Compositio Mathematica **117:** 223–241, 1999.

Characteristic Classes for Difference Operators

A. H. M. LEVELT[1]* and A. FAHIM[2]**

[1]*Department of Mathematics, University of Nijmegen, The Netherlands; e-mail: ahml@sci.kun.nl*
[2]*Département de Mathématiques, Université de La Rochelle, France; e-mail: afahim@cri.univ-lr.fr*

(Received: 4 July 1997; accepted in final form: 18 February 1998)

Abstract. We introduce and describe the *characteristic class* of a difference operator over the difference field $(k((t)), \tau)$. Here k is an algebraically closed field of characteristic zero and τ is the k-linear automorphism of $k((t))$ defined by $\tau(t) = t/(1+t)$. The approach is based on the characterization of simple difference operators in terms of their eigenvalues.

Mathematics Subject Classifications (1991): 39A10, 39A70, 47B39.

Key words: difference operator, difference field, characteristic class.

1. Introduction

Let k be a field of characteristic 0, $\mathcal{O}$ the ring of formal power series in t with coefficients in k and K the field of fractions of $\mathcal{O}$. In many respects differential operators and difference operators with coefficients in K resemble linear transformations in finite dimensional vector spaces. For instance such operators have eigenvectors $\neq 0$ over a finite extension of K. In case of linear transformations the existence of eigenvectors is proved by means of the characteristic polynomial. However, for linear differential and difference operators, the existence of eigenvectors is proved in a different way, because there is no good replacement for characteristic polynomials.

In his thesis, R. Sommeling [4] has introduced a natural formal invariant called the *characteristic class* of a differential operator over a differential field of characteristic zero. This characteristic class is not an element of a polynomial ring but of a newly constructed characteristic ring $\mathcal{C}$. It has a certain number of useful properties:

(i) It classifies semi-simple differential operators up to equivalence.
(ii) The characteristic of the sum of two differential operators equals the sum of the characteristic classes of the operators.
(iii) The characteristic class of the tensor product of two differential operators (more precisely: differential modules) equals the product of their characteristic classes.

* The final version of this paper was prepared during a stay at the University of Angers, France.
** Partially supported by the EEC within the framework of the Human Capital and Mobility Program, Project EXB-CHB-GCT 93-0416.

This paper presents the study of the analogous notion for difference operators. We shall define characteristic classes for difference operators with the properties (i), (ii), (iii). The theory is similar to the one for differential operators, though the treatment of normalized eigenvalues is more complicated.

In order to make the subject as transparent as possible we have made one concession: the basic field of constants k is assumed to be algebraically closed. This assumption is not essential and we know how to handle the general case. Since the technicalities of the general case would eclipse the intuitive ideas, we believe that our concession is justified.

2. Definitions and Notations

Throughout this paper we shall use the following notations and definitions.

- k is an algebraically closed field of characteristic 0.
- $K = k((t))$, the field of fractions of $k[[t]]$, the ring of formal power series in t with coefficients in k.
- Let $\bar{K} = \cup_{l=1}^{\infty} k((t^{1/l}))$ be the field of Puiseux series over k. $\bar{K}$ is the algebraic closure of K. Moreover, for any $l \in \mathbb{N}^*$ the only subfield of $\bar{K}$ of degree l over K is $K(t^{1/l})$. In the sequel finite field extensions $K \subset L$ will often appear. By the foregoing for such an extension there exists a K-isomorphism ϕ of L onto the subfield $K(t^{1/l})$ of $\bar{K}$, where $l = [L : K]$ is the degree of L over K. Hence $L = K(s)$, where $\phi(s) = t^{1/l}$ and $s^l = t$. Identification of L with $K(t^{1/l})$ doesn't lead to confusion in most cases. Note that $K \subset L$ is a Galois extension with (cyclic) Galois group $\mathrm{Gal}(L/K)$ generated by $\sigma : s \mapsto \zeta s$, where ζ is a primitive lth root of 1.
- τ is the k-algebra automorphism $K \to K$ such that $\tau(t) = t/(1+t)$ and which is continuous in the t-adic topology. τ extends (uniquely) to an automorphism on $\bar{K}$ by defining $u_l = (1+t)^{-1/l}$, $\tau_{K(t^{1/l})}(t^{1/l}) = u_l t^{1/l}$ for all $l \in \mathbb{N}^*$.
- $\mathrm{End}_k(K)$ has a K-vector space structure defined by $(z\,f)(w) = z\,f(w)$ for all $z, w \in K$, $f \in \mathrm{End}_k(K)$. Let $\mathcal{D}$ be the smallest K-subalgebra generated by K and τ. $\mathcal{D}$ is called the *ring of difference operators* with respect to K and τ.
- V is a K-vector space of finite dimension. A *difference operator on* V with respect to K, τ is a k-linear map $\Phi : V \to V$ satisfying $\Phi(a\,v) = \tau(a)\Phi(v)$ for all $a \in K$, $v \in V$.
- In the above situation one can define $\tau v = \Phi(v)$ for all $v \in V$. This makes V into a left $\mathcal{D}$-module. Conversely, if V is a left $\mathcal{D}$-module and a finite-dimensional K-vector space, then by $\Phi(v) = \tau\, v$ for all $v \in V$ a difference operator on V is defined.
- Let Φ be a difference operator on V and $(v_1, \ldots, v_n)$ a K-basis of V. Then there exist $a_{i,j} \in K$ such that $\Phi(v_i) = \sum_{j=1}^{n} a_{j,i} v_j$ for all $i \in \{1, \ldots, n\}$. $\mathrm{Mat}(\Phi, (v_1, \ldots, v_n))$ denotes the matrix $(a_{i,j})$.
- Tensor product. Let (V, Φ) and (W, Ψ) be two $\mathcal{D}$-modules. Then the map $v \otimes w \mapsto \Phi(v) \otimes \Psi(w)$ defines a difference operator on $V \otimes_K W$ which will

be denoted by $\Phi \otimes \Psi$. $(V \otimes_K W, \Phi \otimes \Psi)$ is called the *tensor product* of (V, Φ) and (W, Ψ).

- $\mathcal{D}_L$, the ring of difference operators with coefficients in L, can be defined in an obvious way. The above correspondence between difference operators with respect to K and (finite-dimensional) $\mathcal{D}$-modules can be generalized to a correspondence between difference operators with respect to L and $\mathcal{D}_L$-modules.
- $G = \mathrm{Gal}(L/K)$ operates on V_L by $\rho : a \otimes v \mapsto \rho(a) \otimes v$ for all $\rho \in G, a \in L, v \in V$. Note that τ_L is G-equivariant, i.e. $\rho \circ \tau_L = \tau_L \circ \rho$ for all $\rho \in G$.
- When Z is a G-invariant L-subspace of V_L, then there exists a (unique) K-subspace W of V such that the multiplication map $L \otimes_K W \to V_L$ defined by $a \otimes w \mapsto a\,w$ $(a \in L, w \in W)$ is an L-isomorphism of $L \otimes_K W$ onto Z. If, moreover, $\Phi_L(Z) \subset Z$ then $\Phi(W) \subset W$. This means that $\Phi_{|W}$ is a difference operator on W.

3. Eigenvectors and Simple $\mathcal{D}$-Modules

Our analysis is based upon the following theorem of Turrittin [1, 3, 5]:

THEOREM 1. *To any difference operator Φ on a K-vector space V there exists a finite field extension $K \subset L$, $a \in L$ and $v \in V_L$ such that $\Phi_L(v) = av$ and $v \neq 0$.*

DEFINITION 1. For $a, b \in \bar{K}$ one defines

$$a \sim b \overset{\mathrm{def}}{=} \text{There exists } z \in \bar{K} \setminus \{0\} \text{ such that } a = \frac{\tau(z)}{z} b.$$

Remark 1. $\sim$ is an equivalence relation as can easily be shown.

