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NEW MARTINGALE INEQUALITIES IN REARRANGEMENT-INVARIANT FUNCTION SPACES

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Abstract We establish various martingale inequalities in a rearrangement-invariant (RI) Banach function space. If X is an RI space that is not too small, we associate with it RI spaces $\mathcal{H}_p(X)$ $(1 \leqslant p < \infty)$ and $K(X)$, and discuss martingale inequalities in these spaces. One of our results is as follows. Let $1 \leqslant p < \infty$, let $f = (f_n)$ be an L_p-bounded martingale, and let $|f|^p = g + h$ be the Doob decomposition of the submartingale $|f|^p = (|f_n|^p)$ into a martingale $g = (g_n)$ and a predictable non-decreasing process $h = (h_n)$ with $h_0 = 0$. Then, in the case where $1 < p < \infty$, we obtain the inequalities

$$\|h_\infty^{1/p}\|_X \leqslant 2\|f_\infty\|_{\mathcal{H}_p(X)} \quad \text{and} \quad \Big\|\sup_n |g_n|^{1/p}\Big\|_X \leqslant 4\|f_\infty\|_{\mathcal{H}_p(X)},$$

and, in the case where $p = 1$, we obtain the inequalities

$$\|h_\infty\|_X \leqslant \sup_{n\in\mathbb{Z}_+} \|f_n\|_{K(X)} \quad \text{and} \quad \sup_{n\in\mathbb{Z}_+} \|g_n\|_X \leqslant 2 \sup_{n\in\mathbb{Z}_+} \|f_n\|_{K(X)}.$$

For some specific choices of X, we can give explicit expressions for $\mathcal{H}_p(X)$ and $K(X)$. For example, $\mathcal{H}_1(L_1) = L \log L$, $\mathcal{H}_p(L_{p,\infty}) = L_{p,1}$, and so on. Furthermore, if the Boyd indices of X satisfy $0 < \alpha_X \leqslant \beta_X < 1/p$ (respectively, $0 < \alpha_X$), then $\mathcal{H}_p(X) = X$ (respectively, $K(X) = X$). In any case, $\mathcal{H}_p(X)$ is embedded in $K(X)$, and $K(X)$ is embedded in X.

Keywords: martingale inequality; rearrangement-invariant space; Boyd indices

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1. Introduction

The purpose of this paper is to establish various martingale inequalities in rearrangement-invariant (RI) Banach function spaces. Roughly speaking, a rearrangement-invariant Banach function space (or simply, an RI space) is a Banach lattice X of measurable functions (or random variables) such that $\|x\|_X = \|y\|_X$ whenever x and y have the same distribution. Some martingale inequalities in such a space X have been studied by several authors. In 1988 Johnson and Schechtman [**10**] gave a necessary and sufficient condition on X for the inequality

$$c_X\|Sf\|_X \leqslant \|Mf\|_X \leqslant C_X\|Sf\|_X \tag{1.1}$$

to hold for any martingale $f = (f_n)_{n\in\mathbb{Z}_+}$, where Mf denotes the maximal function of f and Sf denotes the square function of f. The same result was proved independently by Antipa [**1**] and by Novikov [**14**]. As an application of our main result, we can obtain an extension of this inequality.

In order to prove inequality (1.1), the authors of papers [**10**], [**1**] and [**14**] used a standard method: they derived (1.1) from a distribution function inequality for Mf and Sf. (Hitczenko [**9**] also gave another proof of (1.1). His method also needs a distribution function inequality.) Our approach differs from their method completely. We will use Boyd's theorem on the boundedness of averaging operators to derive some norm inequalities for processes. The advantage of our approach, which may not be simpler, is that it enables us to derive inequalities involving the norms of two random variables in different RI spaces, such as

$$\|Mf\|_X \leqslant C\|Sf\|_Y \quad \text{or} \quad \|Sf\|_X \leqslant C\|Mf\|_Y.$$

In fact, we can prove that these inequalities hold whenever Y is continuously embedded in an RI space $K(X)$, which will be defined in the next section.

In addition, we can prove some other inequalities. Among them, the inequalities for the Doob decomposition of certain submartingales (Theorems 4.1 and 4.2) may be useful. They are described as follows. Let $1 \leqslant p < \infty$, let $f = (f_n)$ be an L_p-bounded martingale, and let $|f|^p = g + h$ be the Doob decomposition of the submartingale $|f|^p = (|f_n|^p)$ into a martingale $g = (g_n)$ and a predictable non-decreasing process $h = (h_n)$ with $h_0 = 0$. Then, in the case where $1 < p < \infty$, we obtain the inequalities

$$\|h_\infty^{1/p}\|_X \leqslant 2\|f_\infty\|_{\mathcal{H}_p(X)} \quad \text{and} \quad \Big\|\sup_n |g_n|^{1/p}\Big\|_X \leqslant 4\|f_\infty\|_{\mathcal{H}_p(X)},$$

and, in the case where $p = 1$, we obtain the inequalities

$$\|h_\infty\|_X \leqslant \sup_{n\in\mathbb{Z}_+} \|f_n\|_{K(X)} \quad \text{and} \quad \sup_{n\in\mathbb{Z}_+} \|g_n\|_X \leqslant 2 \sup_{n\in\mathbb{Z}_+} \|f_n\|_{K(X)}.$$

Here $\mathcal{H}_p(X)$ is an RI space embedded in X (see the next section for the definition of this space). For some specific choices of X we can give explicit expressions for $\mathcal{H}_p(X)$ and $K(X)$. For example, if X is isomorphic to the Lorentz space $L_{p,\infty}$, then $\mathcal{H}_p(X)$ is isomorphic to $L_{p,1}$. Hence we have the inequalities

$$\|h_\infty^{1/p}\|_{p,\infty} \leqslant C_p\|f_\infty\|_{p,1} \quad \text{and} \quad \Big\|\sup_n |g_n|^{1/p}\Big\|_{p,\infty} \leqslant C_p'\|f_\infty\|_{p,1}.$$

As another example, let $0 < a < 1$ and suppose that X is isomorphic to the RI space $L_{\exp:a}$ consisting of all x such that $\exp(\lambda|x|^a)$ is integrable for some $\lambda > 0$. Then $K(X)$ is isomorphic to $L_{\exp:(a/(1-a))}$. Thus we obtain

$$\|h_\infty\|_{\exp:a} \leqslant C_a \sup_{n\in\mathbb{Z}_+} \|f_n\|_{\exp:(a/(1-a))}$$

and

$$\sup_{n\in\mathbb{Z}_+} \|g_n\|_{\exp:a} \leqslant C_a' \sup_{n\in\mathbb{Z}_+} \|f_n\|_{\exp:(a/(1-a))},$$

where $\|\cdot\|_{\exp:a}$ denotes the norm of $L_{\exp:a}$. In addition, $K(L_{\exp:1})$ is isomorphic to L_∞. Hence

$$\|h_\infty\|_{\exp:1} \leqslant C_a \sup_{n\in\mathbb{Z}_+} \|f_n\|_\infty \quad \text{and} \quad \sup_{n\in\mathbb{Z}_+} \|g_n\|_{\exp:a} \leqslant C'_a \sup_{n\in\mathbb{Z}_+} \|f_n\|_\infty.$$

Moreover, it is worth pointing out that if the Boyd indices of X satisfy $0 < \alpha_X \leqslant \beta_X < 1/p$, then $\mathcal{H}_p(X) = X$, and that if $\alpha_X > 0$, then $K(X) = X$.

We conclude this introductory section by giving a brief overview of the contents of this paper.

Section 2 contains preliminary definitions and results. The definition and basic properties of the spaces $\mathcal{H}_p(X)$ and $K(X)$ are given there.

Section 3 is devoted to studying norm inequalities for non-decreasing processes. The proof of the main theorem (Theorem 3.3) is given there.

In § 4, we use Theorem 3.3 to derive various norm inequalities for martingales. Besides the inequalities established there, one may be able to derive some useful inequalities from Theorem 3.3.

The final section contains some explicit expressions for $\mathcal{H}_p(X)$ and $K(X)$ in the case where X is a Lorentz space, Zygmund space, Lorentz–Zygmund space, etc. Then we can spell out the inequalities established in § 4.

2. Preliminaries

In this paper, we work with a complete probability space $(\Omega, \Sigma, \mathbb{P})$, and assume that it is *non-atomic*. Besides this, we consider the canonical probability space $(I, \mathfrak{M}, \mu)$, where I denotes the interval $(0, 1]$, $\mathfrak{M}$ denotes the σ-algebra of Lebesgue measurable sets in I, and μ denotes Lebesgue measure. Throughout the paper, we distinguish these two probability spaces; the reader who is not comfortable with this setting may assume that $\Omega = I$, $\Sigma = \mathfrak{M}$ and $\mathbb{P} = \mu$.

Let x be a random variable on Ω. The *non-increasing rearrangement* of x, denoted by x^*, is a (unique) non-increasing right-continuous function on $I = (0, 1]$ such that

$$\mathbb{P}\{|x| > \lambda\} = \mu\{x^* > \lambda\} \quad (\lambda > 0).$$

Note that x^* is represented as

$$x^*(t) = \inf\{\lambda > 0 \mid \mathbb{P}\{|x| > \lambda\} \leqslant t\} \quad (t \in I),$$

with the convention that $\inf \emptyset = \infty$.

We also define the non-increasing rearrangement ϕ^* of a function ϕ on I by regarding ϕ as a random variable on the probability space I.

If ϕ and ψ are measurable functions on I, we write $\phi \prec \psi$ to mean that

$$\int_0^t \phi^*(s)\,\mathrm{d}s \leqslant \int_0^t \psi^*(s)\,\mathrm{d}s \quad \text{for all } t \in I.$$

Furthermore, if x and y are random variables on Ω and if $x^* \prec y^*$, then we write $x \prec y$. We will frequently use the following facts (see [**3**, pp. 44, 56]).

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Fact 1. *Let x and y be random variables on Ω. If $x^* y^* \in L_1(I)$, then $xy \in L_1(\Omega)$ and*

$$\mathbb{E}[|xy|] \leqslant \int_0^1 x^*(s) y^*(s)\,\mathrm{d}s.$$

In particular,

$$\mathbb{E}[|x| 1_A] \leqslant \int_0^{\mathbb{P}(A)} x^*(s)\,\mathrm{d}s \quad (A \in \Sigma),$$

where 1_A denotes the indicator function of $A \in \Sigma$. Analogous estimates hold for any measurable functions on I.

Fact 2. *Let ϕ_1, ϕ_2 and ψ be non-negative measurable functions on I. If $\phi_1 \prec \phi_2$ and if ψ is non-increasing, then*

$$\int_0^1 \phi_1^*(s)\psi(s)\,\mathrm{d}s \leqslant \int_0^1 \phi_2^*(s)\psi(s)\,\mathrm{d}s.$$

Suppose that X and Y are normed linear spaces of random variables on Ω (or measurable functions on I). We write $Y \hookrightarrow X$ if Y is continuously embedded in X.

