ldots \geq 0 .$$

4.

The definition of $M(n)$ as an upper limit implies that there exists a subsequence H of H_0 such that

$$\lim_{\substack{h\to\infty \\ h\in H}} M_h(n) = M(n) .$$

Here, to each suffix h in H, we can find a vector x_h in S_n such that

$$M_h(n) = \|L_h(x_h)\| \quad (h \in H) ;$$

note that x_h need not be the same as the vector x_h constructed in §3.

As h runs over H, x_h is restricted by the condition of belonging to the finite set S_n. Therefore, if necessary, H can be replaced by an infinite subsequence which we call again H such that, without loss of generality,

$$x = x_h \quad \text{for all} \quad h \in H$$

is a fixed vector in S_n independent of h; naturally,

$$x \neq 0 .$$

Since this vector has the basic property that

(3) $$\lim_{\substack{h\to\infty \\ h\in H}} \|L_h(x)\| = M(n) ,$$

the following result has been established.

LEMMA 1. *For every positive integer* n *there exist an infinite subsequence* H *of* H_0 *and a constant vector* x *in* S_n *with the property* (3).

5.

In this lemma, H will in general be a proper subsequence of H_0 as the following example shows.

Fix n and choose $r = 1$ so that $\mathbf{a}_h$ and $\mathbf{x}$ are now scalars a_h and x. As the linear forms take

$$L_h(x) = \begin{cases} x & \text{if } h \text{ is even,} \\ x\sqrt{2} & \text{if } h \text{ is odd.} \end{cases}$$

In this example, $M_h(n)$ evidently vanishes for even h (we may put $x = 1$), but is positive and independent of h for odd h. Hence also $M(n)$ is positive. Thus, if H_0 is the set of all positive integers h, H in (3) essentially (that is, except for possibly finitely many even numbers) is the sequence of all odd integers.

6.

Consider again the general case, but assume that, for a certain n, $M(n) = 0$. Since $M_h(n) \geq 0$ for all $h \in H_0$, it is clear that now the *upper limit* in the definition of $M(n)$ becomes the *limit*, hence that (3) takes the form

$$\lim_{\substack{h\to\infty \\ h\in H_0}} \|L_h(\mathbf{x})\| = 0 . \tag{4}$$

Denote by α an arbitrary real number which is not an integer. The relation (4) implies that

$$\lim_{\substack{h\to\infty \\ h\in H_0}} \|L_h(\mathbf{x})-\alpha\| = \|\alpha\| > 0 .$$

This formula suggests the problem whether there exist an infinite subsequence H of H_0 and an integral vector $\mathbf{X}$ distinct from $\mathbf{x}$ such that

$$\lim_{\substack{h\to\infty \\ h\in H}} \|L_h(\mathbf{X})-\alpha\| = 0 .$$

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The answer to this problem depends very much on the special forms L_h and the sequences H_0 and H.

A positive answer can be given in the following trivial example. Let $r = 1$ and $n = 2$; let H_0 and H be the sequences of all positive integers and of all odd positive integers, respectively; and let further

$$L_h(x) = \tfrac{1}{2}x \quad \text{for} \quad h \in H_0 .$$

Since $L_h(2) = 1$, evidently

$$M_h(2) = M(2) = 0 .$$

On the other hand,

$$\|L_h(1) - \tfrac{1}{2}\| = 0 \quad \text{for all} \quad h \in H .$$

A negative answer holds in the following rather more interesting example. Let again $r = 1$, and let H_0 be again the sequence of all positive integers. Assume that the forms L_h have the property

$$\lim_{\substack{h\to\infty \\ h\in H_0}} \|L_h(1)\| = 0 . \tag{5}$$

Then obviously also

$$\lim_{\substack{h\to\infty \\ h\in H_0}} \|L_h(x)\| = 0 \quad \textit{for every integer } x , \tag{6}$$

and hence there cannot exist a subsequence H of H_0 and an integer X satisfying

$$\lim_{\substack{h\to\infty \\ h\in H}} \|L_h(X) - \alpha\| = 0 \tag{7}$$

unless α is an integer.

7.

A simple example in which the condition (5) is satisfied and therefore also the conclusion about (7) is given by the linear forms

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$$L_h(x) = h!ex \quad \text{for} \quad h \in H_0$$

where H_0 still denotes the sequence of all positive integers.

Of much greater interest is, however, the sequence of forms

$$(8) \qquad L_h(x) = \lambda\theta^n x \quad \text{for} \quad h \in H_0$$

where $\theta > 1$ is a fixed *algebraic* number, and $\lambda > 0$ is a constant. A theorem due to Pisot [1] (see also Salem [2]) asserts that the limit equation

$$\lim_{h\to\infty} \|\lambda\theta^h\| = 0 \ ,$$

that is, the condition (5), is satisfied if and only if the following two properties hold.

