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ON GRAPHS OF PRIME VALENCY ADMITTING A SOLVABLE ARC-TRANSITIVE GROUP

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Abstract

Let X be a simple, connected, p-valent, G-arc-transitive graph, where the subgroup $G \leq \mathrm{Aut}(X)$ is solvable and $p \geq 3$ is a prime. We prove that X is a regular cover over one of the three possible types of graphs with semi-edges. This enables short proofs of the facts that G is at most 3-arc-transitive on X and that its edge kernel is trivial. For pentavalent graphs, two further applications are given: all G-basic pentavalent graphs admitting a solvable arc-transitive group are constructed and an example of a non-Cayley graph of this kind is presented.

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1. Introduction

A large part of the systematic study of symmetric graphs with prime valency is rooted in the fundamental results of Weiss [12, 13], who applied group theoretical methods to investigate their local properties. Recently, general results on the order of arc-stabilisers that extend his work were obtained by Potočnik *et al.* (see [8, 10, 11]). On the other hand, Lorimer [3] observed that a normal quotient can reduce a symmetric graph to a smaller one with similar properties.

In this paper, we use the concept of generalised graphs with *semi-edges* to prove the following reduction theorem. (See Section 2 for definitions and Section 3 for the proof.)

THEOREM 1.1. *Let X be a simple, connected, p-valent, G-arc-transitive graph, where the group $G \leq \mathrm{Aut}(X)$ is solvable and $p \geq 3$ is a prime. Then there exists a normal subgroup $K \leq G$ such that the following hold.*

(i) *The quotient projection $X \to X/K$ is a regular covering projection and the quotient group $G/K \leq \mathrm{Aut}(X/K)$ is arc-transitive on X/K.*

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(ii) *The quotient graph X/K is isomorphic to the dipole* Dip_p, *the semistar* Semi_p *or the complete bipartite graph* $K_{p,p}$.

Moreover, the quotient projection $X \to X/K$ is a composition of regular elementary abelian covering projections.

While this result can be seen as a refinement of the work by Lorimer [3, Theorem 9], we want to emphasise the point of view that graphs with semi-edges have certain advantages over the usual graphs when studying symmetry, as they form a class of graphs that is closed under taking quotients for some equivalence relation. To illustrate this, we present several applications of Theorem 1.1.

In the corollaries that follow, X is a finite connected simple p-valent graph, $p \geq 3$ is a prime and $G \leq \mathrm{Aut}(X)$ is a solvable group. Cubic graphs admitting a solvable edge-transitive group were first studied by Malnič *et al.* [6], where, among other results, they also proved that the action of a solvable group of automorphisms is at most 3-arc-transitive. We show that this can be generalised to any prime $p \geq 3$.

Corollary 1.2. *Suppose that X is (G, s)-arc-transitive. Then $s \leq 3$ and $s = 3$ is possible only if X is a regular cover of $K_{p,p}$.*

Next, recall that the edge kernel $G_e^{[1]}$ consists of all automorphisms from $G \leq \mathrm{Aut}(X)$ that fix the edge $e \in E(X)$ and all its adjacent edges. Using Theorem 1.1, an elementary proof of the following result is easily derived.

Corollary 1.3. *Suppose that X is (G, s)-arc-transitive with $s \geq 1$. Then the edge kernel $G_e^{[1]}$ is trivial.*

We note that for $p \geq 5$, Corollary 1.3 and the first part of Corollary 1.2 are actually contained in [13, Theorem] under a more general setting with the local group $G_v^{X(v)}$ primitive and containing a regular abelian normal subgroup. However, under the assumption that G is solvable, our method yields an elegant proof that is interesting in its own right.

As another easy application, the following sufficient condition for X to be a Cayley graph is obtained.

Corollary 1.4. *Suppose that X is (G, s)-arc-transitive with $s \geq 1$ and X is not bipartite. Then X is a Cayley graph over some regular subgroup of G and $s \leq 2$.*

The paper is organised as follows. The proofs of Theorem 1.1 and the corollaries are given in Section 3. In Section 4, we focus on the case $p = 5$. Recently, the class of pentavalent symmetric graphs has been a focus of interest. Results on vertex stabilisers by Tutte and later Weiss have been extended to a complete classification of all primitive $(5, 2)$-amalgams by Morgan [7]. In [2], a simple graph X is called *G-basic* if it is G-arc-transitive and G has no normal subgroup K such that the quotient projection $X \to X/K$ is a regular covering projection of connected simple graphs. The G-basic cubic graphs with a solvable group G were essentially determined in [6] and further

TABLE 1. The irreducible $\mathbb{Z}_q^r$-voltage assignments ζ on Dip_5, which are H-admissible for some maximal s-arc-transitive solvable group $H \le \mathrm{Aut}(\mathrm{Dip}_5)$.

Condition	r	$\zeta(x_1)$	$\zeta(x_2)$	$\zeta(x_3)$	$\zeta(x_4)$	s	H
$q \equiv 1 \pmod 5$, $\xi^5 = 1 \in \mathbb{Z}_q, \xi \neq 1$	1	(1)	$(1+\xi)$	$(1+\xi+\xi^2)$	$(1+\xi+\xi^2+\xi^3)$	1	C_{10}
$q \equiv -1 \pmod 5$, $\eta^2+\eta = 1 \in \mathbb{Z}_q$	2	$\begin{pmatrix}1+\eta\\1\end{pmatrix}$	$\begin{pmatrix}1+\eta\\1+\eta\end{pmatrix}$	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}-1\\0\end{pmatrix}$	1	D_{10}
$q \equiv \pm 2 \pmod 5$	4	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	2	$F_{20} \times C_2$

studied by Feng *et al.* [2]. We obtain a similar classification for the pentavalent case by constructing elementary abelian covers of quotients from Theorem 1.1.