Definition 2.1. A real Banach space $(X, \|\cdot\|_X)$ of random variables on Ω (respectively, measurable functions on I) is called a *rearrangement-invariant space*, or simply an *RI space*, over Ω (respectively, I) if X satisfies the following conditions:

(B1) $L_\infty \hookrightarrow X \hookrightarrow L_1$;

(B2) if $|y| \leqslant |x|$ a.s. and $x \in X$, then $y \in X$ and $\|y\|_X \leqslant \|x\|_X$;

(B3) if $0 \leqslant x_n \uparrow x$ a.s., $x_n \in X$ for all n, and $\sup_n \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \sup_n \|x_n\|_X$;

(R) if x and y are identically distributed and if $x \in X$, then $y \in X$ and $\|x\|_X = \|y\|_X$.

For the sake of convenience, we adopt the convention that $\|x\|_X = \infty$ unless $x \in X$.

Strictly speaking, each element of an RI space is an equivalence class of random variables.

Since the underlying probability space Ω (or I) is non-atomic, condition (R) can be replaced by the following condition (cf. [**3**, Exercise 16, p. 90]):

(R′) if $y \prec x$ and $x \in X$, then $y \in X$ and $\|y\|_X \leqslant \|x\|_X$.

If X is a normed linear space satisfying (B1), (B2) and (B3), then X has the Riesz–Fischer property, and hence it is a Banach space (cf. [**3**, Theorem 1.6, p. 5]).

Now let us recall the Luxemburg representation theorem. For any RI space $(X, \|\cdot\|_X)$ *over* Ω, there exists an RI space $(\hat{X}, \|\cdot\|_{\hat{X}})$ *over* I such that

(i) $x \in X$ if and only if $x^* \in \hat{X}$;

(ii) $\|x\|_X = \|x^*\|_{\hat{X}}$ for all $x \in X$.

Such an RI space $\hat{X}$ is unique. We call $(\hat{X}, \|\cdot\|_{\hat{X}})$ the *Luxemburg representation* of X (see [**3**, pp. 62–64] for details).

We consider some linear operators acting on the space of measurable functions on I. For each $p \in [1, \infty]$ the operators $\mathcal{P}_p$, $\mathcal{Q}_p$ and $\mathcal{R}_p$ are defined by

$$(\mathcal{P}_p\phi)(t) = \frac{1}{t^{1/p}} \int_0^t \phi(s) s^{1/p} \frac{\mathrm{d}s}{s} \quad (t \in I),$$
$$(\mathcal{Q}_p\phi)(t) = \frac{1}{t^{1/p}} \int_t^1 \phi(s) s^{1/p} \frac{\mathrm{d}s}{s} \quad (t \in I),$$

and

$$(\mathcal{R}_p\phi)(t) = \int_0^1 \frac{\phi(s) s^{1/p}}{t^{1/p} + s^{1/p}} \frac{\mathrm{d}s}{s} \qquad (t \in I),$$

provided that the respective integrals are finite for almost all $t \in I$. Here we let $1/p = 0$ if $p = \infty$. We write $\mathcal{P}$ for $\mathcal{P}_1$, $\mathcal{Q}$ for $\mathcal{Q}_\infty$, and $\mathcal{R}$ for $\mathcal{R}_1$.

For each $s \in (0, \infty)$, the *dilation operator* D_s is defined by

$$(D_s\phi)(t) = \begin{cases} \phi(st), & \text{if } st \in I, \\ 0, & \text{if } st \notin I, \end{cases} \qquad (t \in I).$$

If Y is an RI space over I, then each D_s is a bounded linear operator from Y into Y and $\|D_s\|_{B(Y)} \leqslant 1 \vee s^{-1}$, where $\|D_s\|_{B(Y)}$ stands for the operator norm of $D_s : Y \to Y$. If we set

$$\alpha_Y = \sup_{0<s<1} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s} \quad \text{and} \quad \beta_Y = \inf_{1<s<\infty} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s},$$

then

$$0 \leqslant \lim_{s\to 0+} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s} = \alpha_Y \leqslant \beta_Y = \lim_{s\to\infty} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s} \leqslant 1$$

(see [**3**, p. 149] for details). The numbers α_Y and β_Y, which are determined by the structure of Y, are called the *lower* and *upper Boyd indices* of Y, respectively. Moreover, if X is an RI space over Ω, then the Boyd indices of X are defined by $\alpha_X = \alpha_{\hat{X}}$ and $\beta_X = \beta_{\hat{X}}$. For example,

$$\alpha_{L_p} = \beta_{L_p} = 1/p \qquad (1 \leqslant p \leqslant \infty)$$

and

$$\alpha_{L_{p,q}} = \beta_{L_{p,q}} = 1/p \quad (1 < p < \infty,\ 1 \leqslant q \leqslant \infty),$$

where $L_{p,q}$ denotes the Lorentz space (see [**3**, pp. 216–220]).

Now let us recall Boyd's theorem.

Boyd's theorem. *Let Y be a rearrangement-invariant space over I, and let $B(Y)$ denote the space of bounded linear operators on Y into Y.*

(i) *Suppose that $1 \leqslant p < \infty$. Then $\mathcal{P}_p \in B(Y) \iff \beta_Y < 1/p$.*

(ii) *Suppose that $1 < p \leqslant \infty$. Then $\mathcal{Q}_p \in B(Y) \iff \alpha_Y > 1/p$.*

(iii) *Suppose that $1 \leqslant p < \infty$. Then $\mathcal{R}_p \in B(Y) \iff 0 < \alpha_Y \leqslant \beta_Y < 1/p$.*

Boyd proved statements (i) and (ii) in [**4**] (see also [**3**, p. 150]). Statement (iii) is an immediate consequence of the first two statements and the following estimate:

$$\tfrac{1}{2}\{(\mathcal{P}_p\phi) + (\mathcal{Q}\phi)\} \leqslant (\mathcal{R}_p\phi) \leqslant (\mathcal{P}_p\phi) + (\mathcal{Q}\phi). \tag{2.1}$$

With each RI space X over Ω, we associate some new spaces of random variables.

Definition 2.2. Let $1 \leqslant p < \infty$, and let X be a rearrangement-invariant space over Ω. For each random variable x, we let

$$\|x\|_{H_p(X)} = \|\mathcal{P}_p x^*\|_{\hat{X}},$$
$$\|x\|_{\mathcal{H}_p(X)} = \|\mathcal{R}_p x^*\|_{\hat{X}}$$

and

$$\|x\|_{K(X)} = \|\mathcal{Q} x^*\|_{\hat{X}}.$$

The space $H_p(X)$ (respectively, $\mathcal{H}_p(X)$, $K(X)$) is defined to be the set of random variables x for which $\|x\|_{H_p(X)}$ (respectively, $\|x\|_{\mathcal{H}_p(X)}$, $\|x\|_{K(X)}$) is finite.

Lemma 2.3. *Let X be a rearrangement-invariant space over Ω. Then*

(i) *$H_p(X)$ is a rearrangement-invariant space and $H_p(X) \hookrightarrow X$ for any $p \in [1, \infty)$;*

(ii) *if the function $t \mapsto -\log t$ belongs to $\hat{X}$, then $K(X)$ is a rearrangement-invariant space and $K(X) \hookrightarrow X$;*

(iii) *if the function $t \mapsto -\log t$ belongs to $\hat{X}$, then $\mathcal{H}_p(X)$ is a rearrangement-invariant space and $\mathcal{H}_p(X) \hookrightarrow X$ for any $p \in [1, \infty)$;*

(iv) *if the function $t \mapsto -\log t$ does not belong to $\hat{X}$, then both $K(X)$ and $\mathcal{H}_p(X)$ consist of the zero function only.*

In view of this lemma, *we will assume that $-\log t \in \hat{X}$ whenever we consider the space $K(X)$ or $\mathcal{H}_p(X)$.*

Proof of Lemma 2.3. (i) We first prove that the functional $\|\cdot\|_{H_p(X)}$ is a norm. It suffices to show that this functional satisfies the triangle inequality. To this end, we

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use the fact that $(x+y)^* \prec x^* + y^*$ (see [**3**, p. 55]). Since the function $s \mapsto s^{(1/p)-1}$ is decreasing,

$$(\mathcal{P}_p(x+y)^*)(t) = \frac{1}{t^{1/p}} \int_0^t (x+y)^*(s) s^{1/p} \frac{\mathrm{d}s}{s}$$
$$\leqslant \frac{1}{t^{1/p}} \int_0^t \{x^*(s) + y^*(s)\} s^{1/p} \frac{\mathrm{d}s}{s}$$
$$= (\mathcal{P}_p x^*)(t) + (\mathcal{P}_p y^*)(t)$$

by Fact 2. This establishes the triangle inequality for $\|\cdot\|_{H_p(X)}$. Next we verify that $H_p(X)$ is an RI space. It is clear that $H_p(X)$ satisfies conditions (B2), (B3) and (R). Hence it suffices to show that $H_p(X)$ satisfies condition (B1). Since $x^* \leqslant p^{-1}\mathcal{P}_p x^*$ on I,

$$\|x\|_X = \|x^*\|_{\hat{X}} \leqslant \frac{1}{p}\|\mathcal{P}_p x^*\|_{\hat{X}} = \frac{1}{p}\|x\|_{H_p(X)},$$

i.e. $H_p(X) \hookrightarrow X$. Let $\mathbf{1}$ denote the constant function taking the value 1. Then $\mathcal{P}_p\mathbf{1} = p\mathbf{1}$ and hence

$$\|x\|_{H_p(X)} \leqslant \|\mathcal{P}_p\mathbf{1}\|_{\hat{X}}\|x\|_\infty = p\|\mathbf{1}\|_{\hat{X}}\|x\|_\infty$$

whenever $x \in L_\infty$. Thus $L_\infty \hookrightarrow H_p(X) \hookrightarrow X \hookrightarrow L_1$, as was to be shown.

(ii) In order to show that the functional $\|\cdot\|_{K(X)}$ is a norm, we have only to prove the triangle inequality. Observe that

$$\int_0^t (\mathcal{Q}\phi)(s)\,\mathrm{d}s = \int_0^1 \phi(s)\frac{s \wedge t}{s}\,\mathrm{d}s \quad (t \in I). \tag{2.2}$$

Using this we have by Fact 2 that, for any $t \in I$,

$$\int_0^t (\mathcal{Q}(x+y)^*)(s)\,\mathrm{d}s = \int_0^1 (x+y)^*(s)\frac{s \wedge t}{s}\,\mathrm{d}s$$
$$\leqslant \int_0^1 \{x^*(s) + y^*(s)\}\frac{s \wedge t}{s}\,\mathrm{d}s$$
$$= \int_0^t \{(\mathcal{Q}x^*)(s) + (\mathcal{Q}y^*)(s)\}\,\mathrm{d}s.$$

In other words, $\mathcal{Q}(x+y)^* \prec \mathcal{Q}x^* + \mathcal{Q}y^*$. From (R′) it follows that, if $x, y \in K(X)$, then $x + y \in K(X)$ and

$$\|x+y\|_{K(X)} = \|\mathcal{Q}(x+y)^*\|_{\hat{X}} \leqslant \|\mathcal{Q}x^*\|_{\hat{X}} + \|\mathcal{Q}y^*\|_{\hat{X}} = \|x\|_{K(X)} + \|y\|_{K(X)}.$$

This proves the triangle inequality for $\|\cdot\|_{K(X)}$.