(i) $\theta = \theta^{(1)}$ *is an algebraic integer of some degree* $m \geq 1$ *such that all its algebraic conjugates* $\theta^{(2)}, \ldots, \theta^{(m)}$ *are less than* 1 *in absolute value.*

(ii) λ *lies in the algebraic number field* $Q(\theta)$ *generated by* θ .

Call $\{\theta, \lambda\}$ a *Pisot pair* whenever these two properties are satisfied. By (7), such pairs have the following further property.

(iii) If α *is any real number,* H *any subsequence of* H_0 *, and* X *any integer, then the equation*

$$\lim_{\substack{h\to\infty \\ h\in H}} \|\lambda\theta^h X-\alpha\| = 0$$

implies that α *is an integer.*

If $\{\theta, \lambda\}$ is a Pisot pair, then by (6) the forms (8) satisfy

$$(9) \qquad M(n) = 0 \quad \text{for all} \quad n \geq 1 \ .$$

This result has a converse. For assume that $\{\theta, \lambda\}$ is not necessarily a Pisot pair, but that (9) is true. This equation (9) is equivalent to

$$(10) \qquad \lim_{\substack{h\to\infty \\ h\in H_0}} \min_{x=\pm1,\pm2,\ldots,\pm n} \|\lambda\theta^h x\| = 0 .$$

Now for every real number α and for every integer g,

$$\|g\alpha\| \le |g|.\|\alpha\| ,$$

hence

$$\|n!\lambda\theta^h\| \le n! \min_{x=\pm1,\pm2,\ldots,\pm n} \|\lambda\theta^h x\| ,$$

because all factors x are divisors of $n!$. The equation (10) implies then that

$$\lim_{h\to\infty} \|n!\lambda\theta^h\| = 0 .$$

This, however, means that $\{\theta, n!\lambda\}$ and hence also $\{\theta, \lambda\}$ are Pisot pairs. Thus the following result holds.

LEMMA 2. *Let $\theta > 1$ be an algebraic number and λ a positive number, let again H_0 be the sequence of all positive integers, and let*

$$L_h(x) = \lambda\theta^h x \quad \textit{for} \quad h \in H_0 .$$

Then $\{\theta, \lambda\}$ is a Pisot pair if and only if

$$M(n) = 0 \quad \textit{for all} \quad n \ge 1 .$$

8.

We return to the general case of §2, but assume now that for a certain value of n,

$$M(n) > 0 .$$

Denote by X the constant vector in S_n given by Lemma 1 and for which

$$(3) \qquad \lim_{\substack{h\to\infty \\ h\in H}} \|L_h(\mathrm{X})\| = M(n) .$$

It follows that there exists an infinite subsequence of H which we call

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again H such that

$$\tfrac{2}{3}M(n) < \|L_h(\mathrm{x})\| < \tfrac{4}{3}M(n) \quad \text{for all} \quad h \in H$$

In explicit form, $\mathrm{x} = (x_1, \ldots, x_r)$, and there exists to each $h \in H$ an integer y_h such that the sum

$$s_h = a_{h1}x_1 + \ldots + a_{hr}x_r - y_h$$

satisfies the equation

$$|s_h| = \|L_h(\mathrm{x})\|$$

and therefore also the inequality

(11) $$\tfrac{2}{3}M(n) < |s_h| < \tfrac{4}{3}M(n) \quad \text{for all} \quad h \in H .$$

9.

Next let α be an arbitrary real number, and let y be the unique integer for which the real number

$$\beta = \alpha + y$$

satisfies the inequality

(12) $$\tfrac{2}{3} < \beta \leq \tfrac{5}{3} .$$

The integral multiples

$$s_h z \quad (z = 0, \pm 1, \pm 2, \ldots)$$

of s_h form an arithmetic progression of distance $|s_h| > 0$. By (11), every open interval of length $\frac{4}{3}M(n)$ contains then at least one element of this progression.

We apply this property to the open interval

$$\text{from} \quad \beta - \tfrac{2}{3}M(n) \quad \text{to} \quad \beta + \tfrac{2}{3}M(n)$$

of this length and deduce that

for every $h \in H$ *there exists an integer* z_h *such that*

$$-\tfrac{2}{3}M(n) < s_h z_h - \beta < \tfrac{2}{3}M(n) .$$

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Here $\beta \le \frac{5}{3}$ and $M(n) \le \frac{1}{2}$, so that by (11),

$$|z_h| < \frac{\beta + \frac{2}{3}M(n)}{\frac{2}{3}M(n)} \le \frac{5+2M(n)}{2M(n)}$$

and therefore

$$(13) \qquad |z_h| < \frac{3}{M(n)} .$$

On the other hand, $\beta > \frac{2}{3}$, and so again by (11),

$$s_h z_h > \beta - \tfrac{2}{3}M(n) \ge \tfrac{2}{3} - \tfrac{1}{3} > 0 ,$$

whence also

$$z_h \ne 0 .$$

In this construction, z_h is a function of $h \in H$ which, by (13), has only finitely many possible values. Since H may, if necessary, once more be replaced by a suitable infinite subsequence, we may without loss of generality assume that

$$z_h = z \quad \text{for all} \quad h \in H$$

has a *fixed* integral value independent of h, where by (13) and (14)

$$(15) \qquad 0 < |z| < \frac{3}{M(n)} .$$

10.