THEOREM 1.5. *Let X be a connected, simple, pentavalent graph admitting a solvable arc-transitive group $G \le \mathrm{Aut}(X)$. Suppose that X is G-basic. Then one of the following holds.*

(i) *X is isomorphic to the complete bipartite graph $K_{5,5}$.*
(ii) *X is isomorphic to the Clebsch graph Cl_{16}.*
(iii) *X is isomorphic to a $\mathbb{Z}_q^r$-cover of Dip_5, $q \neq 5$ a prime, defined by the voltage assignment $\zeta : D(\mathrm{Dip}_5) \to \mathbb{Z}_q^r$, where $\zeta(x_0) = 0$ and the values of $\zeta(x_i)$, $i = 1, 2, 3, 4$, are as shown in Table 1.*

The proof is given in Section 4. As a final application, we show that Corollary 1.4 does not generalise to bipartite graphs by employing computational tools to construct a specific $\mathbb{Z}_2^8$-cover of $K_{5,5}$.

PROPOSITION 1.6. *There exists a connected simple pentavalent bipartite graph X on 2560 vertices such that $\mathrm{Aut}(X)$ is solvable and arc-transitive and X is not a Cayley graph.*

2. Preliminaries

Let X be a finite, simple, connected graph and let $\mathrm{Aut}(X)$ denote its automorphism group. An *s-arc* of a graph is a sequence of vertices $v_0, \ldots, v_s$ such that any two consecutive vertices are adjacent and $v_{i-1} \neq v_{i+1}$. In particular, a 0-arc is just a vertex and a 1-arc is simply an arc. A graph is called *(G, s)-arc-transitive* if G is transitive on the set of s-arcs, and it is *locally (G, s)-arc-transitive* if, for any vertex $v \in V(X)$, the stabiliser G_v is transitive on the set of s-arcs originating at v. A subgroup $G \le \mathrm{Aut}(X)$ is called *vertex-*, *edge-* or *arc-transitive* if the action of G is transitive on the set of vertices $V(X)$, edges $E(X)$ or arcs $A(X)$, respectively.

For a vertex $v \in V(X)$, we denote by $N(v)$ the set of vertices adjacent to v. If G is (locally) arc-transitive, the vertex stabiliser G_v acts transitively on the neighbourhood

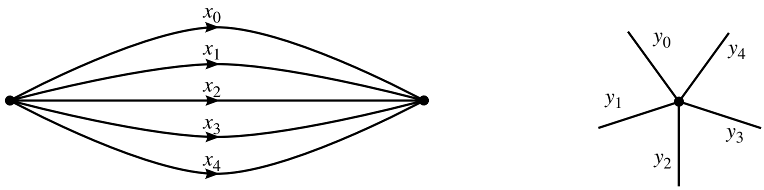

FIGURE 1. The 5-dipole graph Dip_5 and the 5-semistar graph Semi_5.

$N(v)$. The kernel of this action is called the *vertex kernel* and denoted $G_v^{[1]}$; it consists of all group elements that fix v and all its neighbours pointwise. Similarly, for an edge $e = \{u, v\} \in E(X)$ we define the *edge kernel* $G_e^{[1]}$ as the kernel of the action of the edge stabiliser G_e on the neighbours of u and v, that is, $G_e = G_u^{[1]} \cap G_v^{[1]}$.

In the rest of this section, we outline some essential definitions regarding graphs with possible *semi-edges*, together with basic results about their covering projections. For further details of these topics, we refer the reader to [4, 5].

A *graph* is an ordered 4-tuple $X = (D, V; \mathrm{ini}, ^{-1})$, where $V \neq \emptyset$ and D are disjoint finite sets of *vertices* and *darts*, the mapping $\mathrm{ini} : D \to V$ assigns to each dart its *initial vertex* and the mapping $^{-1} : D \to D$ interchanges each dart x with its *inverse dart* x^{-1}. The *edge set* $E(X)$ is then obtained by defining *edges* as orbits of the inverse mapping $^{-1}$ on darts. In this way, the two element orbits $\{x, x^{-1}\}$, $x \neq x^{-1}$, represent either the usual edges (also called *links*) if $\mathrm{ini}(x) \neq \mathrm{ini}(x^{-1})$ or *loops* if $\mathrm{ini}(x) = \mathrm{ini}(x^{-1})$, while a single element orbit $\{x = x^{-1}\}$ is called a *semi-edge*.

It is not difficult to see that the above definition and the usual definition of a graph are equivalent for graphs $X = (V, E)$ without semi-edges: letting $D = \{(u, v) \mid u \sim v;\ uv \in V\}$, we define $(u, v)^{-1} = (v, u)$ and $\mathrm{ini}(u, v) = u$ to obtain $X = (V, D; \mathrm{ini}, ^{-1})$. A graph is called *simple* if it has no semi-edges, loops or parallel edges (that is, different edges with the same initial vertices).

EXAMPLE 2.1. A *p-dipole* Dip_p is the graph with two vertices, connected by p parallel edges, that is, $V = \{u, v\}$, $D = \{x_1^{\pm 1}, \ldots, x_p^{\pm 1}\}$, $x_i^{-1} \neq x_i$, $\mathrm{ini}(x_i) = u$ and $\mathrm{ini}(x_i^{-1}) = v$ for all i. A *p-semistar* Semi_p is the graph with a single vertex and p semi-edges; hence, $D = \{y_1, \ldots, y_p\}$, where $y_i^{-1} = y_i$ and $\mathrm{ini}(y_i) = v$ for all i. See Figure 1 for the case $p = 5$.