We now prove that $L_\infty \hookrightarrow K(X) \hookrightarrow L_1$. Since $(\mathcal{Q}\mathbf{1})(t) = -\log t \in \hat{X}$ by hypothesis, $\|x\|_{K(X)} \leqslant \|\mathcal{Q}\mathbf{1}\|_{\hat{X}}\|x\|_\infty$ for any $x \in L_\infty$, i.e. $L_\infty \hookrightarrow K(X)$. On the other hand, by (2.2),

$$\int_0^t (\mathcal{Q}x^*)(s)\,\mathrm{d}s = \int_0^1 x^*(s)\frac{s \wedge t}{s}\,\mathrm{d}s \geqslant \int_0^t x^*(s)\,\mathrm{d}s \quad (t \in I),$$

i.e. $x^* \prec \mathcal{Q}x^*$. From (R′) it follows that, if $x \in K(X)$, then $x \in X$ and

$$\|x\|_X \leqslant \|\mathcal{Q}x^*\|_{\hat{X}} = \|x\|_{K(X)}.$$

Thus $K(X) \hookrightarrow X(\hookrightarrow L_1)$. It is clear that $K(X)$ satisfies (B2), (B3) and (R). This completes the proof of (ii).

(iii) As in the proof of (i), we can show that $\mathcal{R}_p(x+y)^* \leqslant \mathcal{R}_p x^* + \mathcal{R}_p y^*$, which establishes the triangle inequality for $\|\cdot\|_{\mathcal{H}_p(X)}$. Since $(\mathcal{R}_p \mathbf{1})(t) \leqslant p - \log t$ by (2.1), we find that $L_\infty \hookrightarrow \mathcal{H}_p(X)$. On the other hand, since $(\mathcal{R}_p x^*) \geqslant 2^{-1}\mathcal{Q}x^*$ by (2.1), we find that $\mathcal{H}_p(X) \hookrightarrow K(X) \hookrightarrow X$. Thus (iii) is proved.

(iv) It suffices to prove the statement for $K(X)$, because $\mathcal{H}_p(X) \hookrightarrow K(X)$. Suppose that there is a non-zero element x in $K(X)$. Then there are positive numbers ε and δ such that $x^*(t) \geqslant \varepsilon$ for all $t \in (0,\delta)$. Hence $\varepsilon(\log\delta - \log t) \leqslant (\mathcal{Q}x^*)(t)$ for $t \in (0,\delta)$. Since $\mathcal{Q}x^* \in \hat{X}$, the function $t \mapsto -\log t$ must belong to $\hat{X}$. This proves (iv). □

As shown above, $H_p(X)$ is an RI space for each $p \in [1,\infty)$, and hence we can consider the Luxemburg representation $\widehat{H_p(X)}$ of $H_p(X)$. Let ϕ be a non-negative measurable function on I. Then

$$\phi \in \widehat{H_p(X)} \iff \mathcal{P}_p\phi^* \in \hat{X}. \tag{2.3}$$

Indeed, since Ω is non-atomic, there is a random variable x_ϕ such that $x_\phi^* = \phi^*$ on I (see [**7**, p. 44]). Then

$$\phi \in \widehat{H_p(X)} \iff x_\phi \in H_p(X) \iff \mathcal{P}_p\phi^* = \mathcal{P}_p x_\phi^* \in \hat{X}.$$

In the same way, we find that

$$\phi \in \widehat{K(X)} \iff \mathcal{Q}\phi^* \in \hat{X}, \tag{2.4}$$
$$\phi \in \widehat{\mathcal{H}_p(X)} \iff \mathcal{R}_p\phi^* \in \hat{X}.$$

These facts will be used in the proof of the following lemma.

Lemma 2.4. *Let X be a rearrangement-invariant space over Ω.*

(i) *If $1 \leqslant p < q < \infty$, then*

$$H_q(X) \hookrightarrow H_p(X) \hookrightarrow X \quad \text{and} \quad \mathcal{H}_q(X) \hookrightarrow \mathcal{H}_p(X) \hookrightarrow X.$$

(ii) *For any $p \in [1,\infty)$,*

$$\mathcal{H}_p(X) = H_p(X) \cap K(X) = H_p(K(X)) = K(H_p(X)).$$

Proof. (i) Suppose that $1 \leqslant p < q < \infty$. Since $s^{(1/p)-1} \prec pq^{-1}s^{(1/q)-1} \prec s^{(1/q)-1}$, Fact 2 yields that, for any $t \in I$,

$$(\mathcal{P}_p x^*)(t) = \int_0^1 x^*(st)s^{1/p}\frac{\mathrm{d}s}{s} \leqslant \int_0^1 x^*(st)s^{1/q}\frac{\mathrm{d}s}{s} = (\mathcal{P}_q x^*)(t). \tag{2.5}$$

Hence $\|x\|_{H_p(X)} \leqslant \|x\|_{H_q(X)}$, i.e. $H_q(X) \hookrightarrow H_p(X)$. Moreover, since $\mathcal{R}_p x^* \leqslant 2\mathcal{R}_q x^*$ by (2.1) and (2.5), we see that $\mathcal{H}_q(X) \hookrightarrow \mathcal{H}_p(X)$.

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(ii) It is clear from (2.1) that $\mathcal{H}_p(X) = H_p(X) \cap K(X)$. To prove that $\mathcal{H}_p(X) = H_p(K(X)) = K(H_p(X))$, we use the following formulae:

$$\mathcal{Q}\mathcal{P}_p\phi = p(\mathcal{P}_p\phi + \mathcal{Q}\phi - (\mathcal{P}_p\phi)(1)) \quad (1 \leqslant p < \infty); \tag{2.6}$$

$$\mathcal{P}_p\mathcal{Q}\phi = p(\mathcal{P}_p\phi + \mathcal{Q}\phi) \quad (1 \leqslant p < \infty). \tag{2.7}$$

These formulae are valid for functions ϕ in $L_1(\mathrm{d}u^{1/p})$, in particular, for functions ϕ such that $\mathcal{P}_p\phi \in L_1$. Using (2.1), (2.4) and (2.6) we have

$$x \in H_p(K(X)) \iff \mathcal{P}_p x^* \in \widehat{K(X)} \iff \mathcal{Q}\mathcal{P}_p x^* \in \hat{X} \iff \mathcal{R}_p x^* \in \hat{X} \iff x \in \mathcal{H}_p(X).$$

Thus $H_p(K(X)) = \mathcal{H}_p(X)$. Moreover, from (2.3), (2.4), (2.6) and (2.7), we see that

$$x \in H_p(K(X)) \iff \mathcal{P}_p x^* \in \widehat{K(X)} \iff \mathcal{Q}\mathcal{P}_p x^* \in \hat{X} \iff \mathcal{P}_p\mathcal{Q}x^* \in \hat{X} \iff \mathcal{Q}x^* \in \widehat{H_p(X)} \iff x \in K(H_p(X)).$$

This completes the proof. □

Using Lemma 2.3, Lemma 2.4 and Boyd's theorem, we have the following lemma.

Lemma 2.5. *Let X be a rearrangement-invariant space over Ω.*

(i) *If $\beta_X < 1/p$, then*

- $H_p(X) = X$ *and* $\|\cdot\|_{H_p(X)} \approx \|\cdot\|_X$*;*
- $\mathcal{H}_p(X) = K(X)$ *and* $\|\cdot\|_{\mathcal{H}_p(X)} \approx \|\cdot\|_{K(X)}$*.*

(ii) *If $\alpha_X > 0$, then*

- $K(X) = X$ *and* $\|\cdot\|_{K(X)} \approx \|\cdot\|_X$*;*
- $\mathcal{H}_p(X) = H_p(X)$ *and* $\|\cdot\|_{\mathcal{H}_p(X)} \approx \|\cdot\|_{H_p(X)}$*.*

(iii) *If $0 < \alpha_X \leqslant \beta_X < 1/p$, then*

- $\mathcal{H}_p(X) = X$ *and* $\|\cdot\|_{\mathcal{H}_p(X)} \approx \|\cdot\|_X$*.*

3. Rearrangement inequalities and norm inequalities

In this section, we establish some norm inequalities for non-decreasing processes. For basic facts that we will use here, see, for instance, [**8**], [**12**], [**13**] or [**15**].

Proposition 3.1. *Let $1 \leqslant p < \infty$ and let x, y be non-negative random variables. If x and y satisfy the inequality*

$$\mathbb{E}[(x-\lambda)^p 1_{\{x>\lambda\}}] \leqslant \mathbb{E}[y^p 1_{\{x>\lambda\}}] \quad (\lambda > 0), \tag{3.1}$$

then $\mathcal{P}x^* \leqslant \mathcal{P}_p y^* + \mathcal{Q}y^*$ on I. Moreover, if x and y satisfy (3.1) for $p = 1$, then $x^* \prec \mathcal{Q}y^*$.

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For the proof of the proposition, we need the following lemma.

Lemma 3.2. *If $1 \leqslant p < \infty$ and if $\phi : I \to \mathbb{R}$ is a non-negative non-increasing function, then*

$$\left(\frac{1}{t}\int_0^t \phi(s)^p \,\mathrm{d}s\right)^{1/p} \leqslant \frac{1}{p}(\mathcal{P}_p\phi)(t) \quad (t \in I). \tag{3.2}$$

Proof. Suppose first that ϕ is of the form

$$\phi = \sum_{i=1}^n a_i 1_{(0,t_i)}, \quad \text{where } a_i > 0 \text{ and } 0 < t_1 < t_2 < \cdots < t_n \leqslant 1. \tag{3.3}$$

Then, by Minkowski's inequality,

$$\left(\int_0^t \phi(s)^p \,\mathrm{d}s\right)^{1/p} \leqslant \sum_{i=1}^n a_i (t \wedge t_i)^{1/p} = \frac{t^{1/p}}{p}(\mathcal{P}_p\phi)(t) \quad (t \in I).$$

Thus (3.2) holds for this ϕ. If ϕ is an arbitrary non-negative non-increasing function, then there exists a sequence $\{\phi_i\}$ of functions of the form (3.3) such that $0 \leqslant \phi_i \uparrow \phi$ a.e. on I. Hence we may use the monotone convergence theorem to complete the proof. □

Proof of Proposition 3.1. Setting $\lambda = x^*(t)$ in (3.1) and using Fact 1, we deduce that

$$\begin{aligned}\int_{\{s\in I \mid x^*(s)\geqslant x^*(t)\}} (x^*(s) - x^*(t))^p \,\mathrm{d}s &= \mathbb{E}[(x - x^*(t))^p 1_{\{x > x^*(t)\}}] \\ &\leqslant \mathbb{E}[y^p 1_{\{x>x^*(t)\}}] \leqslant \int_0^{\mathbb{P}\{x>x^*(t)\}} y^*(s)^p \,\mathrm{d}s.\end{aligned}$$

