# STONE ALGEBRAS FORM AN EQUATIONAL CLASS

## (REMARKS ON LATTICE THEORY III)

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To prove the statement given in the title take a set $\Sigma_1$ of identities characterizing distributive lattices $\langle L; \vee, \wedge, 0, 1\rangle$ with 0 and 1, and let

$$\Sigma = \Sigma_1 \cup \{x \wedge x^* = 0,\ 0^* = 1,\ 1^* = 0,\ x \vee x^{**} = x^{**},\ (x \wedge y)^* = x^* \vee y^*,\ (x \vee y)^* = x^* \wedge y^*,\ x^* \vee x^{**} = 1\}.$$

Then $\Sigma$ is a redundant set of identities characterizing Stone algebras $\mathfrak{L} = \langle L; \vee, \wedge, {}^*, 0, 1\rangle$. To show that we only have to verify that for $a \in L$, $a^*$ is the pseudo-complement of $a$. Indeed, $a \wedge a^* = 0$; now, if $a \wedge x = 0$, then $a^* \vee x^* = 0^* = 1$, and $a^{**} \wedge x^{**} = 1^* = 0$; since $a^{**}$ is the complement of $a^*$, the last identity implies $x^{**} \leqq a^*$, thus $x \leqq x^{**} \leqq a^*$, which was to be proved.

It was noted by B. Jónsson, that the subdirectly irreducible Stone algebras are the two and three element chains. Indeed, if $|L| > 3$, and $\mathfrak{L} = \langle L; \vee, \wedge, *, 0, 1\rangle$ is subdirectly irreducible, then $\{0, 1\}$ in the center of $\mathfrak{L}$, hence $L = \{0\} \cup D$, where $D = \{x \mid x^* = 0\}$. By assumption, $|D| > 2$, hence $D = \langle D; \vee, \wedge\rangle$ is subdirectly reducible, in fact there are two congruence relations $\Theta$ and $\Phi$ of $D$ such that $\Theta \neq \omega$, $\Phi \neq \omega$ and $\Theta \wedge \Phi = \omega$. Define $\overline{\Theta}$ on $L$ by the rule: $x \equiv y(\overline{\Theta})$ iff $x = y = 0$ or $x \equiv y(\Theta)$ and define $\overline{\Phi}$ similarly. Then $\overline{\Theta} \wedge \overline{\Phi} = \omega$. Thus $\mathfrak{L}$ is subdirectly-reducible.

We conclude:

THEOREM. Every Stone algebra is a subdirect product of two and three element Stone algebras.

This gives the result that every Stone lattice is isomorphic to a *-sublattice of a direct product of two and three element chains. Since all chains are dense lattices, this is a strong version of Theorem 5.6 of T.P. Speed [3].

Probably it is more surprising that the representation theorem of Grätzer [2] also follows from the above observations. The result states that every Stone algebra $\mathfrak{L}$ can be represented as a subalgebra of the Stone algebra of all ideals of $\mathfrak{P}(H)$ (the Boolean algebra of all subsets of $H$) for a suitable set $H$.

Now, the class of Stone lattices having such a representation is obviously closed under the formation of direct products and subalgebras. Hence, if it contains the two and three element Stone algebras, by the above theorem, it contains all Stone algebras. The two element Stone algebra has such a representation with $|H| = 1$; to find a representation for the three element Stone algebra $\mathfrak{L}$ ($L = \{0, a, 1\}$), take an $H$ with $|H| = \aleph_0$ and let $0 \to \{\phi\}$, $1 \to P(H)$, $a \to I$, where $I$ is the ideal consisting of all finite subsets of $H$.

## References

[1] G. Birkhoff, 'Lattice theory', *Amer. Math. Soc. Colloq. Publ.* 25, (1940, 1948, 1967).

[2] G. Grätzer, 'A generalization of Stone's representation theorem for Boolean algebras', *Duke Math. J.* 30 (1963), 469—474.

[3] T. P. Speed, 'On Stone lattices', *Journ. Aust. Math. Soc.* 9 (1969), 297—307.

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