All the usual notions are naturally extended to graphs with semi-edges. The *degree* or *valency* of a vertex is the size of its dart neighbourhood, which we define here as $N_D(v) = \{x \in D(X) \mid \mathrm{ini}(x) = v\}$. An *s-arc* is a sequence of vertices and darts $v_0, x_0, \ldots, x_{s-1}, v_s$ such that $\mathrm{ini}(x_k) = v_k$ and $\mathrm{ini}(x_k^{-1}) = v_{k+1}$, while $x_{k+1} \neq x_k^{-1}$ for all relevant k. In particular, 1-arcs can be identified with darts; hence, arc-transitive is the same as dart-transitive for graphs without semi-edges.

A *graph homomorphism* $f : X \to X'$ is a function that maps $V(X)$ to $V(X')$, and $D(X)$ to $D(X')$, such that it commutes with ini and $^{-1}$. For any subgroup $G \leq \mathrm{Aut}(X)$, the natural quotient graph X/G is defined by taking vertex orbits Gv as vertices

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of X/G, dart orbits Gx as darts of X/G and by defining $\mathrm{ini}(Gx) = G(\mathrm{ini}(x))$ and $(Gx)^{-1} = G(x^{-1})$.

A graph epimorphism $f : \tilde{X} \to X$ is called a *covering projection* if it is locally bijective, that is, the neighbourhood $N_D(\tilde{u})$ of $\tilde{u} \in V(\tilde{X})$ is mapped bijectively to the neighbourhood $N_D(u)$ of $u = f\tilde{u} \in V(X)$. The graph $\tilde{X}$ is called a *covering graph* of the *base graph* X, and the set of pre-images $f^{-1}(v) \subseteq V(\tilde{X})$ and $f^{-1}(x) \subseteq D(\tilde{X})$ are called a *vertex fibre* and a *dart fibre*, respectively. The subgroup $K \leq \mathrm{Aut}(\tilde{X})$ of all automorphisms fixing each vertex fibre and dart fibre setwise is called the *group of covering transformations*, and $\tilde{X}$ is called a K-cover. If the covering graph $\tilde{X}$ is connected, the action of K is semiregular on each fibre. If it is also transitive and hence regular, we call $\tilde{X}$ a *regular cover* and f a *regular covering projection*. Note that in this case, the K-orbits of $V(\tilde{X})$ coincide with vertex fibres, so the quotient graph $\tilde{X}/K$ is isomorphic to X.

Alternatively, regular covers are derived combinatorially by assigning voltages to base graphs. For K an abstract group, a function $\zeta : D(X) \to K$ with property $\zeta(x^{-1}) = \zeta(x)^{-1}$ is called a *K-voltage assignment*; the pair (X, ζ) is called a *voltage graph* and the values of ζ are *voltages*. The *derived graph* $X \times_\zeta K$ is defined by taking the vertex set $V(X) \times K$, dart set $D(X) \times K$, $\mathrm{ini}(x, g) = (\mathrm{ini}(x), g)$ and $(x, g)^{-1} = (x^{-1}, g\zeta(x))$. Note that with our definition of graphs, the voltages on semi-edges must necessarily be involutions. The derived graph is connected whenever $\mathrm{Im}\,\zeta$ generates K, and every regular K-covering projection $f : \tilde{X} \to X$ is in fact equivalent to some *voltage projection* $f_\zeta : X \times_\zeta K \to X$ (where ζ can be further assumed to be trivial on the darts of an arbitrarily chosen spanning tree of X).

An automorphism $\alpha \in \mathrm{Aut}(X)$ *lifts along* a covering projection $f : \tilde{X} \to X$ if there exists an automorphism $\tilde{\alpha} \in \mathrm{Aut}(\tilde{X})$ such that $f\tilde{\alpha} = \alpha f$. The subgroup $G \leq \mathrm{Aut}(X)$ lifts if every $\alpha \in G$ lifts, in which case we say that f is *G-admissible*. Similarly, we say that $\tilde{\alpha} \in \mathrm{Aut}(\tilde{X})$ or $\tilde{G} \leq \mathrm{Aut}(\tilde{X})$ *projects along* f. Note that $\tilde{G}$ projects if and only if the vertex fibres are $\tilde{G}$-invariant, and that $\tilde{G}$ is s-arc-transitive if and only if G is.

The problem of characterising all voltage assignments ζ such that $\alpha \in \mathrm{Aut}(X)$ lifts along a given voltage projection $f_\zeta : X \times_\zeta K \to X$, where K is elementary abelian, was extensively studied in [5]. We summarise the results as follows. Consider the cycle space of X as a β-dimensional vector space over some prime field $\mathbb{Z}_q$, where $\beta = |E(X)| - |V(X)| + 1$ is the Betti number of the graph. A basis $\mathcal{B} = \{x_1, \ldots, x_\beta\}$ of the cycle space is determined by choosing a single dart from each edge of $X \setminus T$, where T is a spanning tree of X. Then every $\alpha \in \mathrm{Aut}(X)$ induces an invertible linear transformation, which we represent in this basis by a matrix $\alpha^\# \in \mathbb{Z}_q^{\beta\times\beta}$.