Since $\{s \in I \mid x^*(s) \geqslant x^*(t)\} \supset (0, t]$ and $\mathbb{P}\{x > x^*(t)\} \leqslant t$, it follows from the estimates proved above and Jensen's inequality that

$$\begin{aligned}\{(\mathcal{P}x^*)(t) - x^*(t)\}^p &= \left\{\frac{1}{t}\int_0^t (x^*(s) - x^*(t)) \,\mathrm{d}s\right\}^p \\ &\leqslant \frac{1}{t}\int_0^t (x^*(s) - x^*(t))^p \,\mathrm{d}s \\ &\leqslant \frac{1}{t}\int_0^t y^*(s)^p \,\mathrm{d}s.\end{aligned}$$

Thus, by Lemma 3.2,

$$(\mathcal{P}x^*)(t) - x^*(t) \leqslant \left(\frac{1}{t}\int_0^t y^*(s)^p \,\mathrm{d}s\right)^{1/p} \leqslant \frac{1}{p}(\mathcal{P}_p y^*)(t) \quad (t \in I),$$

and hence, by (2.6),

$$\mathcal{P}x^* - (\mathcal{P}x^*)(1) = \mathcal{Q}(\mathcal{P}x^* - x^*) \leqslant \frac{1}{p}\mathcal{Q}\mathcal{P}_p y^* = \mathcal{P}_p y^* + \mathcal{Q}y^* - (\mathcal{P}_p y^*)(1). \tag{3.4}$$

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Furthermore, inequality (3.1) together with Lemma 3.2 implies that

$$(\mathcal{P}x^*)(1) = \|x\|_1 \leqslant \|x\|_p \leqslant \|y\|_p \leqslant \frac{1}{p}(\mathcal{P}_p y^*)(1) \leqslant (\mathcal{P}_p y^*)(1).$$

Hence we conclude from (3.4) that $\mathcal{P}x^* \leqslant \mathcal{P}_p y^* + \mathcal{Q}y^*$.

Now suppose that $p = 1$. Then $\mathcal{P}x^* \leqslant \mathcal{P}y^* + \mathcal{Q}y^* = \mathcal{P}\mathcal{Q}y^*$ by (2.7). This shows that $x^* \prec \mathcal{Q}y^*$, completing the proof. □

Now we consider some norm inequalities for processes. Given a process $\xi = (\xi_n)_{n\in\mathbb{Z}_+}$, we adopt the convention that $\xi_{-1} \equiv 0$. We say that a process $\xi = (\xi_n)$ is *non-decreasing* if $\xi_{n+1} \geqslant \xi_n \geqslant 0$ a.s. for all $n \in \mathbb{Z}_+$. For a non-decreasing process ξ, we let $\xi_\infty = \lim_{n\to\infty} \xi_n$ a.s.

Given a *filtration* $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, we denote by $\mathbb{S}(\mathcal{F})$ the collection of all $\mathcal{F}$-stopping times.

Theorem 3.3. *Let $\xi = (\xi_n)_{n\in\mathbb{Z}_+}$ be a non-decreasing process adapted to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+}$, let γ be a non-negative random variable, and let X be a rearrangement-invariant space over Ω.*

(i) *If $1 < p < \infty$ and if*

$$\mathbb{E}[(\xi_\infty - \xi_{n-1})^p \mid \mathcal{F}_n] \leqslant \mathbb{E}[\gamma^p \mid \mathcal{F}_n] \text{ a.s.} \quad (n \in \mathbb{Z}_+), \tag{3.5}$$

then $\mathcal{P}(\xi_\infty)^* \leqslant 2\mathcal{R}_p\gamma^*$ on I, and $\|\xi_\infty\|_X \leqslant \|\xi_\infty\|_{H_1(X)} \leqslant 2\|\gamma\|_{\mathcal{H}_p(X)}$.

(ii) *If (3.5) holds for $p = 1$, then $(\xi_\infty)^* \prec \mathcal{Q}\gamma^*$ and $\|\xi_\infty\|_X \leqslant \|\gamma\|_{K(X)}$.*

From Theorem 3.3 and Lemma 2.5, we derive the following corollary.

Corollary 3.4. *Let $\xi = (\xi_n)$, γ and X be as in Theorem 3.3.*

(i) *Suppose that $1 < p < \infty$. If (3.5) holds and if $0 < \alpha_X \leqslant \beta_X < 1/p$, then*

$$\|\xi_\infty\|_X \leqslant C_X\|\gamma\|_X, \tag{3.6}$$

where C_X is a positive constant depending only on X.

(ii) *If (3.5) holds for $p = 1$ and if $\alpha_X > 0$, then (3.6) holds.*

Proof of Theorem 3.3. It is routine to deduce from (3.5) that, if $\sigma \in \mathbb{S}(\mathcal{F})$, then

$$\mathbb{E}[(\xi_\infty - \xi_{\sigma-1})^p \mid \mathcal{F}_\sigma] \leqslant \mathbb{E}[\gamma^p \mid \mathcal{F}_\sigma] \text{ a.s.} \tag{3.7}$$

Let $\lambda > 0$ and let $\sigma = \inf\{n \in \mathbb{Z}_+ \mid \xi_n > \lambda\}$. Then $\{\xi_\infty > \lambda\} = \{\sigma < \infty\} \in \mathcal{F}_\sigma$ and $\xi_{\sigma-1} \leqslant \lambda$ (since $\xi_{\sigma-1} = 0$ on the set $\{\sigma = 0\}$). Therefore, by (3.7),

$$\mathbb{E}[(\xi_\infty - \lambda)^p 1_{\{\xi_\infty > \lambda\}}] \leqslant \mathbb{E}[\gamma^p 1_{\{\xi_\infty > \lambda\}}].$$

Using Proposition 3.1, we obtain that $\mathcal{P}(\xi_\infty)^* \leqslant \mathcal{P}_p\gamma^* + \mathcal{Q}\gamma^* \leqslant 2\mathcal{R}_p\gamma^*$, and hence that $\|\xi_\infty\|_{H_1(X)} \leqslant 2\|\gamma\|_{\mathcal{H}_p(X)}$.

If $p = 1$, then $(\xi_\infty)^* \prec \mathcal{Q}\gamma^*$. This implies that $\|\xi_\infty\|_X \leqslant \|\gamma\|_{K(X)}$. □

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Remark 3.5. Clearly, (3.1) implies that

$$(\lambda_1 - \lambda_2)^p \mathbb{P}\{x > \lambda_1\} \leqslant \mathbb{E}[y^p 1_{\{x > \lambda_2\}}] \quad (\lambda_1 > \lambda_2 \geqslant 0).$$

We can show that if this inequality holds for $p \in (1, \infty)$, then

$$\mathcal{P}x^* \leqslant p'(\mathcal{P}_p y^* + \mathcal{Q}y^*),$$

where $p' = p/(p-1)$. Using this fact, we can derive a result analogous to Theorem 3.3: if $\xi = (\xi_n)$ is a (not necessarily non-decreasing) process and if

$$\mathbb{E}[|\xi_\tau - \xi_{\sigma-1}|^p 1_{\{\tau<\infty\}} \mid \mathcal{F}_\sigma] \leqslant \mathbb{E}[\gamma^p \mid \mathcal{F}_\sigma] \quad (\sigma, \tau \in \mathbb{S}(\mathcal{F}),\ \sigma \leqslant \tau),$$

then

$$\mathcal{P}(\sup_n \mid \xi_n|)^* \leqslant p'(\mathcal{P}_p \gamma^* + \mathcal{Q}\gamma^*) \leqslant 2p' \mathcal{R}_p \gamma^*$$

and

$$\Big\| \sup_n |\xi_n| \Big\|_X \leqslant 2p' \|\gamma\|_{\mathcal{H}_p(X)}.$$

We omit the details.

4. Norm inequalities for martingales

In this section, we will consider some norm inequalities for martingales. Throughout this section, X is an RI space over Ω.

We begin by introducing (or recalling) some notation that will be used in this section. Given a martingale $f = (f_n)_{n \in \mathbb{Z}_+}$, we denote by Mf the *maximal function* of f, and denote by Sf the *square function* of f, i.e.

$$Mf = \sup_{n \in \mathbb{Z}_+} |f_n| \quad \text{and} \quad Sf = \left\{ \sum_{n=0}^{\infty} (\Delta_n f)^2 \right\}^{1/2},$$

where $\Delta_n f = f_n - f_{n-1} (n \in \mathbb{Z}_+)$. (Note that $\Delta_0 f = f_0$ by convention.) We denote by s the operator defined by

$$sf = \left\{ \sum_{n=0}^{\infty} \mathbb{E}[(\Delta_n f)^2 \mid \mathcal{F}_{n-1}] \right\}^{1/2},$$

where $\mathcal{F}_{-1} = \{\emptyset, \Omega\}$. More generally, we define

$$s^{(p)} f = \left\{ \sum_{n=0}^{\infty} \mathbb{E}[|\Delta_n f|^p \mid \mathcal{F}_{n-1}] \right\}^{1/p} \quad (1 \leqslant p < \infty).$$

Furthermore, we deal with operators $m^{(p)}$ and $\theta^{(p)}$, acting on *the space of uniformly integrable martingales*, defined, respectively, by

$$m^{(p)} f = \sup_{n \in \mathbb{Z}_+} \mathbb{E}[|f_\infty|^p \mid \mathcal{F}_n]^{1/p} \quad (1 \leqslant p < \infty)$$

and

$$\theta^{(p)} f = \sup_{n\in\mathbb{Z}_+} \mathbb{E}[|f_\infty - f_{n-1}|^p \mid \mathcal{F}_n]^{1/p} \quad (1 \leqslant p < \infty),$$

where $f_\infty = \lim_{n\to\infty} f_n$ a.s.

Now let $1 \leqslant p < \infty$ and let $f = (f_n)_{n\in\mathbb{Z}_+}$ be a martingale such that $f_n \in L_p$ for all $n \in \mathbb{Z}_+$. By the Doob decomposition theorem, the process $|f|^p = (|f_n|^p)_{n\in\mathbb{Z}_+}$ is uniquely decomposed as the sum of a martingale $g = (g_n)_{n\in\mathbb{Z}_+}$ and a *predictable* non-decreasing process $h = (h_n)_{n\in\mathbb{Z}_+}$ with $h_0 = 0$ (see, for example, [**13**, p. 145] or [**15**, p. 153]). If f is an L_p-bounded martingale, then the limits $f_\infty = \lim_{n\to\infty} f_n$, $g_\infty = \lim_{n\to\infty} g_n$ and $h_\infty = \lim_{n\to\infty} h_n$ exist a.s.