PROPOSITION 1. *Let (V, Φ) be a $\mathcal{D}$-module and $K \subset L$ a Galois extension with group G. Let $v \in V_L \setminus \{0\}$ satisfy the following conditions*:

(I) $\Phi_L(v) = av$ *for some* $a \in L$.
(II) *For all* $\rho \in G$ $\rho(a) \sim a$ *implies* $\rho = 1$.
(III) $V_L = \sum_{\rho \in G} L\rho(v)$.

Then the following statements are valid:

(i) $\sum_{\rho \in G} L\rho(v)$ *is a direct sum.*
(ii) *The* $\mathcal{D}_L$*-modules* $L\rho(v)$ *are non-isomorphic.*
(iii) (V, Φ) *is a simple* $\mathcal{D}$*-module.*

Proof. (i) We shall prove that $(\rho(v))_{\rho \in G}$ is an L-basis of V_L. If not, $(\rho(v))_{\rho \in G}$ is linearly dependent over L. Let $S \subset G$, $S \neq \emptyset$ be minimal with the property that there exist $b_\sigma \in L$, not all $= 0$, such that

$$\sum_{\sigma \in S} b_\sigma \sigma(v) = 0. \tag{1}$$

https://doi.org/10.1023/A:1000914223243 Published online by Cambridge University Press

We may assume that $1 \in S$ and $b_1 = 1$. By applying Φ_L one gets

$$\sum_{\sigma \in S} \sigma(a)\tau(b_\sigma)\sigma(v) = 0. \tag{2}$$

Multiplying (1) by a and subtracting from (2) one checks that

$$\sum_{\sigma \in S\setminus\{1\}} (\tau(b_\sigma)\sigma(a) - ab_\sigma)\sigma(v) = 0.$$

Because of the minimality of S one now has $\tau(b_\sigma)\sigma(a) - ab_\sigma = 0$ for all $\sigma \in S$. Hence for some $\sigma \in S \setminus \{1\}$ one has $\sigma(a) \sim a$. This is a contradiction. This completes the proof of the fact that $(\rho(v))_{\rho\in G}$ is an L-basis of V_L.

(iii) Let W a Φ-invariant K-subspace of V. Assume $W \neq \{0\}$. We shall prove that $W = V$ and it will follow that (V, Φ) is simple. There exists a Galois extension $K \subset M$ such that $L \subset M$ and W_M contains a nonzero eigenvector w of Φ_M, $\Phi_M(w) = cw$ for some $c \in M$. There exist $b_\rho \in M$ such that

$$w = \sum_{\rho\in G} b_\rho \rho(v). \tag{3}$$

Apply Φ_M to both sides of (3)

$$cw = \sum_{\rho\in G} \tau(b_\rho)\rho(a)\rho(v). \tag{4}$$

Multiply (3) by c and subtract from (4) $\sum_{\rho\in G}(\tau(b_\rho)\rho(a) - cb_\rho)\rho(v) = 0$. Since $(\rho(v))_{\rho\in G}$ is an M-basis of V_M it follows that

$$\tau(b_\rho)\rho(a) - cb_\rho = 0 \text{ for all } \rho. \tag{5}$$

Assume that there exist $\rho, \sigma \in G$ such that $\rho \neq \sigma$, $b_\rho \neq 0$, $b_\sigma \neq 0$. Then (5) implies

$$\rho(a)\frac{\tau(b_\rho)}{b_\rho} = \sigma(a)\frac{\tau(b_\sigma)}{b_\sigma}.$$

Hence $\rho(a) \sim \sigma(a)$ for $\rho \neq \sigma$. This contradicts (II). We have proved that in (3) only one b_ρ differs from 0, i.e. $w = b\rho(v)$ for some $b \in M \setminus \{0\}$. It follows that $V_M = \sum_{\rho\in G} M\rho(w)$. On the other hand, the right-hand side is contained in W_M. We conclude that $W_M = V_M$, hence $W = V$. So we have shown that (V, Φ) is a simple module.

(ii) For $\rho, \sigma \in G$ let $\phi\colon L\rho(v) \to L\sigma(v)$ be an isomorphism of $\mathcal{D}_L$-modules. Then there exists $z \in L^*$ such that $\phi(\rho(v)) = z\sigma(v)$. The relation $\phi \circ \Phi_L = \Phi_L \circ \phi$

applied to $\rho(v)$ yields $\rho(a)z\sigma(v) = \tau(z)\sigma(a)\sigma(v)$. Hence, $\rho(a) \sim \sigma(a)$. Now (II) gives $\rho = \sigma$.

PROPOSITION 2 (*with the notations and hypotheses of the preceding proposition*). *Write* $d = [L : K] = \mathrm{ord}(G)$, $L = K(s)$ *where* $s = t^{1/d}$. *Then there exist* $a_0, \ldots, a_{d-1} \in K$, $v_0, \ldots, v_{d-1} \in V$ *such that*

$$a = a_0 + \frac{a_1}{s} + \cdots + \frac{a_{d-1}}{s^{d-1}}, \tag{6}$$

$$v = v_0 + v_1 s + \cdots + v_{d-1} s^{d-1} \tag{7}$$

and the following statements hold:

(i) $(v_0, \ldots, v_{d-1})$ *is a* K*-basis of* V *and the matrix of* Φ *w.r.t. this basis is* $A\,(1+t)^{J_d}$, *where*

$$A = \begin{pmatrix} a_0 & a_{d-1}/t & \cdots & a_1/t \\ a_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & a_{d-1}/t \\ a_{d-1} & \cdots & a_1 & a_0 \end{pmatrix} \tag{8}$$

and J_d *the diagonal matrix*

$$J_d = \begin{pmatrix} \frac{0}{d} & & & \\ & \frac{1}{d} & & \\ & & \ddots & \\ & & & \frac{d-1}{d} \end{pmatrix}. \tag{9}$$

(ii) *The characteristic polynomial of* A *coincides with the minimal polynomial of* a *over* K.

Proof. The existence (and uniqueness) of a_i, v_j with the required properties is obvious. For a proof of (i) we express the relation $\Phi(v) = a\,v$ in terms of a_i and v_j. One has on the one hand

$$\Phi(v) = \Phi\left(\sum_{i=0}^{d-1} v_i s^i\right) = \sum_{i=0}^{d-1} u_d^i \Phi(v_i) s^i \tag{10}$$

and on the other hand

$$a\,v \;=\; \sum_{j=0}^{d-1}\sum_{h=0}^{d-1} a_j v_h s^{h-j}$$

$$= \sum_{i=0}^{d-1}\left(\sum_{h=i}^{d-1} a_{h-i} v_h\right) s^i + \sum_{i=0}^{d-1}\left(\sum_{h=0}^{i-1} a_{d+h-i}/t v_h\right) s^i.$$

Comparing equal powers of s in the last member and (10) one finds for all $i \in \{0, \ldots, d-1\}$

$$\Phi(v_i) = (1+t)^{i/d}\left(\sum_{h=i}^{d-1} a_{h-i}\, v_h + \sum_{h=0}^{i-1} a_{d+h-i}/t v_h\right)$$

from which (i) immediately follows.

Let y be the column vector with entries $y_0, \ldots, y_{d-1}$ which will be interpreted as coordinate vectors with respect to $v_0, \ldots, v_{d-1}$. In terms of coordinate vectors the application Φ_L is $y \mapsto A(1+t)^{J_d}\tau(y)$. Now v is an eigenvector of Φ_L with eigenvalue a and has coordinates $1, s, \ldots, s^{d-1}$. It follows that this eigenvector is also an eigenvector of A. Hence, a is a zero of the characteristic polynomial p_A. a has at least d different conjugates over K and $\deg(p_A) = d$. Hence, p_A is the minimal polynomial of a over K. This completes the proofs of (ii) and the proposition.