Theorem 2.2 [5, Theorem 6.2 and Corollary 6.3]. *Let X be a connected graph and let $\alpha \in \mathrm{Aut}(X)$. Fixing a spanning tree T and a basis $\mathcal{B} = \{x_1, \ldots, x_\beta\} \subseteq D(X \setminus T)$ yields a matrix $\alpha^\# \in \mathbb{Z}_q^{\beta\times\beta}$. Suppose that $\zeta : D(X) \to \mathbb{Z}_q^d$ is a voltage assignment, which is trivial on $D(T)$, and denote by $M_\zeta \in \mathbb{Z}_q^{d\times\beta}$ the matrix with columns $\zeta(x_i)$. Then α lifts along the voltage projection $f_\zeta : X \times_\zeta \mathbb{Z}_q^d \to X$ if and only if the rows of M_ζ form a basis of an $(\alpha^\#)^t$-invariant d-dimensional subspace V_ζ of $\mathbb{Z}_q^\beta$. If ζ' is another*

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voltage assignment satisfying these conditions, then $f_{\zeta'}$ is equivalent to f_ζ if and only if $V_\zeta = V_{\zeta'}$. Moreover, two projections f_ζ and $f_{\zeta'}$ (and hence the corresponding covering graphs) are isomorphic if and only if there exists an automorphism $\varphi \in \mathrm{Aut}(X)$ such that the matrix $(\varphi^\#)^t$ maps V_ζ onto $V_{\zeta'}$.

For G a group and an inverse-closed subset $S = S^{-1} \subseteq G \setminus \{1\}$ that generates G, the *Cayley graph* $\mathrm{Cay}(G, S)$ is defined by the vertex set $V(X) = G$, dart set $D(X) = G \times S$ and functions $\mathrm{ini}(g, s) = g$ and $(g, s)^{-1} = (gs, s^{-1})$. Again, this definition coincides with the usual one for a graph without semi-edges, where the edges are given by the adjacency relation $g \sim sg$ for $g \in G, s \in S$. Moreover, every Cayley graph $\mathrm{Cay}(G, S)$ is a regular G-cover of a single vertex graph with dart set $D(X) = S$ and, *vice versa*, every regular cover of a single vertex graph is a Cayley graph. Also, the Sabidussi theorem holds: a graph X is isomorphic to some Cayley graph $\mathrm{Cay}(G, S)$ if and only if $\mathrm{Aut}(X)$ has a vertex-regular subgroup isomorphic to G.

3. Proof of main theorem and corollaries

Proof of Theorem 1.1. Temporarily, we denote a minimal nontrivial normal subgroup in G by K. Since G is solvable, K is elementary abelian and hence $K \cong \mathbb{Z}_q^r$ for some prime q and integer r. We study the quotient graphs X/K according as the number m of K-orbits on $V(X)$ is 1, 2 or at least 3.

Case 1: $m \geq 3$. Since K is normal and G is arc-transitive, the K-orbits of vertices are equal in size and are also blocks of imprimitivity for the action of G. Hence, no two vertices in the same K-orbit are adjacent. This implies that all nontrivial intersections of K-orbits with the neighbourhood $N(v)$ of some vertex v are of the same size. As p is prime, it follows that each K-orbit contains at most one element from $N(v)$. Hence, X/K is a p-valent graph and there are at least $p + 1$ orbits. Moreover, $\mathrm{Stab}_K(v) = 1$. Indeed, if $av = v$ for some $a \in K$ and $u \in N(v)$, then $au \in N(v)$. But u and au are in the same K-orbit (of size 1); hence, $au = u$ and therefore $a = 1$ by connectedness. Thus, $X \to X/K$ is a regular covering projection of X onto a p-valent simple graph with G/K being a solvable and arc-transitive group of automorphisms of X/K. By induction, the new graph is further reduced until at most two vertex orbits remain.

Case 2: $m = 2$. As in the previous case, no two vertices in the same K-orbit are adjacent and hence X is bipartite. Note that since K is abelian, an element of K fixing one vertex fixes its full K-orbit: $av = v$ for $a \in K$ implies $au = a(kv) = kav = kv = u$ for all $u = kv \in Kv$. Suppose first that $\mathrm{Stab}_K(v)$ acts transitively on $N(v)$. If $u \in N(v)$ and $w \sim u$, then $N(w) = N(v)$ since any element fixing v fixes w as well. By connectedness, we have $X \cong K_{p,p}$. Now suppose that $\mathrm{Stab}_K(v)$ is not transitive on $N(v)$. Then any element $a \in \mathrm{Stab}_K(v)$ has at least two orbits on $N(v)$. Denote the size of the minimal orbit of a by s. Then a^s fixes the elements of this orbit. Since K is abelian, a^s fixes the full K-orbit and hence $a^s = 1$. But then all orbits of a have equal size s, so s divides p and hence $s = 1$. Therefore, $\mathrm{Stab}_K(v)$ is trivial and $X \to X/K$ is a regular covering projection onto a p-dipole.

Case 3: $m = 1$. Since K is transitive and also abelian, its action is regular. Hence, $X \to X/K$ is a regular covering projection onto a p-valent graph on a single vertex (observe that X is a Cayley graph in this case). Since X/K is arc-transitive and p is an odd prime, X/K has no loops; therefore, it is isomorphic to a p-semistar. As $K \cong \mathbb{Z}_q^r$, this also implies $q = 2$ and $r \leq p$, since the voltages on semi-edges are involutions and also generate K in this case.

Thus, a sequence of elementary abelian covering projections is obtained in each case. Their composition is a regular covering projection. We may denote its group of covering transformations by K. □

Proof of Corollary 1.2. By Theorem 1.1, X/K is isomorphic to one of the three possible graphs. If $X/K \cong K_{p,p}$, which is 3-arc-transitive, then G/K and hence G is at most 3-arc-transitive, with $s = 3$ possibly attained.