Theorem 4.1. *Let $1 < p < \infty$, let $f = (f_n)_{n\in\mathbb{Z}_+}$ be an L_p-bounded martingale, and let $|f|^p = g + h$ be the Doob decomposition of $|f|^p = (|f_n|^p)$ into a martingale $g = (g_n)$ and a predictable non-decreasing process $h = (h_n)$ with $h_0 = 0$. Then*

$$\|h_\infty^{1/p}\|_X \leqslant 2\|f_\infty\|_{\mathcal{H}_p(X)} \quad \text{and} \quad \|(Mg)^{1/p}\|_X \leqslant 4\|f_\infty\|_{\mathcal{H}_p(X)}.$$

Moreover, if $0 < \alpha_X \leqslant \beta_X < 1/p$, then

$$\|h_\infty^{1/p}\|_X \leqslant C_{p,X}\|f_\infty\|_X \quad \text{and} \quad \|(Mg)^{1/p}\|_X \leqslant C_{p,X}\|f_\infty\|_X,$$

where $C_{p,X}$ is a positive constant depending only on p and X.

Theorem 4.2. *Let $f = (f_n)_{n\in\mathbb{Z}_+}$ be an L_1-bounded martingale, and let $|f| = g + h$ be the Doob decomposition of $|f| = (|f_n|)$ into a martingale $g = (g_n)$ and a predictable non-decreasing process $h = (h_n)$ with $h_0 = 0$. Then*

$$\|h_\infty\|_X \leqslant \sup_{n\in\mathbb{Z}_+} \|f_n\|_{K(X)} \quad \text{and} \quad \sup_{n\in\mathbb{Z}_+} \|g_n\|_X \leqslant 2 \sup_{n\in\mathbb{Z}_+} \|f_n\|_{K(X)}.$$

Moreover, if $\alpha_X > 0$, then

$$\|h_\infty\|_X \leqslant C_X \sup_{n\in\mathbb{Z}_+} \|f_n\|_X \quad \text{and} \quad \sup_{n\in\mathbb{Z}_+} \|g_n\|_X \leqslant C_X \sup_{n\in\mathbb{Z}_+} \|f_n\|_X.$$

Remark 4.3. In addition to the hypotheses of Theorem 4.2, suppose that $f = (f_n)_{n\in\mathbb{Z}_+}$ is uniformly integrable. Then $\sup_n \|f_n\|_{K(X)}$ (respectively, $\sup_n \|f_n\|_X$) can be replaced by $\|f_\infty\|_{K(X)}$ (respectively, $\|f_\infty\|_X$). To see this, recall (see [**3**, p. 74]) that

$$\int_0^t y^*(s)\,\mathrm{d}s = \inf\{\|y_1\|_1 + t\|y_2\|_\infty \mid y = y_1 + y_2,\ y_1 \in L_1,\ y_2 \in L_\infty\}.$$

From this identity, we can derive that $\mathbb{E}[x|\mathcal{G}] \prec x$ for any $x \in L_1$ and any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$. Hence, if $f = (f_n)$ is uniformly integrable, then

$$f_n = \mathbb{E}[f_\infty \mid \mathcal{F}_n] \prec f_\infty \quad \text{for each } n \in \mathbb{Z}_+.$$

Thus $\sup_n \|f_n\|_{K(X)} \leqslant \|f_\infty\|_{K(X)}$ and $\sup_n \|f_n\|_X \leqslant \|f_\infty\|_X$ by (R′).

Proof of Theorem 4.1. We prove the first statement only, because the second statement follows from the first statement and Lemma 2.5 (iii).

Since $Mf \in L_p$ by the Doob maximal inequality and since

$$\mathbb{E}[h_\infty] \leqslant -\mathbb{E}[g_0] + \sup_n \|f_n\|_p^p < \infty,$$

we see that $Mg \leqslant h_\infty + (Mf)^p \in L_1$. Hence $g = (g_n)$ is uniformly integrable, i.e. $g_n = \mathbb{E}[g_\infty \mid \mathcal{F}_n]$ for each $n \in \mathbb{Z}_+$. This implies that

$$\mathbb{E}[(h_\infty^{1/p} - h_n^{1/p})^p \mid \mathcal{F}_n] \leqslant \mathbb{E}[h_\infty - h_n \mid \mathcal{F}_n] = \mathbb{E}[|f_\infty|^p - |f_n|^p \mid \mathcal{F}_n] \leqslant \mathbb{E}[|f_\infty|^p \mid \mathcal{F}_n].$$

If we set $\xi_n = h_{n+1}^{1/p} (n \in \mathbb{Z}_+)$ and $\gamma = |f_\infty|$, then inequality (3.5) holds. From Theorem 3.3 we conclude that $\|h_\infty^{1/p}\|_X \leqslant 2\|f_\infty\|_{\mathcal{H}_p(X)}$.

Now we estimate the norm of $(Mg)^{1/p}$ in X. The Doob maximal inequality, together with Fact 1, shows that

$$\begin{aligned}
\lambda &\leqslant \frac{1}{\mathbb{P}\{Mf \geqslant \lambda\}} \mathbb{E}[|f_\infty| 1_{\{Mf \geqslant \lambda\}}] \\
&\leqslant \frac{1}{\mathbb{P}\{Mf \geqslant \lambda\}} \int_0^{\mathbb{P}\{Mf \geqslant \lambda\}} (f_\infty)^*(s)\, \mathrm{d}s \\
&= (\mathcal{P}(f_\infty)^*)(\mathbb{P}\{Mf \geqslant \lambda\}).
\end{aligned} \tag{4.1}$$

Let $t \in I$ and set $\lambda = (Mf)^*(t)$ in (4.1). Since $\mathbb{P}\{Mf \geqslant (Mf)^*(t)\} \geqslant t$ and since the function $\mathcal{P}(f_\infty)^*$ is non-increasing, it follows that

$$(Mf)^*(t) \leqslant (\mathcal{P}(f_\infty)^*)(t) \quad (t \in I). \tag{4.2}$$

Therefore,

$$\begin{aligned}
\|(Mg)^{1/p}\|_X &\leqslant \|\{(Mf)^p + h_\infty\}^{1/p}\|_X \\
&\leqslant \|Mf + h_\infty^{1/p}\|_X \\
&\leqslant \|Mf\|_X + \|h_\infty^{1/p}\|_X \\
&\leqslant \|\mathcal{P}(f_\infty)^*\|_{\hat{X}} + 2\|f_\infty\|_{\mathcal{H}_p(X)}.
\end{aligned}$$

Since $\mathcal{P}(f_\infty)^* \leqslant \mathcal{P}_p(f_\infty)^* \leqslant 2\mathcal{R}_p(f_\infty)^*$ by (2.1) and (2.5), we obtain that

$$\|(Mg)^{1/p}\|_X \leqslant 4\|f_\infty\|_{\mathcal{H}_p(X)},$$

as desired. □

Proof of Theorem 4.2. Let $N > 0$ be a fixed integer. Then

$$\mathbb{E}[h_N - h_n \mid \mathcal{F}_n] = \mathbb{E}[|f_N| - |f_n| \mid \mathcal{F}_n] \leqslant \mathbb{E}[|f_N| \mid \mathcal{F}_n] \text{ a.s.} \quad (0 \leqslant n \leqslant N).$$

If we set $\gamma = |f_N|$ and $\xi_n = h_{(n+1)\wedge N}$ for each $n \in \mathbb{Z}_+$, then (3.5) holds for $p = 1$. Therefore, $\|h_N\|_X \leqslant \|f_N\|_{K(X)}$ by Theorem 3.3 (ii). Thus we conclude (from the Fatou property (B3) of $K(X)$) that $\|h_\infty\|_X \leqslant \sup_n \|f_n\|_{K(X)}$.

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Furthermore, we see that

$$\sup_n \|g_n\|_X \leqslant \|h_\infty\|_X + \sup_n \|f_n\|_X \leqslant 2 \sup_n \|f_n\|_{K(X)}.$$

This completes the proof of the first statement of Theorem 4.2. The second statement follows from the first statement and Lemma 2.5 (ii). □

Now we consider norm inequalities for $s^{(p)}f$. Burkholder and Gundy proved in [**6**] that if $2 \leqslant r < \infty$, then

$$\|sf\|_r \leqslant C_r \|f_\infty\|_r,$$

with some constant $C_r > 0$ depending only on r. The next theorem extends this.

Theorem 4.4. *Let $f = (f_n)_{n\in\mathbb{Z}_+}$ be a uniformly integrable martingale and let $f_\infty = \lim_{n\to\infty} f_n$ a.s. If $2 \leqslant p < \infty$, then*

$$\|s^{(p)}f\|_X \leqslant C_p \|f_\infty\|_{\mathcal{H}_p(X)}, \tag{4.3}$$

with some constant $C_p > 0$ depending only on p. Moreover, if in addition $0 < \alpha_X \leqslant \beta_X < 1/p$, then

$$\|s^{(p)}f\|_X \leqslant C_{p,X} \|f_\infty\|_X,$$

with some constant $C_{p,X} > 0$ depending only on p and X.

Before proving this theorem, we recall the Burkholder inequality (see [**5**, Theorem 3.2]). For each $p \in (1,\infty)$ there is a constant $c_p > 0$ such that if $f = (f_n)$ is a uniformly integrable martingale, then

$$\mathbb{E}[(Sf)^p] \leqslant c_p \mathbb{E}[|f_\infty|^p].$$

We use the conditional form of this inequality. Given a martingale $f = (f_n)$ and an integer $N \geqslant 0$, we denote by $f^{(N)}$ the stopped martingale $(f_{n\wedge N})_{n\in\mathbb{Z}_+}$. Then

$$\mathbb{E}[S(f - f^{(N)})^p \mid \mathcal{F}_N] \leqslant c_p \mathbb{E}[|f_\infty - f_N|^p \mid \mathcal{F}_N] \text{ a.s.} \quad (N \in \mathbb{Z}_+), \tag{4.4}$$

provided that $1 < p < \infty$. Inequality (4.4) follows by applying the ordinary Burkholder inequality to the martingale $f' = ((f_{N+n} - f_N)1_A)_{n\in\mathbb{Z}_+}$ with respect to the filtration $\mathcal{F}' = (\mathcal{F}_{N+n})_{n\in\mathbb{Z}_+}$, where $A \in \mathcal{F}_N$.