DEFINITION 2. For $c \in k$ the $\mathcal{D}$-module $E(c)$ is defined as the one-dimensional K-vectorspace generated by (the symbol) $e(c)$ such that $\tau(e(c)) = (1+t)^{-c}\, e(c)$.

PROPOSITION 3. *Let (V, Φ) be a $\mathcal{D}$-module, $L = K(s)$ where $s = t^{1/l}$ and $G = \mathrm{Gal}(L/K)$ with generator σ. Let $a \in L$, $v \in V_L \setminus \{0\}$ satisfy*

(a) $\Phi(v) = a\, v$.
(b) *For all $\rho \in G$ the relation $\rho(a) \sim a$ implies $\rho = 1$.*

Finally, define $\tilde{V}$ as the L-subspace of V_L generated by $(\rho(v))_{\rho\in G}$, $h = l/d$, where $d = [K(a) : K]$, and $a_0, \ldots, a_{d-1} \in K$ by $a = a_0 + a_1/s^h + \cdots + a_{d-1}/s^{(d-1)h}$. Then there exist $w_{i,j} \in V$ for all $i \in \{0, \ldots, h-1\}$ and $j \in \{0, \ldots, d-1\}$ such that the following statements hold.

(i) *$(\tilde{V}, \Phi_L)$ is a $\mathcal{D}$-submodule of (V_L, Φ_L) and $(\rho(v))_{\rho\in G}$ is an L-basis of $\tilde{V}$.*
(ii) *$\{w_{i,j} \mid i \in \{0, \ldots, h-1\}, j \in \{0, \ldots, d-1\}\}$ is an L-basis of $\tilde{V}$.*
(iii) *$W_i \overset{\mathrm{def}}{=} \sum_{j=0}^{d-1} K w_{i,j}$ is a simple $\mathcal{D}$-submodule of (V, Φ).*
(iv) *$\mathrm{Mat}(\Phi, (w_{i,0}, \ldots, w_{i,d-1})) = Au_l^i(1+t)^{J_d}$ where A is given by (8). Moreover, the characteristic polynomial of A equals the minimum polynomial of a over K.*
(v) *$w_i = w_{i,0} + w_{i,1}s^h + \cdots + w_{i,d-1}\, s^{(d-1)h} \in V_{K(a)}$ satisfies $\Phi(w_i) = u_l^i\, a\, w_i$ and $\{\sigma^j(w_i) \mid i \in \{0, \ldots, h-1\}, j \in \{0, \ldots, d-1\}\}$ is an L-basis of $\tilde{V}$ consisting of eigenvectors of Φ_L.*
(vi) *For each $i \in \{0, \ldots, h-1\}$ there exists an isomorphism of $\mathcal{D}$-modules*

$$\phi_i \colon E\left(\frac{i}{l}\right) \otimes_K W_0 \xrightarrow{\sim} W_i. \tag{11}$$

https://doi.org/10.1023/A:1000914223243 Published online by Cambridge University Press

(vii) *Define* $W = \sum_{i=0}^{h-1} W_i$. *Then* W *is a* $\mathcal{D}$*-submodule of* V *such that* $W_L \xrightarrow{\sim} \tilde{V}$ *and the map*

$$\left(\bigoplus_{i=0}^{h-1} E\left(\frac{i}{l}\right)\right) \otimes_K W_0 \to W$$

induced by the ϕ_i *is an isomorphism of* $\mathcal{D}$*-modules.*

Proof. Let $\zeta \in k$ be a primitive lth root of 1. $K(a) \subset L$ is a Galois extension with group H. We have $H = \{\sigma^0, \sigma^d, \sigma^{2d}, \ldots, \sigma^{(h-1)d}\}$ and $K(a) = K(s^h)$.

Let us prove (i). When the $\rho(v)$ are linearly dependent over L, then there exist $\rho_1, \ldots, \rho_n \in G$ and $l_1, \ldots, l_n \in L$, not all $= 0$, such that

$$\sum_{i=1}^{n} l_i \rho_i(v) = 0. \tag{12}$$

Now take such ρ_i, l_i with n minimal. We may suppose $l_1 = 1$. Applying Φ_L we find

$$\sum_{i=1}^{n} \tau(l_i)\rho_i(a)\rho_i(v) = 0. \tag{13}$$

Now multiply (12) by $\rho_1(a)$ and subtract from (13). The result is

$$\sum_{i=2}^{n} (\tau(l_i)\rho_i(a) - l_i\rho_1(a))\rho_i(v) = 0.$$

This is an L-linear relation between the $\rho(v)$ with less terms. Hence, it must be the trivial relation, i.e. $\tau(l_i)\rho_i(a) - l_i\rho_1(a)) = 0$ for all i. Since all l_i are different from 0 it follows that $\rho_i(a) \sim \rho_1$. Hence, $\rho_i = \rho_1$, implying $n = 1$. This is a contradiction. So we have proved that the $\rho(v)$ are linearly independent over L.

(ii) In order to prove the existence of the $w_{i,j}$ with the stated properties, we define $H = \{\rho \in G \mid \rho(a) = a\}$ (so $K(a) = L^H$) and

$$w_i \stackrel{\text{def}}{=} \sum_{\rho \in H} \rho(s^i v). \tag{14}$$

Then one readily checks that $\rho(w_i) = w_i$, $\Phi_L(w_i) = u_l^i a w_i$ for all $\rho \in H$ and $i \in \{0, \ldots, h-1\}$. It follows that all w_i are in $V_{K(a)}$ and so there exist $w_{i,0}, \ldots, w_{i,d-1} \in V$ satisfying

$$w_i = w_{i,0} + w_{i,1}s^h + \cdots + w_{i,d-1}s^{(d-1)h}. \tag{15}$$

Note that the $w_{i,j}$ are uniquely determined by w_i.

https://doi.org/10.1023/A:1000914223243 Published online by Cambridge University Press

In order to show that the $w_{i,j}$ are linearly independent over L we first prove the same property for the $\sigma^j(w_i)$. From (14) we derive for all $i \in \{0, \ldots, h-1\}$, $j \in \{0, \ldots, d-1\}$

$$\sigma^j(w_i) = \sum_{r=0}^{h-1} s^i \zeta^{i(rd+j)} \sigma^{rd+j}(v).$$

For fixed j this shows that $\sigma^j(w_0), \ldots, \sigma^j(w_{h-1})$ are L-linear expressions in $\sigma^j(v), \sigma^{j+d}(v), \ldots, \sigma^{j+(h-1)d}(v)$ and that the connecting matrix is of Vandermonde type, constructed from $s\zeta^j, s\zeta^{j+d}, \ldots, s\zeta^{j+(h-1)d}$. Since the latter matrix is invertible we see that $\sigma^j(w_0), \ldots, \sigma^j(w_{h-1})$ are linearly independent over L. Note that for $r \in \{0, \ldots, h-1\}$ and $j \in \{0, \ldots, d-1\}$ the expression $rd + j$ assumes as values all numbers in $\{0, \ldots, l-1\}$ just once. Hence, the $\sigma^{rd+j}(v)$ form a L-basis of $\tilde{V}$ and so the same holds for the $\sigma^j(w_i)$. From (15) we get

$$\sigma^j(w_i) = w_{i,0} + w_{i,1}\sigma^j(s^h) + \cdots + w_{i,d-1}\sigma^j(s^{(d-1)h}).$$

For fixed i we get $w_i, \sigma(w_i), \ldots, \sigma^{d-1}(w_i)$ linearly expressed in

$$w_{i,0}, s^h w_{i,1}, \ldots, s^{(d-1)h} w_{i,d-1}$$

and the connecting matrix is again of Vandermonde type, based now on the sequence $1, \zeta^h, \ldots, \zeta^{(d-1)h}$. It follows that

$$w_i, \sigma(w_i), \ldots, \sigma^{d-1}(w_i) \quad \text{and} \quad w_{i,0}, w_{i,1}, \ldots, w_{i,d-1}$$

span the same L-subspace of $\tilde{V}$. Since the $\sigma^j(w_i)$ form an L-basis of $\tilde{V}$, the same holds for the $w_{i,j}$.