If $X/K \cong \mathrm{Dip}_p$, we may assume that there is a sequence of regular covering projections $X \to X' \to \mathrm{Dip}_p$ such that X' is a simple graph, $X' \to \mathrm{Dip}_p$ is an elementary abelian covering projection and the projection G' of G along $X \to X'$ is the lift of G/K along $X' \to X/K$. Moreover, G' is s-arc-transitive whenever G is. Hence, X' is derived by assigning voltages $\zeta(x_i) \in \mathbb{Z}_q^r$ to darts $x_0, \ldots, x_{p-1}$ of Dip_p. We may assume that $\zeta(x_0) = 0$ and $\langle \zeta(x_i), i = 1, \ldots, p-1 \rangle$ generate $\mathbb{Z}_q^r$. As X' is simple, we may also assume there are two voltages, say $\zeta(x_1)$ and $\zeta(x_2)$, such that $\langle \zeta(x_1) \rangle \neq \langle \zeta(x_2) \rangle$. Hence, the 3-walks $x_0x_1x_2$ and $x_0x_1x_0$ in Dip_p lift to 3-arcs in X'. But if G' were 3-arc-transitive on X' these 3-arcs would project to isomorphic walks, which is a contradiction. Hence, $s \leq 2$.

In similar fashion, we see that if $X/K \cong \mathrm{Semi}_p$, we have $s \leq 2$. □

Corollary 1.3 is obtained from Theorem 1.1 by observing that the edge kernel is preserved by regular quotient projection, as described in Proposition 3.1 below. Note that for graphs with semi-edges, the vertex kernel $G_v^{[1]}$ is defined as the subgroup of elements of G that fix all the darts $x \in D(X)$ with $\mathrm{ini}(x) = v$, and the edge kernel $G_e^{[1]}$ as the intersection $G_v^{[1]} \cap G_u^{[1]}$, where $e = \{x, x^{-1}\}$, $v = \mathrm{ini}(x)$ and $u = \mathrm{ini}(x^{-1})$. For graphs without semi-edges, these definitions coincide with the usual ones.

Proposition 3.1. *Let X be a graph and $G \leq \mathrm{Aut}(X)$. Suppose that the normal subgroup $K \triangleleft G$ is semiregular on $V(X)$, whence the quotient projection $X \to X/K$ is a regular covering projection with $G/K \leq \mathrm{Aut}(X/K)$. Then the vertex and edge kernels of G are preserved by the projection, that is, $G_v^{[1]} \cong (G/K)_{Kv}^{[1]}$ and $G_e^{[1]} \cong (G/K)_{Ke}^{[1]}$.*

Proof. Fix $v \in V(X)$. For $g \in G_v^{[1]}$, we have $gx = x$ for all darts x with $\mathrm{ini}(x) = v$. Denote the induced automorphism by $gK \in \mathrm{Aut}(X/K)$. It follows that $(gK)(Kx) = Kgx = Kx$, so $gK \in (G/K)_{Kv}^{[1]}$. Since K is semiregular, the induced mapping $G_v^{[1]} \to (G/K)_{Kv}^{[1]}$ is also one-to-one, as $gK = hK$ implies $gh^{-1} = 1 \in K \cap G_v^{[1]}$. On the other hand, let $gK \in (G/K)_{Kv}^{[1]}$ for some $g \in G$. For any dart x with $\mathrm{ini}(x) = v$, we have $(gK)(Kx) = Kx$, implying that $(gk)x = x$ for some $k \in K$. Thus, gk fixes v. If another dart y with $\mathrm{ini}(y) = v$ is fixed by gk' for some k', then $(gk)^{-1}(gk') = k^{-1}k' \in K$ fixes v and so $k = k'$ by semiregularity. Hence, $gk \in G_v^{[1]}$ projects to gK, so the mapping is onto.

The isomorphism $G_e^{[1]} \cong (G/K)_{Ke}^{[1]}$ then follows directly from the definition of $G_e^{[1]}$ as the intersection of two (possibly equal) vertex kernels. □

Proof of Corollary 1.3. By Theorem 1.1, X/K is isomorphic to a p-dipole, a p-semistar or a $K_{p,p}$. Since any subgroup of $\mathrm{Aut}(X/K)$ has trivial edge kernel in these three cases, by Proposition 3.1 the same is true for X and G. □

Proof of Corollary 1.4. By Theorem 1.1, if X is not bipartite, then it is a regular cover over a semistar and hence a Cayley graph over some regular subgroup of G. By the proof of Corollary 1.2, $s \leq 2$. □

Remark 3.2. Although several regular covering projections $K_{p,p} \to \mathrm{Dip}_p$ exist, a subgroup $G \leq \mathrm{Aut}(K_{p,p})$ only projects to Dip_p by taking a quotient relative to its semiregular normal subgroup of order p. Similarly, a subgroup of $\mathrm{Aut}(\mathrm{Dip}_p)$ only projects to Semi_p by taking a quotient relative to its semiregular normal subgroup of order two. This is not possible for all choices of G. As an example, all conjugacy classes of arc-transitive and solvable groups $G \leq \mathrm{Aut}(X)$ of the three base graphs for the case $p = 5$ are displayed in Table 2 (computed by Magma [1]). The good choices for $G \leq \mathrm{Aut}(K_{5,5})$ to project to Dip_5 are 1, 2, 3 and 7 (see the last column), but only in cases 1 and 7 do the respective groups project further to Semi_5. Hence, all three cases of Theorem 1.1 along with the information on the respective group action are necessary for a complete reconstruction of all possible graphs. We investigate this phenomenon in further detail for pentavalent graphs in Section 4.

4. Pentavalent case

Proof of Theorem 1.5. By Theorem 1.1, all p-valent graphs admitting a solvable arc-transitive group are obtained by a sequence of elementary abelian covers of one of the three small graphs Y. Moreover, the respective covering projections $X \to Y$ must admit a lift of some solvable arc-transitive group H of the base graph Y. Theorem 2.2 describes a general method for characterising all voltage assignments ζ with values in the elementary abelian group $\mathbb{Z}_q^r$ such that an automorphism $\alpha \in \mathrm{Aut}(Y)$ lifts along the derived covering projection $f : Y \times_\zeta \mathbb{Z}_q^r \to Y$ (see also [5]).