Proof of Theorem 4.4. We prove the first statement only. Let

$$\xi_n = \left\{ \sum_{k=0}^{n+1} \mathbb{E}[|\Delta_k f|^p \mid \mathcal{F}_{k-1}] \right\}^{1/p} \quad (n \in \mathbb{Z}_+) \qquad \text{and} \qquad \xi_\infty = \lim_{n\to\infty} \xi_n = s^{(p)}f.$$

Since $2 \leqslant p < \infty$, we find that

$$\begin{aligned}
\mathbb{E}[(\xi_\infty - \xi_{N-1})^p \mid \mathcal{F}_N] &\leqslant \mathbb{E}[\xi_\infty^p - \xi_{N-1}^p \mid \mathcal{F}_N] \\
&\leqslant \mathbb{E}\biggl[\sum_{k=N+1}^{\infty} \mathbb{E}[|\Delta_k f|^p \mid \mathcal{F}_{k-1}] \biggm| \mathcal{F}_N\biggr] \\
&= \mathbb{E}\biggl[\sum_{k=N+1}^{\infty} |\Delta_k f|^p \biggm| \mathcal{F}_N\biggr] \\
&\leqslant \mathbb{E}\biggl[\biggl(\sum_{k=N+1}^{\infty} |\Delta_k f|^2\biggr)^{p/2} \biggm| \mathcal{F}_N\biggr] \\
&= \mathbb{E}[S(f - f^{(N)})^p \mid \mathcal{F}_N] \\
&\leqslant c_p \mathbb{E}[|f_\infty - f_N|^p \mid \mathcal{F}_N] \\
&\leqslant c_p 2^{p-1}(\mathbb{E}[|f_\infty|^p \mid \mathcal{F}_N] + |f_N|^p) \\
&\leqslant c_p 2^p \mathbb{E}[|f_\infty|^p \mid \mathcal{F}_N],
\end{aligned}$$

where we have used (4.4). Thus, if we set $\gamma = c_p^{1/p} 2|f_\infty|$, then γ satisfies (3.5). From Theorem 3.3 (i), we obtain (4.3) with $C_p = 4c_p^{1/p}$, completing the proof. □

The next theorem is an extension of the Burkholder–Davis–Gundy inequality.

Theorem 4.5. *There are absolute constants k_1 and k_2 such that if $f = (f_n)_{n \in \mathbb{Z}_+}$ is a martingale, then*

$$\|Sf\|_X \leqslant k_1 \|Mf\|_{K(X)} \quad \textit{and} \quad \|Mf\|_X \leqslant k_2 \|Sf\|_{K(X)}.$$

Moreover, if $f = (f_n)_{n \in \mathbb{Z}_+}$ is a uniformly integrable martingale and $f_\infty = \lim_{n\to\infty} f_n$ a.s., then

$$\|Sf\|_X \leqslant 2k_1 \|f_\infty\|_{\mathcal{H}_1(X)}.$$

Suppose that $\alpha_X > 0$. Then it follows from Theorem 4.5 and Lemma 2.5 that

$$c_X \|Sf\|_X \leqslant \|Mf\|_X \leqslant C_X \|Sf\|_X$$

for all martingales f, where c_X and C_X are positive constants depending only on X. This result was established independently by Antipa [**1**], Johnson and Schechtman [**10**] and Novikov [**14**].

Proof of Theorem 4.5. We use the (conditional form of the) Davis inequality (see, for example, [**8**, p. 286] or [**12**, p. 89]). There are constants k_1 and k_2 such that

$$\mathbb{E}[Sf - S_{n-1}f \mid \mathcal{F}_n] \leqslant k_1 \mathbb{E}[Mf \mid \mathcal{F}_n] \text{ a.s.} \quad (n \in \mathbb{Z}_+)$$

and

$$\mathbb{E}[Mf - M_{n-1}f \mid \mathcal{F}_n] \leqslant k_2 \mathbb{E}[Sf \mid \mathcal{F}_n] \text{ a.s.} \quad (n \in \mathbb{Z}_+),$$

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where

$$S_n f = \left\{ \sum_{k=0}^{n} (\Delta_k f)^2 \right\}^{1/2}, \quad M_n f = \sup_{0 \leqslant k \leqslant n} |f_k|, \quad \text{and} \quad S_{-1} f = M_{-1} f \equiv 0.$$

Therefore, the first statement follows from Theorem 3.3 (ii). Moreover, since $k_1^{-1}(Sf)^* \prec \mathcal{Q}(Mf)^*$, inequality (4.2) together with (2.6) gives that if $f = (f_n)$ is uniformly integrable, then

$$k_1^{-1}(Sf)^* \prec \mathcal{Q}\mathcal{P}(f_\infty)^* \leqslant \mathcal{P}(f_\infty)^* + \mathcal{Q}(f_\infty)^* \leqslant 2\mathcal{R}(f_\infty)^*.$$

Therefore,

$$\|Sf\|_X \leqslant 2k_1 \|f_\infty\|_{\mathcal{H}_1(X)},$$

which proves the second statement. □

Now let us consider some norm inequalities for the operators $\theta^{(p)}$ and $m^{(p)}$. Long [**11**] established norm inequalities for these operators in Orlicz spaces (with a different notation). Our aim here is to extend his results. We begin by recalling a basic result. If $f = (f_n)$ is a uniformly integrable martingale, then

$$\tfrac{1}{2}\mathbb{E}[\theta^{(p)} f] \leqslant \mathbb{E}[m^{(p)} f] \leqslant 17\mathbb{E}[\theta^{(p)} f] \quad (1 \leqslant p < \infty). \tag{4.5}$$

Long's proof of this result is based on a 'rearrangement technique'. In Appendix A, however, we will give another proof by means of a usual 'distribution function technique'.

Theorem 4.6. *If $f = (f_n)_{n \in \mathbb{Z}_+}$ is a uniformly integrable martingale, then*

$$\tfrac{1}{2}\|\theta^{(p)} f\|_X \leqslant \|m^{(p)} f\|_X \leqslant 17\|\theta^{(p)} f\|_{K(X)} \quad (1 \leqslant p < \infty). \tag{4.6}$$

In particular, if $\alpha_X > 0$, then $\|\theta^{(p)} f\|_X \approx \|m^{(p)} f\|_X$ for any finite $p \geqslant 1$.

Proof. The first inequality of (4.6) is obvious, since $\theta^{(p)} f \leqslant 2m^{(p)} f$ by Minkowski's inequality. Let us prove the second inequality. Fix $n \in \mathbb{Z}_+$ and let $A \in \mathcal{F}_n$. We consider the martingale $f' = (f'_k)_{k \in \mathbb{Z}_+}$ given by

$$f'_k = \mathbb{E}[(f_\infty - f_{n-1})1_A \mid \mathcal{F}'_k] \quad (k \in \mathbb{Z}_+), \quad \text{where } \mathcal{F}'_k = \mathcal{F}_{n+k}.$$

For each $k \in \mathbb{Z}_+$, let

$$m_k^{(p)} f = \sup_{0 \leqslant j \leqslant k} \mathbb{E}[|f_\infty|^p \mid \mathcal{F}_j]^{1/p}.$$

Then

$$(m^{(p)} f - m_{n-1}^{(p)} f)1_A \leqslant \sup_{k \in \mathbb{Z}_+} \mathbb{E}[|f_\infty - f_{n-1}|^p \mid \mathcal{F}'_k]^{1/p} 1_A = m^{(p)} f'. \tag{4.7}$$

To see this, it suffices to observe that

$$\begin{aligned} m^{(p)} f &= (m_{n-1}^{(p)} f) \vee \Big(\sup_{k \geqslant n} \mathbb{E}[|f_\infty|^p \mid \mathcal{F}_k]^{1/p} \Big) \\ &\leqslant (m_{n-1}^{(p)} f) \vee \Big(\sup_{k \geqslant n} \mathbb{E}[|f_\infty - f_{n-1}|^p \mid \mathcal{F}_k]^{1/p} + |f_{n-1}| \Big) \\ &\leqslant \sup_{k \in \mathbb{Z}_+} \mathbb{E}[|f_\infty - f_{n-1}|^p \mid \mathcal{F}'_k]^{1/p} + m_{n-1}^{(p)} f. \end{aligned}$$

Since $\theta^{(p)} f' \leqslant (\theta^{(p)} f) 1_A$, combining (4.5) and (4.7) yields that

$$\mathbb{E}[(m^{(p)} f - m_{n-1}^{(p)} f) 1_A] \leqslant 17 \mathbb{E}[\theta^{(p)} f'] \leqslant 17 \mathbb{E}[(\theta^{(p)} f) 1_A].$$

Hence

$$\mathbb{E}[m^{(p)} f - m_{n-1}^{(p)} f \mid \mathcal{F}_n] \leqslant 17 \mathbb{E}[\theta^{(p)} f \mid \mathcal{F}_n] \text{ a.s.} \quad (n \in \mathbb{Z}_+).$$

Thus the second inequality of (4.6) follows from Theorem 3.3 (ii). □

We conclude this section with the following theorem.

Theorem 4.7. *Let $1 \leqslant p < \infty$ and let $f = (f_n)_{n \in \mathbb{Z}_+}$ be a uniformly integrable martingale.*

(i) *If $p < q < \infty$, then there is a constant $C_{p,q} > 0$, depending only on p and q, such that*

$$\|m^{(p)} f\|_X \leqslant C_{p,q} \|f_\infty\|_{\mathcal{H}_q(X)}. \tag{4.8}$$

(ii) *If $0 < \alpha_X \leqslant \beta_X < 1/p$, then there is a constant $C_{p,X} > 0$, depending only on p and X, such that*

$$\|m^{(p)} f\|_X \leqslant C_{p,X} \|f_\infty\|_X. \tag{4.9}$$

Proof. (i) Assume first that f_∞ is bounded. Fix $n \in \mathbb{Z}_+$ and let $A \in \mathcal{F}_n$. We consider the martingale $f' = (\mathbb{E}[|f_\infty|^p \mid \mathcal{F}_{n+k}] 1_A)_{k \in \mathbb{Z}_+}$ with respect to the filtration $\mathcal{F}' = (\mathcal{F}_{n+k})_{k \in \mathbb{Z}_+}$. Applying the Doob maximal inequality, we find that

$$\mathbb{E}\Big[\sup_{k \geqslant n} \mathbb{E}[|f_\infty|^p \mid \mathcal{F}_k]^{q/p} 1_A\Big]^{p/q} \leqslant \frac{q}{q-p} \mathbb{E}[|f_\infty|^q 1_A]^{p/q}.$$

Thus

$$\mathbb{E}\Big[\sup_{k \geqslant n} \mathbb{E}[|f_\infty|^p \mid \mathcal{F}_k]^{q/p} \mid \mathcal{F}_n\Big] \leqslant C'_{p,q} \mathbb{E}[|f_\infty|^q \mid \mathcal{F}_n] \text{ a.s.},$$

where $C'_{p,q} = \{q/(q-p)\}^{q/p}$. On the other hand, we have that

$$(m^{(p)} f - m_{n-1}^{(p)} f)^q \leqslant (m^{(p)} f)^q - (m_{n-1}^{(p)} f)^q \leqslant \sup_{k \geqslant n} \mathbb{E}[|f_\infty|^p \mid \mathcal{F}_k]^{q/p}.$$

Therefore,

$$\mathbb{E}[(m^{(p)} f - m_{n-1}^{(p)} f)^q \mid \mathcal{F}_n] \leqslant C'_{p,q} \mathbb{E}[|f_\infty|^q \mid \mathcal{F}_n] \text{ a.s.} \quad (n \in \mathbb{Z}_+).$$

Applying Theorem 3.3 (i), we obtain (4.8) with $C_{p,q} = 2C'_{p,q}$.