It is clear now that (v) holds. Define $\tilde{G} = \mathrm{Gal}(K(a)/K)$. (iii) follows by applying Proposition 1 with L replaced by $K(a)$, G by $\tilde{G}$, V by W_i, a by $u_l^i a$ and v by w_i. (iv) follows from Proposition 2 when we moreover replace s by s^h.

Finally we shall prove (vi) and (vii). Let Ψ_i be the difference operator on $E(i/l) \otimes_K W_0$. One has $\Psi_i(e(i/l) \otimes w) = u_l^i e(i/l) \otimes \Phi(w)$. Hence the matrix of Ψ_i with respect to the basis $(e(i/l) \otimes w_{0,0}, \ldots, e(i/l) \otimes w_{0,d-1})$ of $E(i/l) \otimes_K W_0$ is just u_l^i times that of Φ with respect to the basis $(w_{0,0}, \ldots, w_{0,d-1})$ of W_0, i.e. $u_l^i A(1+t)^{J_d}$. This is also the matrix of Φ with respect to the basis $(w_{i,0}, \ldots, w_{i,d-1})$ of W_i. Define the K-linear isomorphism (11) by $\phi_i(e(i/l) \otimes w_{0,j}) = w_{i,j}$.

We must show $\phi_i \circ \Psi_i = \Phi \circ \phi_i$. Or, it is an immediate consequence of the above matrix description of the two difference operators. (vii) is a trivial consequence.

DEFINITION 3. An element $a \in \bar{K}$ is said to be *in normal form* or *normalized* if either $a = 0$ or a can be written as

$$a = a_0 t^{i/d} (1+t)^{a_1} \exp(\tau(q) - q), \tag{16}$$

where $d = [K(a) : K]$, $a_0, a_1 \in k$, $i \in \mathbb{Z}$ and

$$q = \frac{q_1}{t^{1/d}} + \cdots + \frac{q_{d-1}}{t^{(d-1)/d}}$$

with $q_1, \ldots, q_{d-1} \in k$ ($q = 0$ if $d = 1$). The set of elements in normal form will be denoted by $\mathcal{N}$. $\mathcal{N}^* = \mathcal{N} \setminus \{0\}$. $\mathcal{N}^*$ is a group under multiplication.

PROPOSITION 4. (i) *For any $a \in \bar{K} \setminus \{0\}$ there exists $z \in K(a) \setminus \{0\}$ such that $b =_{\text{def}} a\,\tau(z)/z$ is in normal form. Here b is in $K(a)$.*

(ii) *For all $a, b \in \mathcal{N}$ one has $a \sim b$ if and only if there exists $j \in \mathbb{Z}$ such that $a = u_d^j b$.*

Remark 2. In (i) of the above proposition $b \in K(a)$ may not be replaced by $K(a) = K(b)$ as is evident from the example

$$s = \sqrt{t}, \qquad a = \frac{1+s}{1+u_2 s}, \qquad z = 1+s.$$

Then $K(a) = K(s) \neq K$, $z \in K(a) \setminus \{0\}$ and $b = a\tau(z)/z = 1$.

Remark 3. Note that the relation $a = u_d^j b$ in (ii) of the proposition implies $K(a) = K(b)$.

The proof is based on the next two lemmas. $s = t^{1/d}$ in both lemmas.

LEMMA 1. *For any $b = \sum_{i=d+1}^{\infty} b_i s^i \in k[[s]]$ there exists $y = \sum_{i=1}^{\infty} y_i s^i$ in $k[[s]]$ satisfying $\tau(y) - y = b$.*

Proof. Define

$$y_1 = -d\, b_{d+1}, \qquad \varepsilon(1) = -\frac{b_{d+1}}{s} + y_1 \frac{\tau(s) - s}{s^{d+2}}.$$

Then one easily checks that $\varepsilon(1) \in k[[s]]$ and

$$\tau(y_1 s) - y_1 s = b_{d+1} s^{d+1} + \varepsilon(1) s^{d+2}.$$

Now suppose that for some $m \in \mathbb{N}^*$ we have found $y(m) =_{\text{def}} \sum_{i=1}^{m} y_i s^i$ and $\varepsilon(m)$ satisfying

$$\tau(y(m)) - y(m) = \sum_{i=d+1}^{d+m} b_i s^i + \varepsilon(m) s^{d+m+1}, \quad \varepsilon(m) \in k[[s]]. \tag{17}$$

Then we try to find $y(m+1), \varepsilon(m+1)$ satisfying (17) with m replaced by $m+1$. An easy computation shows that it is sufficient to find y_{m+1} and $\varepsilon(m+1)$ such that

$$\varepsilon(m+1) = \frac{\varepsilon(m) - b_{d+m+1}}{s} + y_{m+1} \frac{\tau(s^{m+1}) - s^{m+1}}{s^{d+m+2}}$$

https://doi.org/10.1023/A:1000914223243 Published online by Cambridge University Press

with $\varepsilon(m+1) \in k[[s]]$. Using the relation

$$\tau(s^{m+1}) - s^{m+1} = -\frac{m+1}{d} s^{m+d+1} + \mathrm{O}(s^{m+d+2}),$$

one checks that

$$y_{m+1} = \frac{d}{m+1}(\varepsilon(m)_0 - b_{d+m-1})$$

solves the problem. Here $\varepsilon(m)_0$ denotes the constant term of $\varepsilon(m)$.

LEMMA 2. *For any* $c = \sum_{i=1}^{\infty} c_i s^i \in k[[s]]$ *there exist* $\gamma \in k$, $y = \sum_{i=1}^{\infty} y_i s^i \in k[[s]]$ *and* $q = q_1/s + \cdots + q_{d-1}/s^{d-1}$, *where* $q_i \in k$, *verifying*

$$c = \gamma \log(1+t) + \tau(q) - q + \tau(y) - y. \tag{18}$$

Proof. For all $q_1, \ldots, q_{d-1}$ the following relation holds

$$\tau(q) - q = \frac{d-1}{d} q_{d-1}\, s + \cdots + \frac{1}{d} q_1 s^{d-1} + \mathrm{O}(s^d)$$

as one easily sees. Defining $q_i = dc_{d-i}/i$, we get $c - \tau(q) + q = \tilde{c}$, where $\tilde{c} = \sum_{i=d}^{\infty} \tilde{c}_i s^i$. Choose $\gamma = \tilde{c}_d$. Then $c - \gamma \log(1+t) - \tau(q) + q = b$, where $b = \sum_{i=d+1}^{\infty} b_i s^i$. By virtue of Lemma 1, there exists $y = \sum_{i=1}^{\infty} y_i s^i$ such that $b = \tau(y) - y$. Putting the above relations together, we get a proof of (18).