We construct the G-basic simple graphs X by applying this method to pairs (Y, H), where Y is isomorphic to Dip_5, Semi_5 or $K_{5,5}$, and $H \leq \mathrm{Aut}(Y)$ is a solvable arc-transitive group. As the graph $K_{5,5}$ is simple, part (i) of Theorem 1.5 is obvious.

Case $Y = \mathrm{Semi}_5$. In order to obtain a connected simple covering graph X, different voltages of order two must be assigned to each semi-edge; hence, the voltage group is $\mathbb{Z}_2^r$ for some $r \leq 5$. Moreover, some automorphism $\rho \in \mathrm{Aut}(Y)$ of order five must lift along the respective covering projection in order to obtain an arc-transitive cover. We may assume that $\rho = (12345)$ is a cyclic rotation of semi-edges and the corresponding

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TABLE 2. The conjugacy classes of all s-arc-transitive and solvable subgroups $G \leq \mathrm{Aut}(X)$, where X is isomorphic to $K_{5,5}$, Dip_5 or Semi_5, together with their respective vertex-, edge- and arc-stabilisers that constitute the primitive (5, 2)-amalgams (see [7]).[1]

			$G \leq \mathrm{Aut}(K_{5,5})$			$\exists H \leq G$	$\exists C_5 \lhd G$
#	$\|G\|$	G_v	G_e	G_a	s	vert. reg.	semireg.
1	50	C_5	C_2	1	1	Yes	Yes
2	100	D_5	C_2^2	C_2	1	Yes	Yes
3	100	D_5	C_4	C_2	1	No	Yes
4	200	D_{10},	D_4	C_2^2	1	Yes	No
5	200	F_{20}	Q	C_4	2	No	No
6	200	F_{20}	D_4	C_4	2	Yes	No
7	200	F_{20}	$C_2 \times C_4$	C_4	2	Yes	Yes
8	200	F_{20},	C_8	C_4	2	No	No
9	400	$C_2 \times F_{20}$	N_{16}	$C_2 \times C_4$	2	Yes	No
10	400	$C_2 \times F_{20}$	M_{16}	$C_2 \times C_4$	2	No	No
11	800	$C_4 \times F_{20}$	$C_4 \wr C_2$	$C_4 \times C_4$	3	Yes	No
			$G \leq \mathrm{Aut}(\mathrm{Dip}_5)$			$\exists H \leq G$	$\exists C_2 \lhd G$
#	G	G_v	G_e	G_a	s	vert. reg.	semireg.
1	C_{10}	C_5	C_2	1	1	Yes	Yes
2	D_5	C_5	C_2	1	1	Yes	No
3	F_{20}	D_5	C_4	C_2	1	No	No
4	D_{10}	D_5	C_2^2	C_2	1	Yes	Yes
5	$F_{20} \times C_2$	F_{20}	$C_2 \times C_4$	C_4	2	Yes	Yes
			$G \leq \mathrm{Aut}(\mathrm{Semi}_5)$			$\exists H \leq G$	
#	G	G_v	G_e	G_a	s	vert. reg.	
1	C_5	C_5	1	1	1	Yes	
2	D_5	D_5	C_2	1	1	Yes	
3	F_{20}	F_{20}	C_4	C_2	1	Yes	

[1]Here v, e, a denote a vertex, edge or arc of graph X, respectively; C_n is the cyclic group of order n, D_n the dihedral group of order $2n$, $F_{20} \cong \mathrm{AGL}(1,5)$ the Frobenius group of order 20 and M_{16} and N_{16} the subgroups of order 16 in Sym(8) generated by $\langle(12345678), (26)(48)\rangle$ and $\langle(1234)(5678), (57)(68), (15)(26)(37)(38)\rangle$, respectively.

matrix is

$$C = (\rho\#)^t = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{Z}_2^{5\times 5}.$$

By Theorem 2.2, the $\langle\rho\rangle$-admissible voltage assignments on Y are obtained from C-invariant subspaces, which, in this case, are exactly the binary cyclic codes of length five. The minimal invariant subspaces correspond to the two irreducible divisors

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$\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$ and $\lambda + 1$ of the polynomial $\lambda^5 + 1 \in \mathbb{Z}_2[\lambda]$. In the first case, the corresponding one-dimensional subspace $\langle (1, 1, 1, 1, 1)^t \rangle$ yields a nonsimple graph Dip_5, as voltage 1 is assigned to each semi-edge. In the second case, the polynomial $\lambda + 1$ corresponds to the invariant subspace determined by the rowspace of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The columns of this matrix then correspond to voltages on semi-edges. The derived graph obtained is the Cayley graph

$$\mathrm{Cay}(\mathbb{Z}_2^4, \{(1, 0, 0, 0), (1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 1, 1), (0, 0, 0, 1)\}).$$

It is easy to check by computer that it is isomorphic to the Clebsch graph of order 16, which is defined by adding edges to antipodal pairs of vertices in the hypercube Q_4 (see [14]). Moreover, the maximal solvable arc-transitive subgroup $\langle \rho, \tau \rangle \cong F_{20}$ of $\mathrm{Aut}(Y)$, where $\tau = (1254)$, also lifts along this covering projection, as the above subspace is also invariant for the respective permutation matrix

$$T = (\tau^{\#})^t = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, by lifting copies of C_5, D_5 or F_{20} in $\mathrm{Aut}(\mathrm{Semi}_5)$, the Clebsch graph admits three conjugacy classes of solvable arc-transitive groups of orders 80, 160 and 320, all of them 1-arc-transitive.

