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THE THREE-ARC AND THREE-SEPARATED-ARC PROPERTIES OF MEROMORPHIC FUNCTIONS

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Let Γ be the unit circle and D be the open unit disk in the complex plane, and denote the Riemann sphere by Ω. Suppose that $f(z)$ is a meromorphic function in D, and that $\zeta \in \Gamma$. The principal cluster set of f at ζ is the set

$$\Pi(f,\zeta) = \bigcap_{\Lambda} C_{\Lambda}(f,\zeta) ,$$

where Λ ranges over all arcs at ζ; the chordal principal cluster set of f at ζ is the set

$$\Pi_{\chi}(f,\zeta) = \bigcap_{X} C_{X}(f,\zeta) ,$$

where X ranges over all chords at ζ; and we define the set

$$\Delta(f,\zeta) = \Pi_{\chi}(f,\zeta) - \Pi(f,\zeta) ,$$

where it is clear that $\Pi(f,\zeta) \subseteq \Pi_{\chi}(f,\zeta)$. We say that f has the three-arc property at ζ, if there exist three arcs $\Lambda_1, \Lambda_2, \Lambda_3$ at ζ such that

$$C_{\Lambda_1}(f,\zeta) \cap C_{\Lambda_2}(f,\zeta) \cap C_{\Lambda_3}(f,\zeta) = \phi ;$$

if, moreover, the three arcs can be taken to be mutually exclusive, we say that f has the three-separated-arc property at ζ; and if the three arcs can be taken to be chords at ζ, we say that f has the three-chord property at ζ.

Gresser [5, p. 145, Theorem 2] has shown that there exists a meromorphic function in D that has the three-chord property (and hence the three-separated-arc property) at each point of a perfect subset of Γ. Belna [4, p. 220] has raised the question of the existence of a meromorphic function in D that has the three-separated-arc property at each point of Γ.

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THEOREM 1. *There exists a normal meromorphic function $f(z)$ in D that has the three-separated-arc property at every point of Γ.*

Proof. Let $f(z)$ be a Schwarzian triangle-function in D whose fundamental triangle has angles $\pi/7, \pi/3, \pi/2$, and let its system of triangles be that displayed in [6, p. 437, Fig. 122]. A vertex of the figure is a common vertex of either fourteen, six, or four of the figure's triangles; we assume that f has the value $\infty, 0, 1$ at a vertex of the first, second, third kind, respectively. Then, as is well known, f is meromorphic and normal in D.

By a 14-star, 6-star, 4-star of the figure, we shall mean the union of the fourteen, six, four triangles having a common vertex of the first, second, third kind, respectively, which vertex will be called the center of the n-star; we consider the interior points as well as the boundary points as belonging to the triangle in question. The hyperbolic diameter of an n-star is independent of its center, and will be denoted by d_n $(n = 14, 6, 4)$. As z describes the frontier of an n-star, $f(z)$ varies over the closed interval I_1 of the real axis between 0 and 1 if $n = 14$, the closed interval I_2 of the positive real axis between 1 and ∞ if $n = 6$, and the closed interval I_3 of the nonpositive real axis between 0 and ∞ if $n = 4$.

Now suppose that ζ is any point of Γ. Denote the radius at ζ by R and the diameter with end points ζ and $-\zeta$ by M. Let G_1 be the lens-shaped region between two hypercycles H_2 and H_3 through ζ and $-\zeta$, one on either side of M and each the same hyperbolic distance greater than d_{14} from M. Denote by G_2, G_3 the remaining two regions into which the hypercycles divide D, where H_2, H_3 is part of the frontier of G_2, G_3, respectively. Take a hypercycle H_2' in G_2 through ζ and $-\zeta$ such that its hyperbolic distance from H_2 is greater than d_6, and let H_3' be a hypercycle in G_3 through ζ and $-\zeta$ such that its hyperbolic distance from H_3 is greater than d_4.

Consider the union U_1 of those 14-stars that intersect the radius R at ζ, the union U_2 of those 6-stars that intersect the arc at ζ that extends along H_2' from some point of H_2' to ζ, and the union U_3 of those 4-stars that intersect the arc at ζ that extends along H_3' from some point of H_3' to ζ. For each (fixed) $n = 14, 6, 4$, it is evident that D is the union of all the n-stars of the figure, and any two of these n-stars are either mutually exclusive or have only frontier points in common. Consequently each of the sets U_1, U_2, U_3 is connected, and there are arcs $\Lambda_1, \Lambda_2, \Lambda_3$ at ζ which lie on the frontiers of the respective sets U_1, U_2, U_3. Since $U_j \subset G_j$ $(j = 1, 2, 3)$, the arcs $\Lambda_1, \Lambda_2, \Lambda_3$ are mutually exclusive. Moreover $C_{\Lambda_j}(f, \zeta) \subsetneqq I_j$ $(j = 1, 2, 3)$, and so f has the three-separated-arc property at ζ. In view of the fact that ζ was an arbitrarily chosen point of Γ, the proof of the theorem is complete.

Although there exists [3, p. 31, Theorem 4] a function of bounded characteristic in D that has the three-arc property at each point of a perfect subset of Γ, the general behavior of a function of bounded characteristic relative to the three-arc property is markedly different from that of a normal function.

THEOREM 2. *The set of points on Γ at which a meromorphic function $f(z)$ of bounded characteristic in D has the three-arc property is of measure zero.*

Proof. According to [7, p. 208], f has a radial limit at almost every point of Γ. If $\zeta \in \Gamma$, and if f has the radial limit ω at ζ, then unless ζ is an ambiguous point of f, we have $\omega \in C_\Lambda(f, \zeta)$ for every arc Λ at ζ, and consequently f cannot have the three-arc property at ζ. Since f has at most enumerably many ambiguous points on Γ [1, p. 380, Theorem 2], the theorem is proved.

The following question was raised in [3, p. 32, Question 4]:

"Let f be of class (A) in D; that is, let f be bounded and holomorphic in D, and let the radial limit of f have modulus 1 at almost all points $e^{i\theta}$. Does f have the three-arc property at all (or almost all) singular points $e^{i\theta}$ that are not isolated singularities?"

Theorem 2 answers this question in the negative, because if we take f to be a Blaschke product such that the set of limit points of its zeros is Γ, then every point of Γ is a nonisolated singularity of f, but f has the three-arc property almost nowhere on Γ.

There exists [3, p. 33] a normal holomorphic function f in D such that $\Delta(f, \zeta) = \Omega$ for almost all $\zeta \in \Gamma$; and there exists [2] a meromorphic function f in D such that $\Delta(f, \zeta) = \Omega$ for all $\zeta \in \Gamma$. The proof of Theorem 2 shows, however, that for functions of bounded characteristic we have the following

COROLLARY. *If $f(z)$ is a meromorphic function of bounded characteristic in D, then $\Delta(f, \zeta) = \phi$ for almost all $\zeta \in \Gamma$.*

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