Proof of Proposition 4. (i) For $a \in \bar{K} \setminus \{0\}$ define $d = [K(a) : K]$ and $s = t^{1/d}$. There exist $a_0 \in k$, $i \in \mathbb{Z}$ and $\tilde{a} = 1 + a_1 s + a_2 s^2 + \cdots \in k[[s]]$ such that $a = a_0 s^i \tilde{a}$. Define $c = c_1 s + c_2 s^2 + \cdots$ by $c = \log(\tilde{a})$. Next take γ, q and y as in Lemma 2 and define $z = \mathrm{e}^{-y}$. Then one has

$$a = a_0 s^i\, \mathrm{e}^c = a_0 s^i\, \mathrm{e}^{\gamma \log(1+t) + \tau(q) - q + \tau(y) - y} = b\, \frac{z}{\tau(z)},$$

where

$$b = a_0 s^i (1+t)^{\gamma}\, \mathrm{e}^{\tau(q)-q} \in K(s) = K(a). \tag{19}$$

Let H be the Galois group of $K(s)/K(b)$. If $h = \mathrm{order}(H)$, then $K(b) = K(v)$ where $v = s^h$. Define $m = d/h \in \mathbb{N}^*$. Then $v = t^{1/m}$ and $[K(b) : K] = m$. H is generated by $\sigma : s \to \zeta s$ where ζ is a primitive hth root of 1.

From (19) and $\sigma(b) = b$, we derive

$$\sigma(s^i)\sigma(\mathrm{e}^{\tau(q)-q}) = s^i\, \mathrm{e}^{\tau(q)-q}. \tag{20}$$

Hence

$$\zeta^i \sigma(\mathrm{e}^{\tau(q)-q}) = \mathrm{e}^{\tau(q)-q}. \tag{21}$$

Note that

$$\tau(q) - q = \sum_{j=1}^{d-1} \frac{q_j}{s^j}((1+t)^{j/d} - 1) = \sum_{j=1}^{d-1} \frac{j}{d} q_j \frac{t}{s^j} f_j, \tag{22}$$

where $f_j = 1 + \mathrm{O}(t)$, $\mathrm{O}(t)$ denoting an element of $tk[[t]]$. Consequently, $\mathrm{e}^{\tau(q)-q} = 1 + \mathrm{O}(s)$, where $\mathrm{O}(s)$ stands for an element of $sk[[s]]$. Now it follows that $\zeta^i = 1$ in virtue of (21). Hence, $s^i \in K(b)$, $s^i = v^l$, say. Because $\sigma(\mathrm{e}^{\tau(q)-q}) = \mathrm{e}^{\sigma(\tau(q)-q)}$, we derive from (21) by taking logarithms $\sigma(\tau(q) - q) = \tau(q) - q$ which implies

$$\sum_{j=1}^{d-1} (\zeta^{-j} - 1) \frac{j}{d} q_j \frac{t}{s^j} f_j = 0.$$

Such a relation can only hold if $q_j = 0$ when j is not a multiple of h. This means that

$$q = \sum_{j=1}^{m-1} \frac{q_{jh}}{s^{jh}} = \sum_{j=1}^{m-1} \frac{q_{jh}}{v^h}.$$

From this relation and (19) we derive

$$b = a_0 v^l (1+t)^\gamma \sum_{j=1}^{m-1} \frac{q_{jh}}{v^h}.$$

Since $[K(b) : K] = m$ and $v = t^{1/m}$, we see that b is in normal form. This completes the proof of (i).

(ii) Since $u_d^j = \tau(s^j)/s^j$, it is evident that $a = u_d^j b$ implies $a \sim b$. Conversely, let $a \sim b$, i.e. $a = b\,\tau(z)/z$ for some $z \in \bar{K} \setminus \{0\}$. Then there exists $d \in \mathbb{N}^*$ such that $K(a, b, z) = K(s)$, where $s = t^{1/d}$. We can write a in the form (16) and $b = b_0 t^{j/d} (1+t)^{b_1} \exp(\tau(r) - r)$, where $b_0, b_1 \in k$, $i \in \mathbb{Z}$,

$$r = \frac{r_1}{t^{1/d}} + \cdots + \frac{r_{d-1}}{t^{(d-1)/d}},$$

and z in the form $z = z_m s^m + z_{m+1} s^{m+1} + \cdots$ with $z_m \neq 0$. Write $c = b/a$. The reader can check that $\tau(z)/z = 1 - (m/d)s^d + \mathrm{O}(s^{d+1})$. Here $\mathrm{O}(s^{d+1})$ stands for an element of $s^{d+1}k[[s]]$. Hence, we have

$$c_0 t^{l/d} (1+t)^{c_1} \mathrm{e}^{\tau(w)-w} = 1 - (m/d)s^d + \mathrm{O}(s^{d+1}), \tag{23}$$

where $c_0 = a_0/b_0$, $l = i - j$, $c_1 = a_1 - b_1$, $w = q - r$. Similar to (22) we have

$$\tau(w) - w = \sum_{j=1}^{d-1} \frac{j}{d} w_j s^{d-j} g_j,$$

where $w_j = q_j - r_j$ and $g_j = 1 + \mathrm{O}(t)$. Also $\mathrm{e}^{\tau(w)-w} = 1 + \mathrm{O}(s)$ and so it follows from (23) that $c_0 = 1$ and $l = 0$. This proves $a_0 = b_0$, $i = j$ and

$$(1+t)^{c_1} e^{\tau(w)-w} = 1 - (m/d)s^d + \mathrm{O}(s^{d+1}).$$

Taking the logarithms at both sides yields

$$c_1 \log(1+t) + (\tau(w) - w) = -\frac{m}{d}s^d + \mathrm{O}(s^{d+1}).$$

Hence

$$\sum_{j=1}^{d-1} \frac{j}{d} w_j s^{d-j} g_j = -\frac{m}{d}s^d + \mathrm{O}(s^{d+1}) - c_1(t + \mathrm{O}(t^2)).$$

From this relation and $g_j = 1 + \mathrm{O}(t)$ it now follows that all w_j vanish. This proves $q = r$, $c_1 = -m/d$, hence $a = u_d^m b$, and terminates the proof of (ii).

COROLLARY 1. *Let $K \subset L$ be a finite Galois extension with a Galois group G. Then for any $a \in L$ in normal form and $\rho \in G$ the relation $\rho(a) \sim a$ implies $\rho = 1$.*

Proof. We may assume $a \neq 0$. Let $l = [L : K]$ and $d = [K(a) : K]$. Then $\rho(a) = u_d^{-j}\, a$ for some $j \in \mathbb{Z}$ in virtue of Proposition 4. One has $a = \rho^l(a) = u_d^{-lj}\, a$. Since $u_d^{-lj} = 1$ it follows that $j = 0$, i.e. $\rho(a) = a$. Hence, $\rho = 1$.

DEFINITION 4. $a \in \mathcal{N}$ is called *normalized eigenvalue* if there exists $v \in V_{K(a)}$, $v \neq 0$, such that $\Phi_{K(a)}(v) = av$.

PROPOSITION 5. *Let (V, Φ) be a $\mathcal{D}$-module, $K \subset L$ a finite field extension, $a \in L$, $v \in V_L \setminus \{0\}$ such that $\Phi_L(v) = a\,v$. Then $K(a)$ contains a normalized eigenvalue.*

Remark 4. In conjunction with Theorem 1 this proposition shows the existence of normalized eigenvalues for any $\mathcal{D}$-module.

Proof. From Lemma 4 we know that $z \in K(a)^*$ exists such that $b =_{\mathrm{def}} a\tau(z)/z$ is in normal form. Define $w = zv$. Then $\Phi(w) = bw$. So we have obtained a nonzero eigenvector and an eigenvalue in normal form. This doesn't guarantee that b is a normalized eigenvalue, because we don't know whether w belongs to $V_{K(b)}$. We shall complete our proof by a Galois argument. Let H be the Galois group of $K(b) \subset L$, $h = [L\colon K(b)]$, $d = l/h$, $s = t^{1/l}$, ζ a primitive hth root of 1 and ρ a generator of H. For $i \in \{0, \ldots, h-1\}$ define

$$w_i = \sum_{j=0}^{h-1} \rho^j(s^{id} w) = s^{id} \sum_{j=0}^{h-1} \zeta^{ij} \rho^j(w).$$

Obviously w_i is fixed by the operations of H and so belongs to $V_{K(b)}$. Using the fact that the matrix $(\zeta_{i,j})$, i, j running through $\{0, \dots, h-1\}$, is nonsingular, one sees that not all w_i can vanish. Because otherwise w would vanish. On the other hand one has

$$\begin{aligned}\Phi(w_i) &= \tau(s^{id}) \sum_{j=0}^{h-1} \zeta^{ij} \rho^j(\Phi(w)) = (u_l s)^{id} \sum_{j=0}^{h-1} \zeta^{ij} \rho^j(bw) \\ &= u_h^i b s^{id} \sum_{j=0}^{h-1} \zeta^{ij} \rho^j(w) = u_h^i b w_i.\end{aligned}$$

Note that $u_h^i b$ is in normal form and that $K(u_h^i b) = K(b)$. This terminates the proof.