Case $Y = \mathrm{Dip}_5$. We remark that the general construction of all edge-transitive elementary abelian covers of Dip_p is described in [5]. Our construction here follows this general case and sharpens the results. Let $G \leq \mathrm{Aut}(Y)$ be arc-transitive and solvable. By inspecting the groups in Table 2, we may assume that G contains a cyclic permutation $\nu = (x_0 x_1 x_2 x_3 x_4)(x_0^{-1} x_1^{-1} x_2^{-1} x_3^{-1} x_4^{-1})$, where $x_i \in D(Y)$ are darts (Figure 1). Let $B = \{b_1, b_2, b_3, b_4\}$ be the homology basis of Dip_5, where $b_i = x_i x_0^{-1}$ are the fundamental cycles. In this basis, ν corresponds to the matrix

$$V = (\nu^{\#})^t = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{Z}_q^{4\times 4}$$

and the invariant subspaces of its transpose matrix determine the G-admissible covers. These, however, depend on the factorisation of its minimal polynomial $m_V(\lambda) =$

$\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1$ over $\mathbb{Z}_q$, which is

$$m(\lambda) = \begin{cases} (\lambda - 1)^4, & q = 5, \\ (\lambda - \xi)(\lambda - \xi^2)(\lambda - \xi^3)(\lambda - \xi^4), & q \equiv 1 \pmod 5, \\ (\lambda^2 - \eta_1\lambda + 1)(\lambda^2 - \eta_2\lambda + 1), & q \equiv -1 \pmod 5, \\ \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1, & q \equiv \pm 2 \pmod 5. \end{cases}$$

Here $\xi \in \mathbb{Z}_q$ is any primitive fifth root of unity, $q \equiv 1 \pmod 5$, and $\eta_{1,2}$ are the two roots of the equation $\eta^2 + \eta - 1 = 0$; hence, $\eta_{1,2} = \frac{1}{2}(-1 \pm \sqrt{5}) \in \mathbb{Z}_q$, $q \equiv -1 \pmod 5$.

For $q = 5$, the Jordan canonical form of V consists of a single cell, and it is easy to compute that the only V-invariant subspaces are $\operatorname{Ker}(V - I)^j = \langle v_1, \ldots, v_j \rangle$, where v_k is the kth column of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 4 & 1 & 4 & 1 \end{bmatrix} \in \mathbb{Z}_5^{4\times 4}$. The minimal V-invariant subspace is the eigenspace $\langle (1, 2, 3, 4)^t \rangle$. However, the corresponding covering graph obtained by the voltage assignment $\zeta(x_i) = i \in \mathbb{Z}_5$ is $K_{5,5}$, which already appears in our list.

For $q \equiv 1 \pmod 5$, the minimal V-invariant subspaces are one dimensional of the form

$$\operatorname{Ker}(V - \xi I) = \operatorname{Ker} \begin{pmatrix} -1-\xi & 1 & 0 & 0 \\ -1 & -\xi & 1 & 0 \\ -1 & 0 & -\xi & 1 \\ -1 & 0 & 0 & -\xi \end{pmatrix} = \langle (1, 1+\xi, 1+\xi+\xi^2, 1+\xi+\xi^2+\xi^3)^t \rangle.$$

Note that if ξ, ξ' are two different nontrivial roots of $\xi^5 - 1$, the respective covering projections give isomorphic covers. To see this, suppose without loss of generality that $\xi' = \xi^2$. The automorphism $\psi \in \operatorname{Aut}(\mathrm{Dip}_5)$, defined by $\psi(x_i) = x_{3i \pmod 5}$, corresponds in basis B to the matrix

$$\Psi = (\psi^\#)^t = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathbb{Z}_q^{4\times 4},$$

and a short computation then shows that $\Psi(\operatorname{Ker}(V - \xi I)) = \operatorname{Ker}(V - \xi^2 I)$, which, by Theorem 2.2, implies that the respective covers are isomorphic.

Similarly, for $q \equiv -1 \pmod 5$, the minimal V-invariant subspaces are two dimensional of the form

$$\begin{aligned} \operatorname{Ker}(V^2 - \eta V + I) &= \operatorname{Ker} \begin{pmatrix} \eta+1 & -\eta-1 & 1 & 0 \\ \eta & 0 & -\eta & 1 \\ \eta & -1 & 1 & -\eta \\ 1+\eta & -1 & 0 & 1 \end{pmatrix} \\ &= \langle (1+\eta, 1+\eta, 0, -1)^t, (1, 1+\eta, 1, 0)^t \rangle. \end{aligned}$$

Again we can easily see that the two solutions of $\eta^2 + \eta = 1$ in $\mathbb{Z}_q$ yield isomorphic covers, as Ψ^2 maps the first subspace onto the second one.

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In the remaining case $q \equiv \pm 2 \pmod 5$, the full space $\mathrm{Ker}(V^4 + V^3 + V^2 + V + I) = (\mathbb{Z}_q)^{4\times 1}$ is the only V-invariant subspace.

The bases of minimal V-invariant subspaces determine the voltage assignments in Table 1. By construction, all of these admit a lift of an edge-transitive group $\langle v\rangle$. Now observe that the involution $\sigma = (x_0x_0^{-1})(x_1x_1^{-1})(x_2x_2^{-1})(x_3x_3^{-1})(x_4x_4^{-1}) \in \mathrm{Aut}(Y)$ corresponds to minus the identity matrix $S = -I$. Therefore, all V-invariant subspaces are also $\langle V, S\rangle$-invariant and the respective covering projections are arc-transitive, as $\langle v, \sigma\rangle \cong C_{10}$ always lifts.

