RIBBON CONCORDANCE OF SURFACE-KNOTS
VIA QUANDLE COCYCLE INVARIANTS

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Abstract

We give necessary conditions of a surface-knot to be ribbon concordant to another, by introducing a new variant of the cocycle invariant of surface-knots in addition to using the invariant already known. We demonstrate that twist-spins of some torus knots are not ribbon concordant to their orientation reversed images.

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1. Introduction

Throughout this paper, a *surface-knot* means a connected, oriented closed surface smoothly embedded in 4-space $\mathbb{R}^4$ up to ambient isotopies. Let F_0 and F_1 be surface-knots of the same genus. We say that F_1 *is ribbon concordant to* F_0 if there is a concordance C in $\mathbb{R}^4 \times [0, 1]$ between $F_1 \subset \mathbb{R}^4 \times \{1\}$ and $F_0 \subset \mathbb{R}^4 \times \{0\}$ such that the restriction to C of the projection $\mathbb{R}^4 \times [0, 1] \to [0, 1]$ is a Morse function with critical points of index 0 and 1 only. We write $F_1 \geq F_0$. Note that if $F_1 \geq F_0$, then there is a set of n 1-handles on a split union of F_0 and n trivial sphere-knots, for some $n \geq 0$, such that F_1 is obtained by surgeries along these handles (Figure 1).

The notion of ribbon concordance was originally introduced by Gordon [8] for classical knots in $\mathbb{R}^3$, and there are several studies found in [7, 13, 12, 17], for example. Note that F is a ribbon surface-knot if and only if F is a ribbon concordant to the trivial sphere-knot.

Given surface-knots F_0 and F_1, it is natural to ask whether F_1 is ribbon concordant to F_0. Cochran [5] gave a necessary condition for a sphere-knot F to be ribbon in
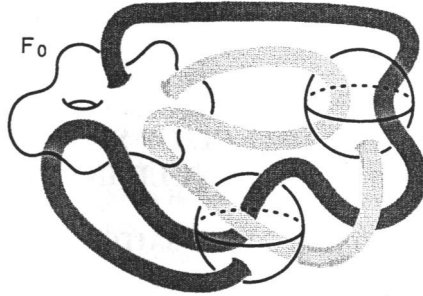
FIGURE 1.

terms of the knot group $\pi_1(\mathbb{R}^4 \setminus F)$. The aim of this paper is to give new necessary conditions for a pair of surface-knots to be ribbon concordant by using quandle cocycle invariants.

A quandle [9, 11] is an algebraic object whose model is a group with conjugation, and its cohomology theory was developed in [4] as a generalization of the theory given in [6]. It is known that each quandle 3-cocycle θ defines an invariant of a surface-knot F, called the *quandle cocycle invariant*, $\Phi_\theta(F)$. The invariant $\Phi_\theta(F)$ is regarded as a multi-set of elements in the coefficient group A of the cohomology where repetitions of the same element are allowed. For two multi-sets A' and A'' of A, we use the notation $A' \overset{m}{\subset} A''$ if for any $a \in A'$ it holds that $a \in A''$. In other words, $A' \overset{m}{\subset} A''$ if and only if $\tilde{A}' \subset \tilde{A}''$ where $\tilde{A}'$ and $\tilde{A}''$ are the subsets of A obtained from A' and A'' by eliminating the multiplicity of elements, respectively. The following is a necessary condition for ribbon concordance.

THEOREM 1.1. *If* $F_1 \geq F_0$, *then* $\Phi_\theta(F_1) \overset{m}{\subset} \Phi_\theta(F_0)$.

By Theorem 1.1, we give many examples of pairs of surface-knots such that one is *not* ribbon concordant to another (Corollary 2.1). For example, we can easily see that the 2-twist-spun trefoil and its mirror image are not ribbon concordant to each other. However, Theorem 1.1 is not effective in the family of ribbon surface-knots; in fact, $\Phi_\theta(F) = \mathbf{0}$ for any ribbon surface-knot F. Here, we use the notation $\mathbf{0}$ to stand for a multi-set consisting of zero elements of A only. In this paper, we define a new variation of cocycle invariants of surface-knots by using a quandle 2-cocycle ϕ (the definition is given in Section 3). The invariant of a surface-knot F is denoted by $\Omega_\phi(F) = \{A_\lambda \mid \lambda \in H_1(F; \mathbb{Z})\}$ which is a family of multi-sets A_λ of the coefficient group A. Note that a 2-cocycle ϕ is originally used to define the invariant, $\Phi_\phi(K)$, of a classical knot K (cf. [4]). The invariant Ω_ϕ gives another necessary condition for ribbon concordance.

THEOREM 1.2. *If $F_1 \geq F_0$, then for any $A' \in \Omega_\phi(F_1)$, there is $A'' \in \Omega_\phi(F_0)$ such that $A' \overset{m}{\subset} A''$.*

As an application of our new invariant Ω_ϕ of a surface-knot, we obtain a result on the cocycle invariant of a classical knot as follows (refer to [9] for the definition of an *involutory* quandle, or see Section 4).

THEOREM 1.3. *If ϕ is a 2-cocycle of an involutory quandle, then $\Phi_\phi(K) = 0$ for any 2-bridge knot K.*

This paper is organized as follows. In Section 2, we review the definition of the original cocycle invariant $\Phi_\theta(F)$. The proof of Theorem 1.1 and its application (Corollary 2.1) are also contained in this section. In Section 3, we introduce a new invariant $\Omega_\phi(F)$ by using a 2-cocycle ϕ, and then prove Theorem 1.2. An application of the theorem is given in Section 4 (Corollary 4.3), where we only sketch the outline of the proof and its completion is left to Appendix. Boyle [3] studied a surface-knot obtained from a twist-spun knot by surgery along a 1-handle. By using his result, we prove Theorem 1.3 also in Section 4.

REMARK. Kawauchi points out that the linking signature of every surface-knot is invariant under ribbon concordance. This result has not appeared in any paper, but can be obtained as a corollary of [10].