Now we remove the restriction that f_∞ is bounded. Let $f = (f_n)_{n \in \mathbb{Z}_+}$ be an arbitrary uniformly integrable martingale. For each $k = 1, 2, \dots$, let $f^{\langle k \rangle} = (f_n^{\langle k \rangle})_{n \in \mathbb{Z}_+}$ denote the martingale defined by

$$f_n^{\langle k \rangle} = \mathbb{E}[f_\infty 1_{\{|f_\infty| \leqslant k\}} \mid \mathcal{F}_n] \quad (n \in \mathbb{Z}_+).$$

Then $m^{(p)} f^{\langle k \rangle} \uparrow m^{(p)} f$ a.s. as $k \uparrow \infty$, and hence $\|m^{(p)} f^{\langle k \rangle}\|_X \uparrow \|m^{(p)} f\|_X$ by (B3). From what we have proved above, it follows that

$$\|m^{(p)} f\|_X = \lim_{k \to \infty} \|m^{(p)} f^{\langle k \rangle}\|_X \leqslant C_{p,q} \lim_{k \to \infty} \|f_\infty^{\langle k \rangle}\|_{\mathcal{H}_q(X)} = C_{p,q} \|f_\infty\|_{\mathcal{H}_q(X)}.$$

This completes the proof of statement (i).

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(ii) Suppose that $0 < \alpha_X \leqslant \beta_X < 1/p$. If we choose a number q so that $p < q < 1/\beta_X$, then (4.8) holds. Hence (4.9) follows from Lemma 2.5 (iii). $\square$

5. Examples of $H_p(X)$, $K(X)$ and $\mathcal{H}_p(X)$

Let us recall the definition of *Lorentz space* $L_{p,q}$. Let $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$. For each measurable function ϕ on I, we let

$$\|\phi\|_{p,q} = \begin{cases} \left(\displaystyle\int_0^1 \{t^{1/p}\phi^*(t)\}^q \frac{\mathrm{d}t}{t}\right)^{1/q} & \text{if } 1 \leqslant q < \infty, \\ \sup\limits_{t \in I}\{t^{1/p}\phi^*(t)\} & \text{if } q = \infty. \end{cases}$$

The *Lorentz space* $L_{p,q} = L_{p,q}(I)$ (over I) consists of all measurable functions ϕ for which $\|\phi\|_{p,q} < \infty$. If $1 \leqslant q \leqslant p < \infty$, then $(L_{p,q}, \|\cdot\|_{p,q})$ is an RI space over I. However, unless $1 \leqslant q \leqslant p < \infty$, the functional $\|\cdot\|_{p,q}$ is not a norm. So we need to consider another functional; let $\|\phi\|_{(p,q)} = \|\mathcal{P}\phi^*\|_{p,q}$. Then $\|\cdot\|_{p,q} \approx \|\cdot\|_{(p,q)}$ and $(L_{p,q}, \|\cdot\|_{(p,q)})$ is an RI space, provided that $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. In any case, we have $\alpha_{L_{p,q}} = \beta_{L_{p,q}} = 1/p$ (see [**3**, pp. 216–220] for details).

More generally, we also consider the *Lorentz–Zygmund spaces*. Let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and $-\infty < a < \infty$. We define

$$\|\phi\|_{p,q:a} = \begin{cases} \left(\displaystyle\int_0^1 \{t^{1/p}(1-\log t)^a\phi^*(t)\}^q \frac{\mathrm{d}t}{t}\right)^{1/q} & \text{if } 1 \leqslant q < \infty, \\ \sup\limits_{t \in I}\{t^{1/p}(1-\log t)^a\phi^*(t)\} & \text{if } q = \infty, \end{cases}$$

and let $\|\phi\|_{(p,q):a} = \|\mathcal{P}\phi^*\|_{p,q:a}$. The *Lorentz–Zygmund space* $L_{p,q}(\log L)^a$ consists of all functions ϕ for which $\|\phi\|_{p,q:a} < \infty$. If $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ and $-\infty < a < \infty$, then

$$\|\cdot\|_{p,q:a} \approx \|\cdot\|_{(p,q):a} \quad \text{and} \quad (L_{p,q}(\log L)^a, \|\cdot\|_{(p,q):a})$$

is an RI space. Moreover, $L_{1,1}(\log L)^a$ is also an RI space whenever $0 \leqslant a < \infty$ (see [**2**]).

Let $1 \leqslant p < \infty$ and $0 \leqslant a < \infty$. The space $(L_{p,p}(\log L)^a, \|\cdot\|_{p,p:a})$ is called the *Zygmund space* and is denoted by $(L_p(\log L)^a, \|\cdot\|_{p:a})$. One can prove that $\phi \in L_p(\log L)^a$ if and only if $|\phi|\{\log(1+|\phi|)\}^a \in L_p$ (cf. [**3**, p. 252]). We write $L(\log L)^a$ for $L_1(\log L)^a$, $L_p(\log L)$ for $L_p(\log L)^1$, and $L(\log L)$ for $L_1(\log L)^1$.

The space $L_{p,q} = L_{p,q}(\Omega)$ *over* Ω is defined to be the space of random variables x for which $\|x\|_{p,q} := \|x^*\|_{p,q} < \infty$. The other spaces over Ω are defined in the same way.

It is a simple matter to verify that

- $H_1(L_1) = \mathcal{H}_1(L_1) = L(\log L)$;
- $H_p(L_1) = \mathcal{H}_p(L_1) = L_{p,1}$ $(1 < p < \infty)$.

Furthermore, we have

- $H_p(L_{p,1}) = \mathcal{H}_p(L_{p,1}) = L_{p,1}(\log L)$ $(1 < p < \infty)$;
- $H_p(L_{p,\infty}) = \mathcal{H}_p(L_{p,\infty}) = L_{p,1}$ $(1 < p < \infty)$;
- $H_p(L_{q,r}) = \mathcal{H}_p(L_{q,r}) = L_{p,1}$ $(1 < q < p < \infty,\ 1 \leqslant r \leqslant \infty)$;
- $H_p(L_{q,r}) = \mathcal{H}_p(L_{q,r}) = L_{q,r}$ $(1 \leqslant p < q < \infty,\ 1 \leqslant r \leqslant \infty)$;
- $H_p(L_{p,1}(\log L)^a) = \mathcal{H}_p(L_{p,1}(\log L)^a) = L_{p,1}(\log L)^{a+1}$ $(1 \leqslant p < \infty,\ 0 \leqslant a < \infty)$.

Indeed, it is straightforward to verify that

$$H_p(L_{p,1}) = L_{p,1}(\log L), \quad H_p(L_{p,\infty}) = L_{p,1},$$

and

$$H_p(L_{p,1}(\log L)^a) = L_{p,1}(\log L)^{a+1}.$$

Since $\alpha_{L_{p,1}} = 1/p > 0$, we have $K(L_{p,1}) = L_{p,1}$ by Lemma 2.5. Hence

$$\mathcal{H}_p(L_{p,1}) = H_p(K(L_{p,1})) = H_p(L_{p,1}) = L_{p,1}(\log L)$$

by Lemma 2.4. In the same way, we see that

$$\mathcal{H}_p(L_{p,\infty}) = L_{p,1} \quad \text{and} \quad \mathcal{H}_p(L_{p,1}(\log L)^a) = L_{p,1}(\log L)^{a+1}.$$

Now, let $1 < q < p < \infty$ and $1 \leqslant r \leqslant \infty$. Then $L_{p,\infty} \hookrightarrow L_{q,r}$ and hence

$$L_{p,1} = H_p(L_{p,\infty}) \hookrightarrow H_p(L_{q,r}).$$

On the other hand, since $L_{q,r} \hookrightarrow L_{1,1} = L_1$, we see that

$$H_p(L_{q,r}) \hookrightarrow H_p(L_1) = L_{p,1}.$$

Thus $H_p(L_{q,r}) = L_{p,1}$. Moreover, we have $H_p(L_{q,r}) = \mathcal{H}_p(L_{q,r})$, because $K(L_{q,r}) = L_{q,r}$.

Next, let $1 \leqslant p < q < \infty$ and $1 \leqslant r \leqslant \infty$. Then, since $\alpha_{L_{q,r}} = \beta_{L_{q,r}} = 1/q \in (0, 1/p)$, it follows from Lemma 2.5 that $H_p(L_{q,r}) = \mathcal{H}_p(L_{q,r}) = L_{q,r}$.

We now give some examples of RI spaces X such that $K(X) \subsetneqq X$. Given $a \in (0,\infty)$, we denote by $L_{\exp:a}$ the collection of random variables x on Ω for which

$$\|x\|_{\exp:a} := \sup_{t\in I} \frac{1}{t(1-\log t)^{1/a}} \int_0^t x^*(s)\,\mathrm{d}s = \sup_{t\in I} \frac{(\mathcal{P}x^*)(t)}{(1-\log t)^{1/a}} < \infty.$$

Then $(L_{\exp:a}, \|\cdot\|_{\exp:a})$ is an RI space. One can show that $x \in L_{\exp:a}$ if and only if $\exp(\lambda|x|^a) \in L_1$ for some $\lambda > 0$ (cf. [**2**, Theorem 10.3]).

Instead of the norm $\|\cdot\|_{\exp:a}$, we may use the functional $N_a(\cdot)$ defined by

$$N_a(x) = \sup_{t\in I} \frac{x^*(t)}{(1-\log t)^{1/a}}.$$

The functional $N_a(\cdot)$ is not a norm. However, for each $a \in (0,\infty)$, there is a constant $k_a > 0$ such that

$$N_a(x) \leqslant \|x\|_{\exp:a} \leqslant k_a N_a(x). \tag{5.1}$$

A proof of this fact will be given in Appendix B.

If $a > 1$, then the function $t \mapsto -\log t$ does not belong to $\hat{L}_{\exp:a} = L_{\exp:a}(I)$, and hence the space $K(L_{\exp:a})$ consists of the zero function only. On the other hand, if $0 < a \leqslant 1$, then

- $K(L_{\exp:1}) = \mathcal{H}_p(L_{\exp:1}) = L_\infty$ $(1 \leqslant p < \infty)$;
- $K(L_{\exp:a}) = \mathcal{H}_p(L_{\exp:a}) = L_{\exp:(a/(1-a))}$ $(1 \leqslant p < \infty, 0 < a < 1)$.

To prove that $K(L_{\exp:1}) = L_\infty$, let $x \in K(L_{\exp:1})$. If $0 < t \leqslant \delta \leqslant 1$, then

$$\frac{x^*(\delta)\log(\delta/t)}{1-\log t} \leqslant \frac{(\mathcal{Q}x^*)(t)}{1-\log t} \leqslant \frac{(\mathcal{P}(\mathcal{Q}x^*))(t)}{1-\log t} \leqslant \|x\|_{K(L_{\exp:1})}.$$

Letting $t \to 0+$ yields that $x^*(\delta) \leqslant \|x\|_{K(L_{\exp:1})}$ for all $\delta \in I$. Therefore, $\|x\|_\infty \leqslant \|x\|_{K(L_{\exp:1})}$, i.e. $K(L_{\exp:1}) \hookrightarrow L_\infty$. Thus we have $K(L_{\exp:1}) = L_\infty$, since the embedding $L_\infty \hookrightarrow K(L_{\exp:1})$ is trivial.