Put finally

$$X_1 = x_1 z, \ldots, X_r = x_r z , \quad Y_h = y_h z + y .$$

Then $X = (X_1, \ldots, X_r)$ is an integral r-vector independent of h such that

$$(16) \qquad 1 \le \max(|X_1|, \ldots, |X_r|) < \frac{3n}{M(n)} ,$$

while Y_h is an integer which in general depends on h. In this new notation, the lower and upper estimates for $s_h z_h - \beta$ take the form

$$-\tfrac{2}{3}M(n) < L_h(X) - Y_h - \alpha < \tfrac{2}{3}M(n) \quad \text{for all} \quad h \in H .$$

Since $\frac{2}{3}M(n) < \frac{1}{2}$, this is equivalent to

(17) $$\|L_h(X)-\alpha\| < \tfrac{2}{3}M(n) \quad \text{for all} \quad h \in H .$$

Thus the following result has been obtained.

LEMMA 3. *For a certain* $n \geq 1$ *let* $M(n) > 0$. *Then, to every real number* α , *there exist an infinite subsequence* H *of* H_0 *and a constant integral vector* X *such that both* (16) *and* (17) *are satisfied.*

This lemma becomes particularly interesting when $M(n)$ is positive for *all* positive integers n . For, by the earlier estimate (2),

$$\lim_{n\to\infty} M(n) = 0 .$$

Therefore, for sufficiently large n , the right-hand side of (17) can be made arbitrarily small, giving the following result.

THEOREM 1. *Let* $r \geq 1$ *be a fixed integer, and let* H_0 *be a strictly increasing infinite sequence of positive integers. Associate with each* h *in* H_0 *a real linear form*

$$L_h(x) = a_{h1}x_1 + \ldots + a_{hr}x_r ,$$

and assume that the upper limit $M(n)$, *as defined in* §2, *is positive for every positive integer* n .

Then, given any real number α *and any positive number* δ , *there exist an infinite subsequence* H *of* H_0 *and an integral vector* $X \neq 0$ *independent of* h *such that*

$$\|L_h(X)-\alpha\| < \delta \quad \textit{for all suffices} \quad h \quad \textit{in} \quad H .$$

11.

We combine this theorem with Lemma 2, taking $r = 1$. Let θ and λ be as in Lemma 2, but assume that $\{\theta, \lambda\}$ is *not* a Pisot pair. Then $M(n)$ is positive for all $n \geq 1$, and Theorem 1 gives the following consequence.

THEOREM 2. *Let* $\theta > 1$ *be an algebraic number, and* $\lambda > 0$ *a*

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constant. Assume that at least one of the following two properties is not satisfied.

(i) $\theta = \theta^{(1)}$ *is an algebraic integer of degree* $m \geq 1$ *such that all its algebraic conjugates* $\theta^{(2)}, \ldots, \theta^{(m)}$ *have absolute values less than* 1 *.*

(ii) λ *lies in the algebraic number field* $Q(\theta)$ *generated by* θ *.*

Then, given any real number α *and any positive number* δ *, there exists a positive integer* X *such that the inequality*

$$\|X\lambda\theta^h - \alpha\| < \delta$$

has infinitely many solutions in positive integers h *.*

By way of example, this theorem can be applied to each of the inequalities

$$\left\|X\sqrt{2}\left(\frac{1+\sqrt{5}}{2}\right)^h - \alpha\right\| < \delta\ , \quad \|Xe(1+\sqrt{2})^h - \alpha\| < \delta\ , \quad \|X\lambda\left(\tfrac{3}{2}\right)^h - \alpha\| < \delta\ ,$$

where in the last inequality λ may be an arbitrary positive number.

12.

We conclude this note with an application of Theorem 1 when r is an arbitrary positive integer. For this purpose, assume that

$$L_h(x) = a_1 x_1 + \ldots + a_r x_r$$

does not depend on h . Any relation $M(n) = 0$ where $n \geq 1$ now implies that the numbers

$$a_1, \ldots, a_r, 1$$

are linearly dependent over the rational field Q . Conversely, if these numbers are linearly independent over Q , then $M(n)$ is positive for all $n > 1$. In this case it follows from Theorem 1 that for every real number α and for every positive number δ there exist r integers $X_1, \ldots, X_r$ not all zero such that

$$\|a_1 X_1 + \ldots + a_r X_r - \alpha\| < \delta .$$

We obtain thus a rather special case of Kronecker's Theorem.

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