2. Invariants by using 3-cocycles

We first review the definition of the quandle 3-cocycle invariants of surface-knots. Refer to [4] for more details. A *quandle* is a set X with a binary operation $(a, b) \mapsto a*b$ satisfying the following three axioms:

- $a * a = a$ for any $a \in X$.
- The map $*a : X \to X$ defined by $x \mapsto x * a$ is bijective for any $a \in X$, and
- $(a * b) * c = (a * c) * (b * c)$ for any $a, b, c \in X$.

For an abelian group A, we say that a map $\theta : X^3 \to A$ is a 3-*cocycle* if it satisfies the conditions that

- $\theta(x_1, x_2, x_3) = 0$ if $x_1 = x_2$ or $x_2 = x_3$, and
- for any $x_1, \ldots, x_4 \in X$,

$$\theta(x_1, x_3, x_4) - \theta(x_1, x_2, x_4) + \theta(x_1, x_2, x_3)$$
$$= \theta(x_1 * x_2, x_3, x_4) - \theta(x_1 * x_3, x_2 * x_3, x_4)$$
$$+ \theta(x_1 * x_4, x_2 * x_4, x_3 * x_4)$$

We denote by $Z^3(X; A)$ the set of such 3-cocycles.

To describe a surface-knot, we use a fixed projection of $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ as well as a description of a classical knot into the plane. Every surface-knot F can be perturbed slightly in $\mathbb{R}^4$ so that the projection image $\pi(F)$ has double point curves, isolated triple points, and isolated branch points as the closures of the multiple point set. Crossing information is indicated in $\pi(F)$ as follows: Along every double point curve, two sheets intersect locally, one of which is under the other relative to the projection direction of π. Then the under-sheet is broken by the over-sheet. A *diagram* of F is the image $\pi(F)$ with such crossing information. Hence a diagram is regarded as a union of disjoint compact, connected surfaces. For a diagram D, we denote by $\Sigma(D)$ the set of such connected surfaces of D. Note that three sheets near a triple point are labeled top, middle, and bottom according to crossing information, and the middle and bottom sheets are divided into two and four pieces, respectively.

For a quandle X, a map $C : \Sigma(D) \to X$ is called an *X-coloring* of D if it satisfies the following condition near every double point d: if $a = C(\alpha_1)$ and $c = C(\alpha_2)$ are the colors of under-sheets α_1 and α_2 separated by the over-sheet β colored by $b = C(\beta)$, where the orientation normal of β points from α_1 to α_2, then $a * b = c$ holds. See the left of Figure 2. We denote the set of such X-colorings of D by $\mathrm{Col}_X(D)$. Also, the pair (a, b) is called the *color* of a double point d, and denoted by $C(d) \in X^2$.
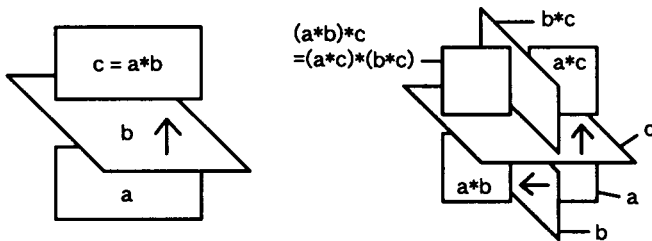

FIGURE 2.

Each triple point t of D is assigned the sign $\varepsilon(t) = \pm 1$ induced from the orientation in such a way that $\varepsilon(t) = +1$ if and only if the ordered triple of the orientation normals of the top, middle, and bottom sheets, respectively, agrees with the orientation of $\mathbb{R}^3$. Given an X-coloring $C \in \mathrm{Col}_X(D)$, the colors of the sheets near t are determined by three colors $a = C(\alpha)$, $b = C(\beta)$, and $c = C(\gamma)$, where γ is the top sheet, β is the middle sheet from which the orientation normal of γ points, and α is the bottom sheet from which the orientation normals of β and γ point both. See the right of Figure 2, where the sheets α, β, and γ are shaded. The ordered triple (a, b, c) is called the *color* of t and denoted by $C(t) \in X^3$.

Let D be a diagram of F colored by $C \in \mathrm{Col}_X(D)$. Given a 3-cocycle $\theta \in Z^3(X; A)$, we define the (*Boltzmann*) *weight* of each triple point t by

$$W_\theta(t; C) = \varepsilon(t) \cdot \theta(a, b, c) \in A,$$

where $C(t) = (a, b, c)$. We denote by $W_\theta(C) \in A$ the sum $\sum_t W_\theta(t; C)$ for all triple points of D. Then the *cocycle invariant* of F by using θ is the multi-set $\Phi_\theta(F) = \{W_\theta(C) \in A \mid C \in \mathrm{Col}_X(D)\}$, where repetitions of the same element are allowed. It is proved in [4] to be an invariant of F which does not depend on the choice of a diagram D of F.

Let F_0 and F_1 be surface-knots with $F_1 \geq F_0$, that is, F_1 is ribbon concordant to F_0. For a diagram D_0 of F_0, we may take a typical diagram D_1 of F_1 as follows: There is a set of sufficiently thin n 1-handles, $h_1, \ldots, h_n$, for some $n \geq 0$, connecting a split union of D_0 and n embedded 2-spheres, $S_1, \ldots, S_n$, such that

- each 1-handle h_j connects D_0 and S_j, and intersects $D_0 \cup \left(\bigcup_{i=1}^n S_i\right)$ with disjoint meridian 2-disks of h_j, and
- D_1 is obtained from $D_0 \cup \left(\bigcup_{i=1}^n S_i\right)$ by surgeries along $\bigcup_{j=1}^n h_j$.

In the following, we use D_1 in the above form unless otherwise stated.