Now let us prove that $K(L_{\exp:a}) = L_{\exp:(a/(1-a))}$ for $a \in (0,1)$. Suppose that $x \in L_{\exp:(a/(1-a))}$. Then, by (2.7),

$$\begin{aligned}
\|x\|_{K(L_{\exp:a})} &= \sup_{t\in I} \frac{(\mathcal{P}(\mathcal{Q}x^*))(t)}{(1-\log t)^{1/a}} \\
&= \sup_{t\in I} \frac{(\mathcal{P}x^*)(t) + (\mathcal{Q}x^*)(t)}{(1-\log t)^{1/a}} \\
&\leqslant \|x\|_{\exp:a} + \sup_{t\in I} \frac{(\mathcal{Q}x^*)(t)}{(1-\log t)^{1/a}} \\
&\leqslant \|x\|_{\exp:a} + \sup_{t\in I} \frac{-x^*(t)\log t}{(1-\log t)^{1/a}} \\
&\leqslant \|x\|_{\exp:a} + \|x\|_{\exp:(a/(1-a))} \\
&\leqslant 2\|x\|_{\exp:(a/(1-a))}.
\end{aligned}$$

Thus $L_{\exp:(a/(1-a))} \hookrightarrow K(L_{\exp:a})$. Conversely, suppose that $x \in K(L_{\exp:a})$. Then

$$\frac{x^*(\delta)\log(\delta/t)}{(1-\log t)^{1/a}} \leqslant \frac{(\mathcal{Q}x^*)(t)}{(1-\log t)^{1/a}} \leqslant \|x\|_{K(L_{\exp:a})} \quad (0 < t \leqslant \delta \leqslant 1). \tag{5.2}$$

Therefore, since

$$\max_{0<t\leqslant\delta} \frac{\log(\delta/t)}{(1-\log t)^{1/a}} = \frac{a(1-a)^{(1-a)/a}}{(1-\log\delta)^{(1-a)/a}},$$

we have by (5.2) that

$$\frac{a(1-a)^{(1-a)/a}x^*(\delta)}{(1-\log\delta)^{(1-a)/a}} \leqslant \|x\|_{K(L_{\exp:a})} \quad (0 < \delta \leqslant 1).$$

Thus $N_{a/(1-a)}(x) \leqslant C_a \|x\|_{K(L_{\exp:a})}$, where $C_a^{-1} = a(1-a)^{(1-a)/a}$. From (5.1) we conclude that

$$\|x\|_{\exp:(a/(1-a))} \leqslant k_{a/(1-a)} C_a \|x\|_{K(L_{\exp:a})},$$

and hence that $K(L_{\exp:a}) \hookrightarrow L_{\exp:(a/(1-a))}$. As a result, $K(L_{\exp:a}) = L_{\exp:(a/(1-a))}$.

Now it remains to show that $K(L_{\exp:a}) = \mathcal{H}_p(L_{\exp:a})$ whenever $0 < a \leqslant 1$ and $1 \leqslant p < \infty$. However, this follows from Lemma 2.5, because $\beta_{L_{\exp:a}} = 0$. We omit the details (cf. [**3**, p. 248]).

Combining the results in the present and preceding sections, we obtain various inequalities for martingales. For example,

$$\begin{aligned}
\|s^{(p)} f\|_{p,1} &\leqslant C_p \|f_\infty\|_{p,1:1} && (2 \leqslant p < \infty),\\
\|m^{(p)} f\|_{q,1:a} &\leqslant C_{p,q,a} \|f_\infty\|_{q,1:a+1} && (1 \leqslant p < q < \infty,\ 0 \leqslant a < 1),\\
\|m^{(p)} f\|_{\exp:a} &\leqslant C_{p,a} \|f_\infty\|_{\exp:(a/(1-a))} && (1 \leqslant p < \infty,\ 0 < a < 1),\\
\|Mf\|_{\exp:a} &\leqslant C_a \|Sf\|_{\exp:(a/(1-a))} && (0 < a < 1),\\
\|Mf\|_{\exp:1} &\leqslant C \|Sf\|_\infty.
\end{aligned}$$

Appendix A.

Proof of (4.5). Let $1 \leqslant p < \infty$ and set

$$\theta_n^{(p)} f = \sup_{0 \leqslant k \leqslant n} \mathbb{E}[|f_\infty - f_{k-1}|^p \mid \mathcal{F}_k]^{1/p} \quad (n \in \mathbb{Z}_+).$$

Suppose that $0 < \delta < 1 < b < \infty$ and $0 < \lambda < \infty$. We define the stopping times ρ, σ and τ, respectively, by

$$\rho = \inf\{n \in \mathbb{Z}_+ \mid \theta_n^{(1)} f > \delta\lambda\},$$
$$\sigma = \inf\{n \in \mathbb{Z}_+ \mid |f_n| > \lambda\}$$

and

$$\tau = \inf\{n \in \mathbb{Z}_+ \mid |f_n| > b\lambda\}.$$

Then

$$\begin{aligned}
\{Mf > b\lambda,\ \theta^{(1)} f \leqslant \delta\lambda\} &= \{\tau < \infty,\ \rho = \infty\}\\
&\subset \{\tau < \infty,\ \sigma < \rho\}\\
&\subset \{|f_\tau - f_{\sigma-1}| \geqslant (b-1)\lambda,\ \sigma < \rho\}. \qquad \text{(A 1)}
\end{aligned}$$

Since the operators $\mathbb{E}[\cdot \mid \mathcal{F}_\sigma]$ and $\mathbb{E}[\cdot \mid \mathcal{F}_n]$ commute each other, we find that

$$\mathbb{E}[|f_\infty - f_{\sigma-1}| \mid \mathcal{F}_\sigma] 1_{\{\sigma=n\}} = \mathbb{E}[|f_\infty - f_{n-1}| \mid \mathcal{F}_n] 1_{\{\sigma=n\}} \leqslant (\theta_n^{(1)} f) 1_{\{\sigma=n\}}.$$

Therefore,

$$\begin{aligned}\mathbb{E}[|f_\infty - f_{\sigma-1}| \mid \mathcal{F}_\sigma]1_{\{\sigma<\rho\}} &= \sum_{n=0}^{\infty} \mathbb{E}[|f_\infty - f_{\sigma-1}| \mid \mathcal{F}_\sigma]1_{\{\sigma=n<\rho\}} \\ &\leqslant \sum_{n=0}^{\infty} (\theta_n^{(1)} f)1_{\{\sigma=n<\rho\}} \\ &\leqslant (\theta_{\rho-1}^{(1)} f)1_{\{\sigma<\rho\}} \\ &\leqslant \delta\lambda 1_{\{\sigma<\rho\}} \\ &\leqslant \delta\lambda 1_{\{Mf>\lambda\}}. \end{aligned} \tag{A 2}$$

Using (A 1) and (A 2), we have that

$$\begin{aligned}\mathbb{P}\{Mf > b\lambda,\ \theta^{(1)} f \leqslant \delta\lambda\} &\leqslant \mathbb{P}\{|f_\tau - f_{\sigma-1}| \geqslant (b-1)\lambda,\ \sigma < \rho\} \\ &\leqslant \frac{1}{(b-1)\lambda}\mathbb{E}[|f_\tau - f_{\sigma-1}|1_{\{\sigma<\rho\}}] \\ &\leqslant \frac{1}{(b-1)\lambda}\mathbb{E}[|\mathbb{E}[f_\infty - f_{\sigma-1} \mid \mathcal{F}_\tau]|1_{\{\sigma<\rho\}}] \\ &\leqslant \frac{1}{(b-1)\lambda}\mathbb{E}[\mathbb{E}[|f_\infty - f_{\sigma-1}| \mid \mathcal{F}_\sigma]1_{\{\sigma<\rho\}}] \\ &\leqslant \frac{\delta}{b-1}\mathbb{P}\{Mf > \lambda\}. \end{aligned}$$

Hence, by Lemma 7.1 of [**5**],

$$\mathbb{E}[Mf] \leqslant \frac{b(b-1)}{\delta(b-b\delta-1)}\mathbb{E}[\theta^{(1)} f],$$

provided $b - b\delta - 1 > 0$. By setting $b = 2$ and $\delta = \frac{1}{4}$, we obtain that

$$\mathbb{E}[Mf] \leqslant 16\mathbb{E}[\theta^{(1)} f] \leqslant 16\mathbb{E}[\theta^{(p)} f].$$

Since $m^{(p)} f \leqslant \theta^{(p)} f + Mf$ by Minkowski's inequality,

$$\mathbb{E}[m^{(p)} f] \leqslant 17\mathbb{E}[\theta^{(p)} f].$$

On the other hand, Minkowski's inequality yields that $\theta^{(p)} f \leqslant m^{(p)} f + Mf \leqslant 2m^{(p)} f$. This completes the proof. □

Appendix B.

Proof of (5.1). Because the first inequality of (5.1) is obvious, we prove the second inequality only. Suppose first that $a \geqslant 1$ and $N_a(x) < \infty$. Then

$$x^*(s) \leqslant N_a(x)(1 - \log s)^{1/a} \quad \text{for all } s \in I,$$

and therefore

$$\begin{aligned}(\mathcal{P}x^*)(t) &\leqslant \frac{N_a(x)}{t}\int_0^t (1-\log s)^{1/a}\,\mathrm{d}s \\ &\leqslant N_a(x)\biggl\{\frac{1}{t}\int_0^t (1-\log s)\,\mathrm{d}s\biggr\}^{1/a} \\ &= N_a(x)(2-\log t)^{1/a} \\ &\leqslant 2^{1/a}N_a(x)(1-\log t)^{1/a}.\end{aligned}$$

Thus $\|x\|_{\exp:a} \leqslant 2^{1/a}N_a(x)$. Suppose now that $0 < a < 1$ and $N_a(x) < \infty$. Then, using Lemma 3.2, we find that

$$\begin{aligned}(\mathcal{P}x^*)(t)^a &\leqslant N_a(x)^a\biggl\{\frac{1}{t}\int_0^t (1-\log s)^{1/a}\,\mathrm{d}s\biggr\}^a \\ &\leqslant N_a(x)^a\frac{a}{t^a}\int_0^t (1-\log s)s^a\frac{\mathrm{d}s}{s} \\ &= N_a(x)^a\biggl(1+\frac{1}{a}-\log t\biggr) \\ &\leqslant \biggl(1+\frac{1}{a}\biggr)N_a(x)^a(1-\log t).\end{aligned}$$

Therefore, $\|x\|_{\exp:a} \leqslant (1+(1/a))^{1/a}N_a(x)$. This completes the proof of (5.1). □

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