PROPOSITION 6. *Let* (V, Φ) *be a simple* $\mathcal{D}$*-module,* $a \in \mathcal{N}$ *a normalized eigenvalue,* $v \in V_{K(a)} \backslash \{0\}$ *such that* $\Phi(v) = a\,v$ *and* G *the Galois group of* $K(a)/K$. *Then the following statements hold*:

(i) $(\rho(v))_{\rho \in G}$ *is a* $K(a)$*-basis of* $V_{K(a)}$.
(ii) $\dim_K(V) = [K(a) : K]$.
(iii) *For any* $b \in \mathcal{N}$ b *is a normalized eigenvalue of* (V, Φ) *if and only if* $b \sim \rho(a)$ *for some* $\rho \in G$.

Proof. (i) This is an immediate consequence of Proposition 3(i) applied with $L = K(a)$. Note that $V_K(a)$ is the direct sum of the $K(a)$-subspaces $K(a)\rho(v)$, $\rho \in G$, invariant under $\Phi_K(a)$, and that $\Phi_{K(a)}(\rho(v)) = \rho(a)v$. Moreover, $\rho(a) \sim \sigma(a)$ if and only if $\rho = \sigma$.

(ii) Trivial consequence of (i).

(iii) If $b \sim \rho(a)$ then obviously b is a normalized eigenvalue for (V, Φ).

Now let b be a normalized eigenvalue. We may assume that $K(a)$ and $K(b)$ are both subfields of $\bar{K}$. Because both have the same degree over K, viz. $\dim_K(V)$, we have $K(a) = K(b)$. Let L denote the latter field. There exists $w \in V_L \setminus \{0\}$ satisfying $\Phi(w) = b\,w$. Because $(\rho(v))_{\rho \in G}$ is an L-basis of V_L, there exist $l_\rho \in L$ such that $w = \sum_{\rho \in G} l_\rho \rho(v)$. Applying Φ_L, we get $bw = \Phi(w) = \sum_{\rho \in G} \tau(l_\rho)\rho(a)\rho(v)$. On the other hand one has $bw = \sum_{\rho \in G} bl_\rho \rho(v)$. The latter two relations yield $\tau(l_\rho)\rho(a) = b\,l_\rho$. Since not all l_ρ vanish, one has $b \sim \rho(a)$ for some $\rho \in G$.

DEFINITION 5. Let (V, Φ) be a simple $\mathcal{D}$-module. Let L be a finite extension field of K. Note that L is called a *splitting field* for (V, Φ) if

(i) There exists $a \in L$ and $v \in V_L \setminus \{0\}$ such that $\Phi(v) = av$.
(ii) L is minimal with respect to the above property.

Obviously, any simple $\mathcal{D}$-module has a splitting field.

https://doi.org/10.1023/A:1000914223243 Published online by Cambridge University Press

PROPOSITION 7. *Let (V, Φ) be a simple $\mathcal{D}$-module, $K \subset L$ and L splitting field for (V, Φ). Then L contains a normalized eigenvalue for (V, Φ) and for any normalized eigenvalue $b \in L$ one has $L = K(b)$.*

Proof. In virtue of Proposition 5 L contains a normalized eigenvalue a. Because of the minimality of L it follows that $L = K(a)$. Now let b be a normalized eigenvalue. We may suppose $K(a) \subset \bar{K}$ and $K(b) \subset \bar{K}$. Then by Proposition 6(ii) we have $[K(a) : K] = [K(b) : K]$. Hence $L = K(a) = K(b)$.

PROPOSITION 8. *For all $a \in \mathcal{N}$ there exists a simple $\mathcal{D}$-module having a as normalized eigenvalue*.

Proof. Define $L = K(a)$, $d = [L : K]$, $s = t^{1/d}$. The idea of the proof is to reverse the constructions in the proof of Proposition 2. For this define $a_0, \ldots, a_{d-1} \in K$ by (6), the matrix A by (8) and the matrix J_d by (9). Moreover, put $V = K^d$ with canonical basis $(v_0, \ldots, v_{d-1})$ and define a difference operator Φ on V by the action of $A(1+t)^{J_d}$ on the basis $(v_0, \ldots, v_{d-1})$. Finally, define $v \in V_L$ by (7). An easy computation shows that $\Phi_L(v) = a\, v$. The only thing we must still prove is that (V, Φ) is a simple $\mathcal{D}$-module. Well, this is an immediate consequence of Proposition 3. The hypotheses of that proposition are satisfied

(a) by the above construction,
(b) because $a \in \mathcal{N}$.

In our situation $L = K(a)$, hence $h = 1$ and $\tilde{V} = V_L$ and from (vii) we conclude that $W_L \xrightarrow{\sim} V_L$ for a simple $\mathcal{D}$-submodule W of V. Hence (V, Φ) itself is simple.

The $\mathcal{D}$-module constructed in the above proof will be called the *canonical module associated to $a \in \mathcal{N}$*. It will be denoted by $(V(a), \Phi)$ (or, shortly, by $V(a)$).

PROPOSITION 9. *For $i = 1, 2$ let (V_i, Φ_i) be a simple $\mathcal{D}$-module with normalized eigenvalue a_i. Let $p_i \in K[T]$ be the minimal polynomial of a_i over K. Then the following statements are equivalent*:

(i) *The $\mathcal{D}$-modules (V_1, Φ_1) and (V_2, Φ_2) are isomorphic*.
(ii) *$a_1 \sim \tilde{a}_2$, where $\tilde{a}_2$ is conjugated with a_2 over K*.
(iii) *There exists $j \in \mathbb{Z}$ satisfying $p_2(T) = (1+t)^j p_1(u_d^j T)$ where d is the degree of a_1 (= degree a_2) over K*.

Proof. (i) $\Rightarrow$ (ii). Let $f\colon (V_1, \Phi_1) \xrightarrow{\sim} (V_2, \Phi_2)$ be a $\mathcal{D}$-isomorphism. Since $\dim_K(V_i) = [K(a_i) : K]$ and f induces an isomorphism of K-vector spaces $V_1 \xrightarrow{\sim} V_2$, one has $K(a_1)$ and $K(a_2)$ have the same degree over K and so they can be identified. We shall write L instead of $K(a_i)$. Let $v \in (V_1)_L \setminus \{0\}$ verify $\Phi_1(v) = a_1 v$. Then it follows that $\Phi_2(f(v)) = a_1 f(v)$. Proposition 6 leads now to (ii).

(ii) $\Rightarrow$ (iii) Note that $\tilde{a}_2$ also belongs to $\mathcal{N}$. Applying Proposition 4 we see that $\tilde{a}_2 = (1+t)^{j/d} a_1$ for some $j \in \mathbb{Z}$ and (iii) follows.