PROOF OF THEOREM 1.1. For any element $a \in \Phi_\theta(F_1)$, there is an X-coloring $C_1 \in \mathrm{Col}_X(D_1)$ with $a = W_\theta(C_1) = \sum_t W_\theta(t; C_1)$ on D_1. Since the intersection of D_0 and each 1-handle h_j consists of small 2-disks, the X-coloring C_1 restricted to the punctured diagram $D_0 \setminus \left(\bigcup_{j=1}^n h_j\right)$ determines the X-coloring of D_0 uniquely, $C_0 \in \mathrm{Col}_X(D_0)$. Since the set of triple points of D_1 is coincident with that of D_0, and since $W_\theta(t; C_0) = W_\theta(t; C_1)$ for any triple point t, we have

$$a = \sum_t W_\theta(t; C_0) = W_\theta(C_0) \in \Phi_\theta(D_0). \qquad \square$$

We present specific examples as an application of Theorem 1.1 in the rest of this section. The set $\{0, 1, \ldots, p-1\}$ becomes a quandle under the operation $a * b = 2b - a$ (mod p), which is called the *dihedral quandle* of order p, and denoted by R_p. For an odd prime p, Mochizuki [14] found a 3-cocycle $\theta_p \in Z^3(R_p, \mathbb{Z}_p)$ given by

$$\theta_p(x_1, x_2, x_3) = (x_1 - x_2)\frac{(2x_3 - x_2)^p + x_2^p - 2x_3^p}{p},$$

where coefficients in the numerator are divisible by p. The reader can check that θ_p satisfies the 3-cocycle conditions by hands (cf. [2]).

In 1965 Zeeman [18] introduced an important family of sphere-knots. We take a tangle (knotted arc) T_K in the 3-ball B^3, whose closure is a classical knot K. For an integer $r \geq 0$, let $\{f_t\}_{t \in [0,1]}$ be the ambient isotopy of B^3 which rotates the tangle T_K a

total of r times about an axis while keeping the boundary of T_K fixed. Furthermore, $f_0(T_K) = f_1(T_K)$. We construct an annulus A properly embedded in $B^3 \times S^1$ from

$$\bigcup_{t \in [0,1]} f_t(T_K) \times \{t\} \subset B^3 \times [0, 1]$$

by identifying the quotient $[0, 1]/(0 = 1)$ with S^1. The *r-twist-spin of K* is a sphere-knot obtained by embedding $(B^3 \times S^1, A)$ in $\mathbb{R}^4$ standardly and capping A with two 2-disks along the boundary of A. We denote the sphere-knot by $\tau^r K$.

Let $T(2, q)$ denote the $(2, q)$-torus knot in $\mathbb{R}^3$. For a surface-knot F, let $-F$ denote the surface-knot F with the reversed orientation. Then we have the following.

COROLLARY 2.1.   (i) *If q and q' are distinct odd primes, then we have*

$$\tau^2 T(2, q) \not\gtrsim \tau^2 T(2, q') \quad and \quad \tau^2 T(2, q') \not\gtrsim \tau^2 T(2, q).$$

(ii) *If q is an odd prime with $q \equiv 3$ (mod 4), then we have*

$$\tau^2 T(2, q) \not\gtrsim -\tau^2 T(2, q) \quad and \quad -\tau^2 T(2, q) \not\gtrsim \tau^2 T(2, q).$$

PROOF. (i) It is proved in [2] that $\Phi_{\theta_p}(\tau^2 T(2, q)) = 0$ for $p \neq q$, and

$$\Phi_{\theta_q}(\tau^2 T(2, q)) = \left\{ \begin{array}{ccc} 0, & \cdots, & 0, \\ -2 \cdot 1^2, & \cdots, & -2 \cdot 1^2, \\ -2 \cdot 2^2, & \cdots, & -2 \cdot 2^2, \\ \cdots & \cdots & \cdots, \\ -2(q-1)^2, & \cdots, & -2(q-1)^2 \end{array} \right\}$$

for $p = q$, where the number of each term of the form $-2k^2$ $(k = 0, 1, \ldots, q - 1)$ is q. In particular, since $\Phi_{\theta_q}(\tau^2 T(2, q)) \neq 0$ and $\Phi_{\theta_q}(\tau^2 T(2, q')) = 0$, we have $\tau^2 T(2, q) \not\gtrsim \tau^2 T(2, q')$ by Theorem 1.1. It is also similarly proved that $\tau^2 T(2, q') \not\gtrsim \tau^2 T(2, q)$.

(ii) It is known that $\Phi_\theta(-F) = -\Phi_\theta(F)$ for any surface-knot F and 3-cocycle θ (see, for example, [4]). On the other hand, we obtain the set $S = \{-2k^2 \mid k = 0, 1, \ldots, (p-1)/2\}$ from $\Phi_{\theta_q}(\tau^2 T(2, q))$ by eliminating the multiplicity of elements. It is not difficult to see that if $q \equiv 3$ (mod 4), then $S \not\subset -S$ and $S \not\supset -S$, and hence, we have the conclusion by Theorem 1.1.   □

3. Invariants by using 2-cocycles

Let X be a quandle and A an abelian group. We say that a map $\phi : X^2 \to A$ is a *2-cocycle* if it satisfies

- $\phi(x_1, x_2) = 0$ if $x_1 = x_2$, and
- $\phi(x_1, x_3) - \phi(x_1, x_2) = \phi(x_1 * x_2, x_3) - \phi(x_1 * x_3, x_2 * x_3)$ for any $x_i \in X$.

We denote by $Z^2(X; A)$ the set of such 2-cocycles.

We define a cocycle invariant of a surface-knot by using a 2-cocycle $\phi \in Z^2(X; A)$. Let D be a diagram of a surface-knot F, and $C \in \mathrm{Col}_X(D)$ an X-coloring of D. Consider an oriented immersed circle L on D intersecting the double point curves transversely, and missing triple points and branch points. Let $d_1, \ldots, d_m$ denote the points on the under-sheet at which L intersects the double point curves. We give the sign $\varepsilon(d_k) = \pm 1$ to d_k such that $\varepsilon(d_k) = +1$ if and only if the orientation of L at d_k agrees with the orientation normal of the over-sheet. We define the Boltzman weight at d_k by $W_\phi(d_k; C) = \varepsilon(d_k) \cdot \phi(a, b) \in A$, where $C(d_k) = (a, b)$. Moreover, we put $W_\phi(L; C) = \sum_{k=1}^{m} W_\phi(d_k; C)$. See Figure 3. We extend these notations for a union of immersed circles L on D naturally.