In order to determine the maximal arc-transitive solvable subgroup of $\mathrm{Aut}(Y)$ that lifts, we further consider the five conjugacy classes of Table 2. Class 3 is represented by the group $\langle v, \pi\rangle \cong F_{20}$, where $\pi = (x_0x_1^{-1}x_3x_2^{-1})(x_1x_3^{-1}x_2x_0^{-1})(x_4x_4^{-1})$, and the respective matrix is

$$P = (\pi^{\#})^t = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \in \mathbb{Z}_q^{4\times 4}.$$

An easy computation now shows that among the above V-invariant subspaces for $q \neq 5$, only the full space $\mathbb{Z}_q^4$ is P-invariant.

Similarly, class 2 is represented by the group $\langle v, \omega\rangle \cong D_5$, where

$$\omega = (x_0x_0^{-1})(x_1x_4^{-1})(x_2x_3^{1})(x_3x_2^{-1})(x_4x_1^{-1}).$$

The respective matrix is

$$W = (\omega^{\#})^t = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{Z}_q^{4\times 4}.$$

In this case, the two-dimensional subspaces of the case $q \equiv -1 \pmod 5$ are W-invariant, while the one-dimensional subspaces of the case $q \equiv 1 \pmod 5$ are not.

This shows that the homological $\mathbb{Z}_q^4$-cover which is minimal for $q \equiv \pm 2 \pmod 5$ admits the lift of a maximal solvable arc-transitive subgroup $F_{20} \times C_2$ of Dip_5, the two-dimensional covers of case $q \equiv -1 \pmod 5$ admit the lift of $\langle v, \sigma, \omega\rangle \cong D_{10}$, while C_{10} is the maximal solvable arc-transitive group that lifts in the remaining case. □

Remark 4.1. For $q = 2$, the respective $\mathbb{Z}_2^4$-cover of Dip_5 of Theorem 1.5 is the 5-cube Q_5. Observe also that the $\mathbb{Z}_q^r$-covers of Dip_5 in Table 1 are also Cayley graphs, since, by Table 2, the maximal groups $H = C_{10}$, D_{10} and $F_{20} \times C_2 \leq \mathrm{Aut}(\mathrm{Dip}_5)$ that lift along the respective covering projection contain a vertex-regular subgroup. More information on the pairs (X, G) of Theorem 1.5 could be obtained by further inspecting the lifts of the respective groups H. For instance, if $X = \mathrm{Cl}_{16}$, then G is a group of order 80, 160 or 320, obtained by lifting a copy of C_5, D_5 or F_{20} along the respective $\mathbb{Z}_2^4$-covering projection to Semi_5 and so on.

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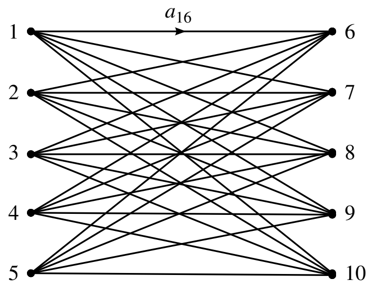

FIGURE 2. A representative graph of class 8 in Table 2.

TABLE 3. Voltage assignment for the graph in Figure 2.

a_{ij}	a_{27}	a_{37}	a_{47}	a_{57}	a_{28}	a_{38}	a_{48}	a_{58}	a_{29}	a_{39}	a_{49}	a_{59}	$a_{2\,10}$	$a_{3\,10}$	$a_{4\,10}$	$a_{5\,10}$
	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
	0	1	0	0	0	0	0	0	1	0	0	1	0	0	1	0
	0	0	1	0	0	0	0	0	1	0	1	1	1	0	0	0
	0	0	0	1	0	0	0	0	1	1	0	1	0	1	0	0
$\zeta(a_{ij})$	0	0	0	0	1	0	0	0	0	0	1	0	1	0	1	1
	0	0	0	0	0	1	0	0	0	0	0	1	1	1	0	1
	0	0	0	0	0	0	1	0	0	1	0	0	1	0	0	1
	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0

PROOF OF PROPOSITION 1.6. The minimal non-Cayley pentavalent graph admitting a solvable automorphism group is constructed as a $\mathbb{Z}_2^8$-cover over $K_{5,5}$ in the following manner. By a well-known theorem of Sabidussi, a graph is a Cayley graph if and only if it admits a vertex-regular subgroup of automorphisms. Hence, a necessary condition for a derived covering graph $X \times_\zeta K$ admitting a lift of some group $G \leq \mathrm{Aut}(X)$ to be non-Cayley is that G contains no vertex-regular subgroup. For the graph $X = K_{5,5}$, a complete list of (conjugacy classes of) subgroups $G \leq \mathrm{Aut}(X)$ that are solvable and arc-transitive is given in Table 2.

As the table shows, there are four conjugacy classes of groups with no vertex-regular subgroups (classes 3, 5, 8 and 10). A representative of class 8 is the group $H = \langle (1, 7, 3, 10, 4, 6, 5, 8)(2, 9), (6, 7, 8, 9, 10)\rangle$ of order 200, where group generators are described as permutations of the vertices (see Figure 2). Using computational tools, we can construct all H-admissible elementary abelian covering projections up to a certain size. In particular, the Magma routines, described in [9], were used for efficient computation of solvable covers. One of these yields the graph in Figure 2, described by the following voltage function $\zeta : A(X) \to \mathbb{Z}_2^8$, where $\zeta(a_{ij})$ is the value on arc a_{ij} from vertex i to vertex j for arcs in Table 3 or is trivial otherwise.

The full automorphism group of this derived graph contains no vertex-regular subgroup and hence the graph is not a Cayley graph, although it admits a solvable and arc-transitive subgroup by construction (in fact, the lifted group is the same as the

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full automorphism group). A similar inspection of other subgroups in $\mathrm{Aut}(K_{5,5})$ and $\mathrm{Aut}(\mathrm{Dip}_5)$ shows that this graph of order 2560 is the smallest example of its kind. □

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