(iii) $\Rightarrow$ (i). We now know that a_2 is conjugated to $u_d^{-j} a_1$. Proposition 6 shows that a_1 is also a normalized eigenvalue of Φ_2. Let $v_i \in (V_i)_L \setminus \{0\}$ satisfy $\Phi_i(v_i) = a_1 v_i$. Define $d = [K(a_1) : K]$, $s = t^{1/d}$ and $v_{i,0}, v_{i,1}, \ldots, v_{i,d-1} \in V_i$ by $v_i = v_{i,0} + v_{i,1}s + \cdots + v_{i,d-1}s^{d-1}$. Then Proposition 2 shows that Mat$(\Phi_1, (v_{1,0}, v_{1,1}, \ldots, v_{1,d-1})$ coincides with that of Φ_2 with respect to $(v_{2,0}, v_{2,1}, \ldots, v_{2,d-1})$. It is now clear how to make an $\mathcal{D}$-isomorphism as needed in (i).

COROLLARY 2. *Let (V, Φ) be a simple $\mathcal{D}$-module. Then there exists $a \in \mathcal{N}$ such that (V, Φ) is isomorphic to the canonical $\mathcal{D}$-module $V(a)$.*

4. Characteristic Classes

In the sequel we will denote by $\mathcal{I}$ the monoid of monic irreducible polynomials of $K[T]$, whereas $\mathcal{M}$ denotes the monoid of all monic polynomials in $K[T]$.

DEFINITION 6. The equivalence relation $\sim$ on $\mathcal{I}$ is defined by as follows. For $f, g \in \mathcal{I}$ one has $f \sim g$ if the following hold:

(1) $\deg(f) = \deg(g)$. Hence, f and g have the same splitting field L.
(2) There exists $j \in \mathbb{Z}$ such that $f(T) = (1+t)^j\, g(u_d^j T)$ where $d = \deg(f)$.

For $f, g \in \mathcal{M}$ the relation $f \sim g$ means the following. When $f = \prod_{i=1}^r f_i$, $g = \prod_{j=1}^s g_j$, where $f_i, g_j \in \mathcal{I}$, then $r = s$ and there exists a permutation π of $\{1, \ldots, r\}$ such that $f_i \sim g_{\pi(i)}$ for all i.

$\mathcal{M}/\sim$ is an Abelian monoid. The associated Abelian group is denoted by $\mathcal{C}$. The operation is written as an addition. For any $f \in \mathcal{M}$ the image in $\mathcal{C}$ is denoted by $[f]$.

DEFINITION 7. Let (V, Φ) be a simple $\mathcal{D}$-module. The *characteristic class* $c((V, \Phi))$ is defined by $[p_a]$, where p_a is the minimal polynomial of a normalized eigenvalue a of (V, Φ).

For an arbitrary $\mathcal{D}$-module (V, Φ) let $V = V_0 \supset V_1 \supset \cdots \supset V_r = \{0\}$ be a Jordan–Hölder sequence in the sense of $\mathcal{D}$-modules. This means that $\Phi(V_i) \subset V_i$ and that the quotients V_{i-1}/V_i with the induced difference operator Φ_i are all simple. Then

$$c((V, \Phi)) \stackrel{\text{def}}{=} \sum_{i=1}^{r} c((V_{i-1}/V_i, \Phi_i)).$$

That the characteristic class of a simple module is well-defined follows from Proposition 9. The correctness in the general case follows from the well-known properties of Jordan–Hölder sequences.

Let us denote by $\mathcal{D}$iff the category of $\mathcal{D}$-modules of finite dimension as K-vector space and by $\mathcal{K}(\mathcal{D})$ the corresponding Grothendieck group. That is the free Abelian group generated by all isomorphism classes $[V]$ of objects V in $\mathcal{D}$iff modulo the subgroup generated by $[V]-[V']-[V'']$, where $0\to V'\to V\to V''\to 0$ is an exact sequence of $\mathcal{D}$-modules.

PROPOSITION 10. *The map* c: $(V,\Phi)\mapsto c((V,\Phi))$ *can be extended in a unique way to an injective group homomorphism of* $\mathcal{K}(\mathcal{D})$ *into* $\mathcal{C}$.

Remark 5. The image of c will be called the *characteristic group* and denoted by $\mathcal{C}_0$.

Proof. The injectivity of c follows from Proposition 9.

5. Tensor Product and Characteristic Ring

In this section (V_1,Φ_1) and (V_2,Φ_2) are simple $\mathcal{D}$-modules with splitting fields M_1, resp. M_2. M_1, M_2 are finite extensions of K and will be identified with subfields of $\bar{K}$. We denote by m_1 (resp. m_2) the degree $[M_1:K]$ (resp. $[M_2:K]$), by L the composition of M_1, M_2 (subfield of $\bar{K}$), by G the Galois group of L over K, by l the degree $[L:K]$ (note that $l=\text{l.c.m.}(m_1,m_2)$), by σ a generator of G, by s an lth root of t and by $\zeta\in k$ a primitive lth root of unity.

We know that there exist $v_1\in(V_1)_{M_1}\backslash\{0\}$, $a_1\in M_1$ (resp. $v_2\in(V_2)_{M_2}\backslash\{0\}$, $a_2\in M_2$) such that $\Phi_1(v_1)=a_1v_1$ (resp. $\Phi_2(v_2)=a_2v_2$) with a_1 and a_2 normalized eigenvalues. It follows that $(\rho(v_i))_{\rho\in\mathrm{Gal}(M_i/K)}$ is linearly independent over M_i for $i=1,2$.

Let Z be the tensor product of the $\mathcal{D}$-modules (V_1,Φ_1) and (V_2,Φ_2). In general (Z,Φ) is not simple. In this section an explicit decomposition of (Z,Φ) will be described as a direct sum of simple modules.

First note that Z_L can be identified with $(V_1)_L\otimes_L(V_2)_L$. Next define elements $z_{i,j}$ of Z_L by

$$z_{i,j}=\sigma^i(v_1)\otimes\sigma^j(v_2)\quad\text{for }0\leqslant i<m_1,\quad 0\leqslant j<m_2.$$

Then $(z_{i,j})_{0\leqslant i<m_1,0\leqslant j<m_2}$ is an L-basis of Z_L. Clearly $\sigma(z_{i,j})=z_{i+1,j+1}$ holds. (Compute modulo m_1 (resp. m_2) with the first (resp. second) index). Define $d=\gcd((m_1,m_2)$ and

$$\mathcal{Z}=\{z_{i,j}\mid 0\leqslant i<m_1,\quad 0\leqslant j<m_2\}$$

and for $i\in\{0,\dots,d-1\}$, $\mathcal{Z}_i=\{\rho(z_{i,0})\mid\rho\in G\}$.

PROPOSITION 11. *The following statements hold*:

(i) *For all* $i\in\{0,\dots,d-1\}$ *the map* $\rho\mapsto\rho(z_{i,0})$ *is a bijection of* G *onto* $\mathcal{Z}_i$.
(ii) $\mathcal{Z}=\coprod_{i=0}^{d-1}\mathcal{Z}_i$.

Proof. (i) We must prove that $\rho\mapsto\rho(z_{i,0})$ is injective. Let $\sigma^m(z_{i,0})=\sigma^n(z_{i,0})$ for some $m,n\in\{0,\dots,l-1\}$. Then

$$\sigma^{m+i}(v_1)\otimes\sigma^m(v_2)=\sigma^{n+i}(v_1)\otimes\sigma^n(v_2).$$

Consequently, $m + i \equiv n + i \bmod m_1$ and $m \equiv n \bmod m_2$. It follows that l divides $m - n$ and so $m = n$.

(ii) We must show that the $\mathcal{Z}_i$ are disjunct. For this assume that i, j are in $\{0, \dots, d-1\}$ and that $\mathcal{Z}_i \cap \mathcal{Z}_j \neq \emptyset$. Then there exists $m, n \in \{0, \dots, l-1\}$ such that $\sigma^m(\sigma^i(v_1) \otimes v_2) = \sigma^n(\sigma^j(v_1) \otimes v_2)$. This leads to the congruences

$$m + i \equiv n + j \bmod m_1, \quad m \equiv n \bmod m_2$$

which imply $d \mid i - j$. Hence $i = j$.