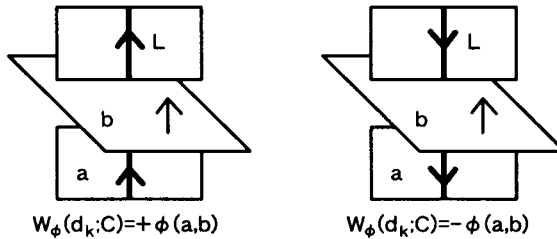$$W_\phi(d_k; C) = + \phi(a, b) \qquad W_\phi(d_k; C) = - \phi(a, b)$$

FIGURE 3.

LEMMA 3.1. *If L and L' are homologous on D, then $W_\phi(L; C) = W_\phi(L'; C)$.*

PROOF. It is sufficient to prove that $W_\phi(L; C)$ does not change under the moves (0)–(3) (and the ones with orientation reversed, or with opposite crossing information) as shown in Figure 4. First, it is clear for the move (0) by the definition of $W_\phi(L; C)$. Since ϕ satisfies $\phi(a, a) = 0$ for any $a \in X$, the move (1) also does not change $W_\phi(L; C)$. In the move (2), the terms $\phi(a, b)$ and $-\phi(a, b)$ cancels in $W_\phi(L'; C)$. Finally, it follows from the 2-cocycle condition of ϕ that $W_\phi(L; C) = W_\phi(L'; C)$ under the move (3). □

For each homology class $\lambda \in H_1(F; \mathbb{Z})$ and its representative curve $L \subset D$, the element $W_\phi(L; C) \in A$ is independent of the choice of L by Lemma 3.1, and hence, we denote it by $W_\phi(\lambda; C)$. Then we assign each class $\lambda \in H_1(F; \mathbb{Z})$ a multi-set $\Omega_\phi(\lambda)$ of A such that $\Omega_\phi(\lambda) = \{W_\phi(\lambda; C) \mid C \in \mathrm{Col}_X(D)\}$. Moreover, we define a family of multi-sets of A by $\Omega_\phi(F) = \{\Omega_\phi(\lambda) \mid \lambda \in H_1(F; \mathbb{Z})\}$.
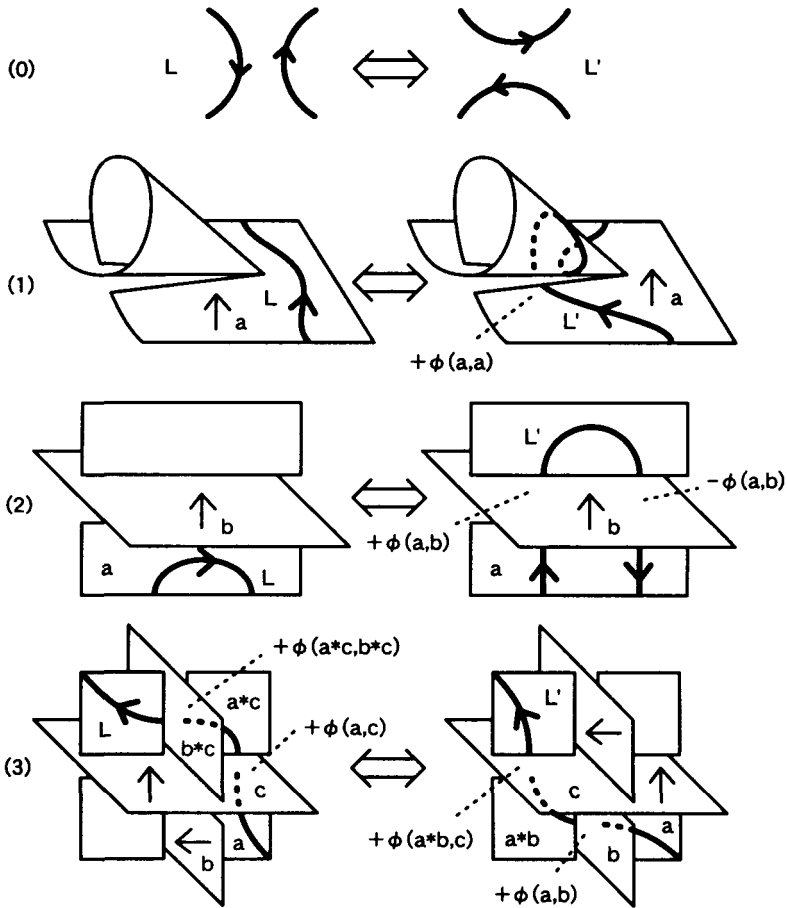
FIGURE 4.

PROPOSITION 3.2. *The family $\Omega_\phi(F)$ does not depend on the choice of a diagram D of F.*

PROOF. It is known that any other diagram D' of F is obtained from D by a finite sequence of Roseman moves [15] up to ambient isotopies of $\mathbb{R}^3$. Assume that D' is obtained from D by a single Roseman move in a sufficiently small 3-ball B^3. For any class $\lambda \in H_1(F; \mathbb{Z})$, we may take its representative curve L on D with $L \cap B^3 = \emptyset$ so that we regard L as a curve on D' also. Moreover, each X-coloring $C \in \mathrm{Col}_X(D)$ induces a coloring $C' \in \mathrm{Col}_X(D')$ uniquely. Hence, any $W_\phi(L; C)$ on D is coincident with $W_\phi(L; C')$ on D'.   □

The following proof is similar to that of Theorem 1.1 in Section 2.

PROOF OF THEOREM 1.2. Let D_i be diagrams of F_i ($i = 0, 1$) as in the beginning of Section 3. For any $A' \in \Omega_\phi(F_1)$, there is a curve L on D_1 with $A' = \Omega_\phi(L)$. Since we can deform L such that $L \cap \left(\bigcup_{j=1}^n h_j \right) = \emptyset$, L is regarded as a curve on D_0.

Put $A'' = \Omega_\phi(L) \in \Omega_\phi(F_0)$. Then $A' \overset{m}{\subset} A''$ can be proved in a similar way to Theorem 1.1.   □

4. Torus-knots with 1-handles

A surface-knot is called a *torus-knot* if it is an embedded torus in $\mathbb{R}^4$. We distinguish it from a classical 'torus knot' in $\mathbb{R}^3$ by inserting the hyphen -. In this section, we use a typical family of torus-knots studied by Boyle [3]. Let K be a classical knot in a 3-ball B^3, and let $D^3 \subset \text{int} B^3$ be a 3-ball such that $D^3 \cap K = T_K$ is the knotting arc for K. For an integer $r \geq 0$, let $\{g_t\}_{t \in [0,1]}$ be the ambient isotopy of B^3 which rotates T_K r times keeping the trivial arc $K \setminus T_K$ fixed. We denote by $\sigma^r K$ the torus-knot obtained from $\bigcup_t g_t(K) \times \{t\} \subset B^3 \times S^1$ by embedding it in $\mathbb{R}^4$ standardly. Note that $\sigma^r K$ is also obtained from the r-twist-spin of K by surgery along a certain 1-handle h.

By definition, $\sigma^r K$ has a diagram D^r in the form $\Delta \times S^1$, where Δ is a knot diagram of K, except the twisting part of T_K. See Figure 5, where we ignore crossing information along double point curves and omit the twisting part. Refer to [2, 16] for the complete figure of the diagram. We take meridional and longitudinal curves α and β on D^r, respectively, such that α can be identified with Δ, and β has no intersection with double point curves.
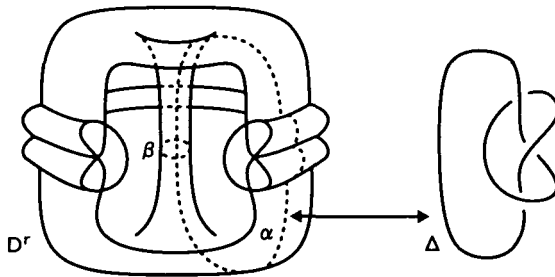

FIGURE 5.

To calculate the invariant Ω_ϕ of $\sigma^r K$, we recall the definition of the cocycle invariant $\Phi_\phi(K)$ of a classical knot K by using a 2-cocycle ϕ. Let $\Delta \subset \mathbb{R}^2$ be a diagram of an oriented classical knot K, and $\Sigma(\Delta)$ the set of arcs separated by over-arcs at crossings. For a quandle X, a map $C : \Sigma(\Delta) \to X$ is called an X-coloring of Δ if it satisfies the following condition near every crossing x: if $a = C(\alpha_1)$ and $c = C(\alpha_2)$ are the colors of under-arcs α_1 and α_2 separated by the over-arc β colored by $b = C(\beta)$, where α_1 is on the right side of β, then $a * b = c$ holds. We denote the set of such X-colorings

of Δ by $\text{Col}_X(\Delta)$. Also, the pair (a, b) is called the color of the crossing x, and denoted by $C(x) \in X^2$. Given a 2-cocycle $\phi \in Z^2(X; A)$, we define the Boltzmann weight at x by $W_\phi(x; C) = \varepsilon(x) \cdot \phi(a, b) \in A$, where $C(x) = (a, b)$. We denote by $W_\phi(C) \in A$ the sum $\sum_x W_\phi(x; C)$ for all crossings of Δ. Then the cocycle invariant of K by using $\cdot$ is the multi-set $\Phi_\phi(K) = \{W_\phi(C) \mid C \in \text{Col}_X(\Delta)\}$ where repetitions of the same element are allowed. It is proved in [4] to be an invariant of K which does not depend on the choice of a diagram Δ of K.

Any X-coloring of D^r determines that of Δ by restricting it to the meridional curve α. Conversely, not any X-coloring of Δ extend to D^r totally; an X-coloring of Δ extends to D^r if and only if $x(*y)^r = x$ for any $x, y \in X$ appeared in Δ; this condition corresponds to the r-twisting of T_K. Refer to [2, 16] for more details. A quandle X is called *of type s* $(s \geq 0)$ if it satisfies that $x(*y)^s = x$ for any $x, y \in X$, and in particular, X is an *involutory* quandle if it is of type 2. The dihedral quandle R_p is an example of involutory quandles; $(x * y) * y \equiv (2y - x) * y \equiv 2y - (2y - x) \equiv x$ (mod p). Then we have the following immediately.

LEMMA 4.1 (cf. [2, 16]). *If X is a quandle of type s, then for any $r=0, s, 2s, 3s, \ldots$, there is a natural one-to-one correspondence between $\text{Col}_X(D^r)$ and $\text{Col}_X(\Delta)$.*

PROPOSITION 4.2. *Assume that X is a quandle of type s, and let $\phi \in Z^2(X; A)$ a 2-cocycle of X. For any $r = 0, s, 2s, 3s, \ldots$, the cocycle invariant $\Omega_\phi(\sigma^r K)$ is given by*

$$\Omega_\phi(\sigma^r K) = \left\{ \begin{array}{ccc} \cdots & \cdots & \cdots \\ -\Phi_\phi(K), & -\Phi_\phi(K), & \cdots \\ 0, & 0, & \cdots \\ \Phi_\phi(K), & \Phi_\phi(K), & \cdots \\ 2\Phi_\phi(K), & 2\Phi_\phi(K), & \cdots \\ \cdots & \cdots & \cdots \end{array} \right\},$$

where the number of each multi-set $k\Phi_\phi(K)$ is infinite $(k \in \mathbb{Z})$. In particular, we have $\Phi_\phi(K) \in \Omega_\phi(\sigma^r K)$.