Now for all $i \in \{0, \dots, d-1\}$ define $c_i = \sigma^i(a_1)a_2$ and $h_i = [L : K(c_i)]$. One has $\Phi_L(z_{i,0}) = c_i z_{i,0}$. Then by virtue of Proposition 3 there exists for every $i \in \{0, \dots, d-1\}$ a simple $\mathcal{D}$-submodule Z_i of Z such that

$$L \otimes_K \left(\left(E(0) \oplus E\left(\frac{1}{l}\right) \oplus \cdots \oplus E\left(\frac{h_i - 1}{l}\right)\right) \otimes_K Z_i \right) \cong L\mathcal{Z}_i$$

is an isomorphism of $\mathcal{D}_L$-modules.

PROPOSITION 12. *There exists an isomorphism of $\mathcal{D}$-modules*

$$V_1 \otimes_K V_2 \cong \bigoplus_{i=0}^{d-1} \bigoplus_{j=0}^{h_i-1} E\left(\frac{j}{l}\right) \otimes_K Z_i.$$

Let $q_i(T) \in K[T]$ be the minimal polynomial of c_i over K for $i = 0, \dots, d-1$. Note that $h_i \deg(q_i) = l$. q_i represents the characteristic class of Z_i. The characteristic class of $E(j/h_i) \otimes_K Z_i$ is represented by $u_l^{j\deg(q_i)} q_i(T/u_l^j)$. Consequently, the characteristic class of $V_1 \otimes_K V_2$ is represented by

$$\prod_{i=0}^{d-1} \prod_{j=0}^{h_i-1} u_{h_i}^j q_i(T/u_l^j). \tag{24}$$

We want to compute the characteristic class of $V_1 \otimes_K V_2$ from the polynomials p_1, p_2, representing the characteristic classes of (V_1, Φ_1), resp. (V_2, Φ_2). For this define $R(T) \in K[T]$ by

$$\begin{aligned} R(T) &= \operatorname{resultant}_S(S^{\deg(p_1)} p_1(T/S), p_2(S)) \\ &= \prod_{i,j} (T - (\sigma^i(a_1)\sigma^j(a_2))), \end{aligned}$$

where i runs through $\{0, \dots, m_1 - 1\}$ and j through $\{0, \dots, m_2 - 1\}$. Note that

$$R(T) = \prod_{i=0}^{d-1} \prod_{\rho \in G} (T - \rho(\sigma^i(a_1)a_2)) = \prod_{i=0}^{d-1} q_i^{h_i} \tag{25}$$

by virtue of Proposition 11.

Note that $R(T)$ and its prime factorization in $K[T]$ can be computed (rationally) from p_1, p_2. However, the right-hand member of (25) is not necessarily the prime factorization of $R(T)$ because the q_i need not be all different. However, when $q_i = q_j$, then obviously $h_i = h_j$. Let us start now from the prime factorization of $R(T)$ in $K[T]$: $R(T) = \prod_{j=1}^{m} f_j^{\varepsilon_j}$. (The f_j are monic, irreducible and different). Define

$$I_j = \{i \mid q_i = f_j\}, \quad \mu_j = \# I_j, \quad \nu_j = h_i \text{ if } i \in I_j.$$

Then $\mu_j \nu_j = \varepsilon_j$ and (24) can be written as

$$\prod_{i=0}^{d-1} \prod_{p=0}^{h_i-1} u_{h_i}^p q_i(T/u_l^p)$$

$$= \prod_{j=1}^{m} \prod_{i \in I_j} \prod_{p=0}^{h_i-1} u_{h_i}^p q_i(T/u_l^j) = \prod_{j=1}^{m} \prod_{i \in I_j} \prod_{p=0}^{\nu_j-1} u_{\nu_i}^p f_j(T/u_l^j)$$

$$= \prod_{j=1}^{m} \left(\prod_{p=0}^{\nu_j-1} u_{\nu_j}^p f_j(T/u_l^p) \right)^{\mu_j}.$$

We have proved

PROPOSITION 13. *Let $(V_1, \Phi_1), (V_2, \Phi_2)$ be simple $\mathcal{D}$-modules. Let p_1,* resp. *p_2 in $K[T]$ represent their characteristic classes and let $\prod_{i=1}^{m} f_i^{\varepsilon_i}$ be the prime factorization in $K[T]$ of* $\mathrm{resultant}_S(S^{\deg(p_1)} p_1(T/S), p_2(S))$. *Then the tensor product of $(V_1, \Phi_1) and (V_2, \Phi_2)$ has the characteristic class which is represented by the polynomial*

$$\prod_{j=1}^{m} \left(\prod_{p=0}^{\nu_j-1} u_{\nu_j}^p f_j(T/u_l^p) \right)^{\mu_j},$$

where

$$\nu_j = l/\deg(f_j), \quad \mu_j = \varepsilon_j/\nu_j, \quad l = \mathrm{lcm}(\deg(f_1)), \ldots, \deg(f_m)).$$

Now using c the product structure on $\mathcal{K}(\mathcal{D})$ can be extented to $\mathcal{C}$. We shall denote this operation by '$*$'. To give a direct description of $*$ it is sufficient to define $[p_1] * [p_2]$ for $p_1, \ p_2 \in \mathcal{I}$:

Let $L = K(t^{1/l})$ be the common splitting field for p_1 and p_2. Let $\prod_{i=1}^{m} f_i^{\varepsilon_i}$ be the prime factorization of $\mathrm{resultant}_S(S^{\deg(p_1)} p_1(T/S), p_2(S))$. Define

$$\phi(p_1, p_2) = \prod_{j=1}^{m} \left(\prod_{p=0}^{\nu_j-1} u_{\nu_j}^p f_j(T/u_l^p) \right)^{\mu_j},$$

https://doi.org/10.1023/A:1000914223243 Published online by Cambridge University Press

where

$$\nu_j = \frac{l}{\deg(f_j)}, \qquad \mu_j = \frac{\varepsilon_j}{\nu_j}.$$

Then $\phi(p_1, p_2) \in \mathcal{I}$, $[\phi(p_1, p_2)]$ only depends on $[p_1]$, $[p_2]$. Define

$$[p_1] * [p_2] \stackrel{\text{def}}{=} [\phi(p_1, p_2)].$$

The operation 'multiplication' $([p_1], [p_2]) \mapsto [p_1] * [p_2]$ then makes $\mathcal{C}$ into a commutative ring with unity. The unity is $[T - 1]$.

The next result summarizes the preceding discussion.

COROLLARY 3. *The characteristic map* $c\colon \mathcal{K}(\mathcal{D}) \to \mathcal{C}$ *is an injective ring homomorphism*.

Remark 6. The image $\mathcal{C}_0 = c(\mathcal{K}(\mathcal{D}))$ is called the *characteristic ring*.

References

1. Duval, A.: Lemmes de Hensel et factorisation formelle pour les opérateurs aux différences, *Funk. Ekv.* **26** (1983), 349–368.
2. Levelt, A. H. M.: Jordan decomposition for a class of singular differential operators, *Ark. Mat.* **13** (1971), 1–27.
3. Praagman, C.: The formal classification of linear difference operators, *Proc. Konink. Nederl. Akad. Wetensch. Ser A* **86** (1983), 249–261.
4. Sommeling, R.: Characteristic classes for irregular singularities, Thesis, University of Nijmegen, 1990.
5. Turrittin, H. L.: The formal theory of systems of irregular homogeneous linear difference and differential equations, *Bol. Soc. Mat. Mexicana* **2** (1960), 225–264.