PROOF. Recall that (α, β) represents a basis of $H_1(\sigma^r K; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. For any class $\lambda = k[\alpha] + l[\beta]$ $(k, l \in \mathbb{Z})$, we have

$$W_\phi(\lambda; C) = k W_\phi(\alpha; C) + l W_\phi(\beta; C) = k W_\phi(\alpha; C)$$

by definition. Hence it follows from Lemma 4.1 that $W_\phi(\lambda) = k\Phi_\phi(K)$.   □

For integers m and n, let $S(m, n)$ be the classical knot represented by the diagram as shown in Figure 6. Note that $S(3, 3)$ is coincident with 8_5 in the knot table. Then we have the following as a corollary of Theorem 1.2. We sketch the outline of the proof here, and a complete proof is given in Appendix.
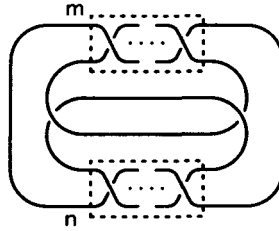
FIGURE 6.

COROLLARY 4.3. *We have $\sigma^r T(2, l) \not\gtrsim \sigma^s S(m, n)$ for any $r, s \equiv 0$ (mod 4) and $l, m, n \equiv 3$ (mod 6).*

PROOF. Let Q_6 be the subset of the permutation group of four letters, consisting of six cyclic elements of length four. Then Q_6 has a quandle structure under conjugation. Note that Q_6 is a quandle of type 4. There exists a 2-cocycle $\phi \in Z^2(Q_6; \mathbb{Z}_4)$ with the coefficient group $\mathbb{Z}_4$ such that the associated invariant of the $(2, k)$-torus knot satisfies

$$\Phi_\phi\bigl(T(2, l)\bigr) = \bigl\{\underbrace{0, 0, \ldots, 0,}_{6} \underbrace{l + 2, l + 2, \ldots, l + 2}_{24}\bigr\}$$

for any $l \equiv 3$ (mod 6), where the values in the invariant are taken in $\mathbb{Z}_4$. On the other hand, the invariant of $S(m, n)$ associated with the same 2-cocycle ϕ satisfies

$$\Phi_\phi\bigl(S(m, n)\bigr) = \bigl\{\underbrace{0, 0, \ldots, 0,}_{30} \underbrace{2, 2, \ldots, 2}_{24}\bigr\}$$

for any $m, n \equiv 3$ (mod 6). By Proposition 4.2, we have $\Phi_\phi\bigl(T(2, l)\bigr) \in \Omega_\phi(\sigma^r T(2, l))$ for $r \equiv 0$ (mod 4). Since

$$\Phi_\phi\bigl(T(2, l)\bigr) \overset{m}{\not\subset} k\Phi_\phi\bigl(S(m, n)\bigr) = \{0, 0, \ldots, 2k, 2k, \ldots\} \in \Omega_\phi\bigl(\sigma^s S(m, n)\bigr)$$

for any $s \equiv 0$ (mod 4) and $k \in \mathbb{Z}$, it follows from Theorem 1.2 that $\sigma^r T(2, l)$ is not ribbon concordant to $\sigma^s S(m, n)$. $\qquad\square$

To prove Theorem 1.3, we prepare the following lemma. We say that a torus-knot is *reducible* if it is obtained from a sphere-knot by surgery along a trivial 1-handle.

LEMMA 4.4. *If a torus-knot F is reducible, then $\Omega_\phi(F) = \{0, 0, \ldots, 0, \ldots\}$ for any 2-cocycle ϕ.*

PROOF. For any class $\lambda \in H_1(F; \mathbb{Z})$, we can choose a representative curve L of λ along the trivial 1-handle which does not meet any double point curves. Hence, we have $W_\phi(\lambda; C) = 0$ by definition. $\qquad\square$

PROOF OF THEOREM 1.3. Consider the invariant $\Omega_\phi(\sigma^2 K)$ of the torus-knot $\sigma^2 K$. Since X is an involutory quandle, that is, of type 2, we have $\Phi_\phi(K) \in \Omega_\phi(\sigma^2 K)$ by Proposition 4.2. On the other hand, Boyle [3] proved that if K is a 2-bridge knot, then $\sigma^2 K$ is a reducible torus-knot. Hence, we have $\Phi_\phi(K) = 0$ by Lemma 4.4.   □

Acknowledgements

Appendix

Let Q_6 be the subset of the permutation group of four letters $1, 2, 3, 4$ consisting of cyclic elements of length four, where the subscript 6 stands for the number of elements belonging to Q_6. Then Q_6 becomes a quandle under conjugation $g * h = h^{-1}gh$; in general, any conjugacy class of a group becomes a quandle under the conjugation. For example, if $g = (1342)$ and $h = (1234)$, then

$$g * h = (1342) * (1234) = (1234)^{-1}(1342)(1234) = (1324).$$

In other words, $g * h$ is obtained from g by replacing the letters in g according to the permutation of h. Note that since

$$g(*h)^4 = h^{-4}gh^4 = g,$$

Q_6 is a quandle of type 4.

The quandle Q_6 can be visualized by using the equilateral octahedron H (Figure 7). First, we number the faces of H by $1, \ldots, 4$ in such a way that each pair of parallel faces admit the same number. At each vertex, we put the element of Q_6 by reading the numbers on faces concentrated at the vertex counterclockwise. Under the identification of Q_6 and the set of vertices of H, the vertex $g * h$ is obtained from g by rotating H quarterly around the diagonal axis through h in the counterclockwise direction; in fact, the permutation of the numbers on faces caused by the rotation is coincident with h as an element of Q_6. Note that for each vertex $g \in Q_6$, the inverse g^{-1} is located on the diagonal vertex. Quandles consisting of rotations of an equilateral polyhedron can be found in [1].
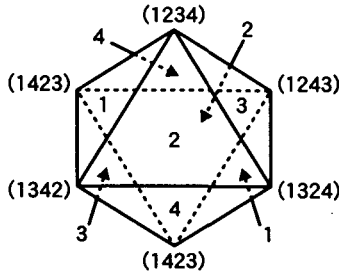
FIGURE 7.

Recall that the 2-cocycle conditions are

(1) $\phi(a, a) = 0$ for any $a \in Q_6$, and
(2) $\phi(a, c) - \phi(a, b) - \phi(a * b, c) + \phi(a * c, b * c) = 0$ for any $a, b, c \in Q_6$.

By using the model of the octahedron H, we will give a way to check whether a given map $\phi : Q_6 \times Q_6 \to A$ satisfies condition (2). For this purpose, we interpret condition (2) visually.

Case 1. Assume that the set $\{a, b, c\}$ contains the same element.

1-i.   If $a = b$ or $b = c$, then condition (2) always holds under (1).

1-ii.   Assume that $a = c \neq b$. If $b = a^{-1}$, then (2) always holds similarly. If $b \neq a^{-1}$, then (2) is equivalent to

(3) $\phi(a, b) + \phi(a * b, a) - \phi(a, b * a) = 0$,

for any pair (a, b) which spans an edge of the octahedron H. We illustrate this condition (3) as in Figure 8, where the black/white arrow $\overrightarrow{xy}$ corresponds to the value $\phi(x, y)$ or $-\phi(x, y)$, respectively.

Case 2. Assume that $\{a, b, c\}$ contains no pair of the same element but a pair of inverse elements.

2-i.   If $b = a^{-1}$, then we have $\phi(a, a^{-1}) = \phi(a * c, a^{-1} * c)$. By changing a and c variously, (2) implies that $\phi(a, a^{-1})$ is constant regardless of $a \in Q_6$, which we denote by $\delta \in A$.

2-ii.   If $c = b^{-1}$, then condition (2) is equivalent to

(4) $\phi(a, b^{-1}) + \phi(a * b^{-1}, b) - \phi(a, b) - \phi(a * b, b^{-1}) = 0$,

for any pair (a, b) which spans an edge of H. We also illustrate condition (4) in Figure 8.

2-iii. If $c = a^{-1}$, then (2) is equivalent to

(5) $\phi(a, b) + \phi(a * b, a^{-1}) - \phi(a, b * a^{-1}) = \delta$,

for any pair (a, b) which spans an edge of H. See Figure 8 again, where $a*b = b*a^{-1}$ holds.

Case 3. Assume that $\{a, b, c\}$ spans a face of the octahedron H.

3-i. If $c = b * a$, then condition (2) is equivalent to (5).
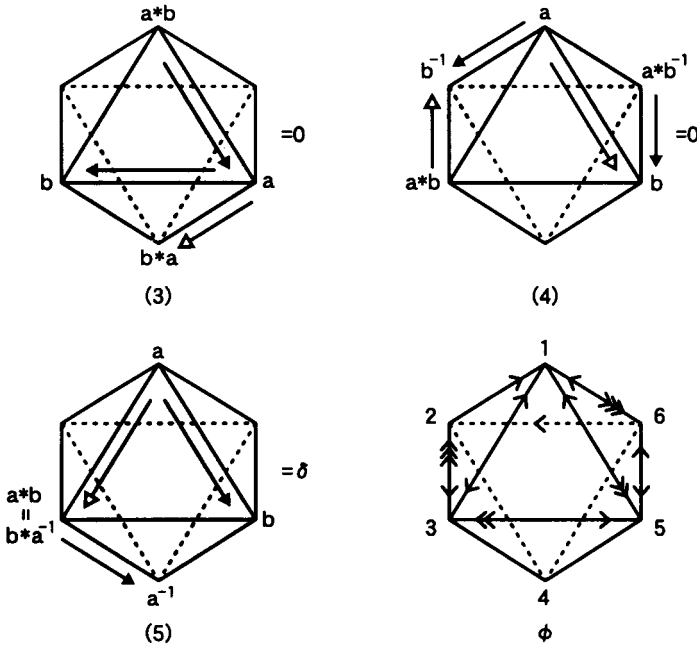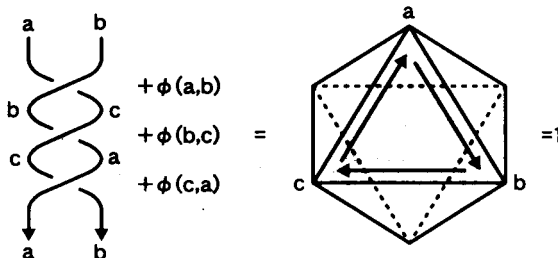3-ii. If $c = a * b$, then condition (2) is equivalent to (3).

FIGURE 8.

FIGURE 9.

We rewrite the elements of Q_6 by

$$1 \leftrightarrow (1234), \quad 2 \leftrightarrow (1423), \quad 3 \leftrightarrow (1342),$$
$$4 \leftrightarrow (1423), \quad 5 \leftrightarrow (1324), \quad 6 \leftrightarrow (1243).$$

Note that $1^{-1} = 4$, $2^{-1} = 5$, and $3^{-1} = 6$. We consider the map

$$\phi : Q_6 \times Q_6 \to \mathbb{Z}_4 = \{0, 1, 2, 3\}$$

such that $\phi(a, a) = 0$ and $\phi(a, a^{-1}) = 1$ for any $a \in Q_6$, and

- $\phi(1, 3) = \phi(2, 1) = \phi(2, 3) = \phi(3, 1) = \phi(3, 5) = \phi(5, 1) = \phi(5, 6) = \phi(6, 1) = \phi(6, 2) = \phi(6, 5) = 1$,
- $\phi(1, 5) = \phi(5, 3) = 2$,
- $\phi(1, 6) = \phi(3, 2) = 3$, and
- $\phi(a, b) = 0$ for other cases.

The value $\phi(a, b)$ for $b \neq a, a^{-1}$ is also indicated in the lower right of Figure 8 by the number of arrows on the edge $\overrightarrow{ab}$. Then the reader can check that ϕ satisfies the conditions (3)–(5), and hence, ϕ is a 2-cocycle in $Z^2(Q_6; \mathbb{Z}_4)$.

For this 2-cocycle ϕ, we calculate the invariant of $T(2, l)$ for $l \equiv 3 \pmod 6$. Consider the diagram of $T(2, l)$ as a closure of the 2-string braid with l half twists. Since each Q_6-coloring of the diagram is determined by the pair of colors (a, b) on the top arcs of the braid, we denote the coloring by $C(a, b)$. There are 6 trivial Q_6-colorings $C(a, a)$ for which we have $W_\phi(C(a, a)) = 0$ by definition. If $b = a^{-1}$, then the bottom arcs of the braid admits the pair of colors (a^{-1}, a); for l is odd. Hence, such a Q_6-coloring does not exist. If $b \neq a, a^{-1}$, that is, $\{a, b\}$ is the boundary of an edge of the octahedron H, then the same pair of colors appears by three half twists. See Figure 9. The number of such Q_6-colorings are $6 \times (6 - 2) = 24$. For each Q_6-coloring $C(a, b)$ with $b \neq a, a^{-1}$, we have

$$W_\phi(C(a, b)) = \frac{l}{3}(\phi(a, b) + \phi(b, c) + \phi(c, a)),$$

where $c = a * b$. On the other hand, we see that $\phi(a, b) + \phi(b, c) + \phi(c, a) = 1$ by the definition of ϕ. Hence, we have $W_\phi(C(a, b)) = l/3 \equiv l + 2 \pmod 4$, and

$$\Phi_\phi(T(2, l)) = \{W_\phi(C(a, b)) \mid a = b \text{ or } b \neq a, a^{-1}\}$$
$$= \{\underbrace{0, 0, \ldots, 0}_{6}, \underbrace{l + 2, l + 2, \ldots, l + 2}_{24}\}.$$

The calculation of $\Phi_\phi(S(m, n))$ for $m, n \equiv 3 \pmod 6$ can be similarly checked, and the details are left to the reader. The classical knot $S(m, n)$ has a diagram as a

closure of the 3-string braid $\sigma_1^m \sigma_2^{-1} \sigma_1^n \sigma_2^{-1}$ for the standard generators σ_1 and σ_2 of the braid group. Let $C(a, b, c)$ be the Q_6-coloring of the diagram such that the colors of the top first, second, and third arcs are $a, b, c \in Q_6$, respectively. Then we have the following three cases:

- $W_\phi\big(C(a, a, a)\big) = 0$ for any $a \in Q_6$;
- $W_\phi\big(C(a, b, b)\big) = m + n + 2$ for any $a, b \in Q_6$ with $b \neq a, a^{-1}$; and
- $W_\phi\big(C(a, b, b^{-1})\big) = m + n$ for any $a, b \in Q_6$ with $b \neq a, a^{-1}$.

Since the numbers of Q_6-colorings in these cases are 6, 24, and 24, respectively, and since $m + n$ is even, we have

$$\Phi_\phi\big(S(m, n)\big) = \big\{ \underbrace{0, 0, \ldots, 0}_{6+24=30}, \underbrace{2, 2, \ldots, 2}_{24} \big\}.$$

References

[1] N. Andruskiewitsch and M. Graña, 'From racks to pointed Hopf algebras', *Adv. Math.* **178** (2003), 177–243.

[2] S. Asami and S. Satoh, 'An infinite family of non-invertible surfaces in 4-space', *Bull. London Math. Soc.* **37** (2005), 285–296.

[3] J. Boyle, 'Classifying 1-handles attached to knotted surfaces', *Trans. Amer. Math. Soc.* **306** (1988), 475–487.

[4] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford and M. Saito, 'Quandle cohomology and state-sum invariants of knotted curves and surfaces', *Trans. Amer. Math. Soc.* **355** (2003), 3947–3989.

[5] T. D. Cochran, 'Ribbon knots in S^4', *J. London Math. Soc.* **28** (1983), 563–576.

[6] R. Fenn, C. Rourke and B. Sanderson, 'James bundles and applications', preprint, available at: http://www.maths.warwick.ac.uk/⌐bjs/.

[7] P. M. Gilmer, 'Ribbon concordance and a partial order on S-equivalence classes', *Topology Appl.* **18** (1984), 313–324.

[8] C. McA. Gordon, 'Ribbon concordance of knots in the 3-sphere', *Math. Ann.* **257** (1981), 157–170.

[9] D. Joyce, 'A classifying invariant of knots, the knot quandle', *J. Pure Appl. Alg.* **23** (1982), 37–65.

[10] A. Kawauchi, 'Torsion linking forms on surface-knots and exact 4-manifolds', in: *Knots in Hellas '98 (Delphi)*, Ser. Knots Everything 24 (World Sci. Publishing, River Edge, NJ, 2000) pp. 208–228.

[11] S. Matveev, 'Distributive groupoids in knot theory', *Math. USSR-Sbornik* **47** (1982), 73–83 in Russian.

[12] K. Miyazaki, 'Ribbon concordance does not imply a degree one map', *Proc. Amer. Math. Soc.* **108** (1990), 1055–1058.

[13] ———, 'Band-sums are ribbon concordant to the connected sum', *Proc. Amer. Math. Soc.* **126** (1998), 3401–3406.

[14] T. Mochizuki, 'Some calculations of cohomology groups of finite Alexander quandles', *J. Pure Appl. Algebra* **179** (2003), 287–330.

[15] D. Roseman, 'Reidemeister-type moves for surfaces in four-dimensional space', in: *Knot Theory (Warsaw, 1995)*, Banach Center Publ. 42 (Polish Acad. Sci., Warsaw, 1998) pp. 275–295.

[16] S. Satoh, 'Surface diagrams of twist-spun 2-knots', *J. Knot Theory Ramifications* **11** (2002), 413–430.

[17] D. S. Silver, 'On knot-like groups and ribbon concordance', *J. Pure Appl. Algebra* **82** (1992), 99–105.

[18] E. C. Zeeman, 'Twisting spun knots', *Trans. Amer. Math. Soc.* **115** (1965), 471–495.

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