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ON PROJECTIVE VARIETIES WITH PROJECTIVELY EQUIVALENT ZERO-DIMENSIONAL LINEAR SECTIONS

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ABSTRACT. Here we give a partial classification of varieties $X \subset \mathbf{P}^n$ such that any two general zero-dimensional linear sections are projectively equivalent. They exist (with $\deg(X) > \operatorname{codim}(X) + 2$) only in positive characteristic.

Funny things occur in the projective geometry of varieties $X \subset \mathbf{P}^n$ when the algebraically closed base field $\mathbf{K}$ has positive characteristic. However, the funny behaviour occurs often only for very particular X, and sometimes it is possible to describe all X with a strange behaviour with respect to a given projective problem. In this paper we consider the following problem. Fix an integral variety $X \subset \mathbf{P}^n$. Of course, if $\deg(X) \leq \operatorname{codim}(X) + 2$, all the general zero-dimensional linear sections $M \cap X$, M a linear space with $\dim(M) = \operatorname{codim}(X)$, are projectively equivalent. Assume $\deg(X) > \operatorname{codim}(X) + 2$. What can be said about $\mathbf{K}$ and X if all the general zero-dimensional linear sections of X are projectively equivalent? Here we give a partial classification (see 0.2 and 0.3). In particular we will see that $\operatorname{char}(\mathbf{K}) > 0$ and if $\operatorname{char}(\mathbf{K}) > 0$ there are many interesting examples. These examples fit in various classes and some of these classes are completely classified (see 0.2). The existence of some examples is classical, going back (as far as I know) to Wallace ([21]).

Recall that an integral variety $X \subset \mathbf{P}^n$ is called *strange* if there is $P \in \mathbf{P}^n$ such that for every smooth point x of X the embedded tangent space T_xX contains P; any such point P is called a strange point of X. To state our results we need the following definition (the name, but not the concept, was introduced in [3]).

DEFINITION 0.1. Let $C \subset \mathbf{P}^n$, $n \geq 3$, be an integral non degenerate curve. C is said to be *very strange* if for a general hyperplane H the points in $C \cap H$ are not in linear general position, i.e. there are n points of $C \cap H$ not spanning H.

A very strange curve is singular and strange ([19], Lemma 4 and Proposition 5). For more on what is known about very strange curves, see subsection 4.1.

The proofs (and statements) of the results of this paper are given by an obvious reduction to the case in which $\dim(X) = 1$. Hence we state first the case of a curve X.

THEOREM 0.2. *Let $X \subset \mathbf{P}^n$ be an integral non degenerate curve with* $\deg(X) \geq n+2$. *Assume that all the general hyperplane sections of X are projectively equivalent. Then* $p := \operatorname{char}(\mathbf{K}) > 0$ *and X belongs to one of the following 6 classes (ai),* $1 \leq i \leq 6$.

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If $n = 2$, either there is a power q of p such that, up to a projective transformation, X has one of the following equations:

(1) (class a1) $$x_0^{q+1} + x_1^{q+1} + x_2^{q+1} = 0$$

(2) (class a2) $$x_0^q + x_1 x_2^{q-1} = 0$$

(3) (class a3) $$x_0^q x_1 + x_2^{q+1} = 0$$

or (class (a4)*) there is an integer $m > 1$ with* $\deg(X) = m + q$ *such that X is a rational strange, the strange point, o, of X has multiplicity m and o is the unique singular point of X.*

Assume $n \geq 3$ and X not very strange. Then $n = 3$, $\deg(X) = q + 2$ *with q a power of p, and for every q the curve X is unique, up to a projective transformation. X is contained in a smooth quadric surface $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$ and choosing homogeneous coordinates z_0, z_1, and w_0, w_1 on the two factors of Q, we may assume that X has the following bihomogeneous equation:*

(4) (class a5) $$z_0 w_1^q = z_1 w_0^q$$

The class (a6) *consists of very strange curves; furthermore for every integer k with $2 \leq k < n$, the projection Y of X from $n - k$ general points of X satisfies the assumptions of 0.2.*

Taking general linear sections, 0.2 implies trivially the first part of the following corollary.

COROLLARY 0.3. *Let $X \subset \mathbf{P}^n$ be an integral non degenerate variety with* $\deg(X) \geq \operatorname{codim}(X) + 3$ *and* $\dim(X) \geq 2$. *Assume that all the general 0-dimensional sections of X are projectively equivalent. Then $p := \operatorname{char}(\mathbf{K}) > 0$ and the general curve sections of X belong to one of the 6 classes (ai), $1 \leq i \leq 6$, considered in the statement of 0.2. Furthermore if X is normal, then X is a hypersurface of degree q, q a power of p, and there is an integer $r \leq n$ and homogeneous coordinates $x_0, \ldots, x_n$ of $\mathbf{P}^n$ such that X is given by the equation*

(5) $$\sum_{i=0}^{r} x_i^{q+1} = 0$$

(i.e. X is a cone over a smooth Fermat hypersurface).

In a first attempt to prove 0.2 we used Mumford's Geometric Invariant Theory, but then we realized that the matter is much more elementary.

In § 1 we will give a few preliminary remarks (building the general set-up). Section 2 is devoted to the proof of 0.2, in the case $n = 2$, in the case $n = 2$. Section 3 contains the proof of 0.2 for $n \geq 3$, under the assumption that X is not very strange. The first part (4.1) of § 4 contains more known results on very strange curves and a proposition (Proposition 4.2) which proves the rationality assertion on the class (a6) stated in 0.2. The second part of § 4 proves 0.3. Section 5 contains a few remarks on varieties $X \subset \mathbf{P}^n$

with $\deg(X) \geq \operatorname{codim}(X) + 3$ and such that all their general hyperplane sections are projectively equivalent (they are all uniruled).

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1. Every scheme will be defined and algebraic over an algebraically closed field $\mathbf{K}$. Set $\mathbf{P} := \mathbf{P}^n$ and let $\mathbf{P}^*$ the set of its hyperplanes. Let $I := \{(x, H) \in \mathbf{P} \times \mathbf{P}^* : x \in H\}$ be the incidence variety. Let $f: I \longrightarrow \mathbf{P}^*$ be the projection on the second factor; f is a locally trivial fibration (indeed it is the projectivization of a vector bundle on $\mathbf{P}^*$); hence for each $[H] \in \mathbf{P}^*$ we may find an open neighbourhood U of $[H]$ in $\mathbf{P}^*$ and an isomorphism (over f) of $f^{-1}(U)$ and $U \times \mathbf{P}^{n-1}$. We choose one such isomorphism and we use it to map $f^{-1}(U)$ onto $\mathbf{P}^{n-1}$. This map allows us to see a (continuous or algebraic or ...) family of subsets $\{X_t\}_{t \in U}$ with $X_t \subset f^{-1}(t)$ for every t as a family of subsets of a fixed $\mathbf{P}^{n-1}$. Of course the map is not unique, but it still allows us to say if X_t and X_u are projectively equivalent. Set $G := \operatorname{Aut}(\mathbf{P}^{n-1})$. Note that the orbits of an algebraic group for an algebraic action are locally closed and the closure of each orbit is a union of orbits, one open in its closure and the other of lower dimension. Fix $a \in U$ and assume the existence of a Zariski dense subset V of U such that X_t is projectively equivalent to X_u if both t and u are in V; then this is true for a Zariski open subset of the set of hyperplane sections; if X_a is not projectively equivalent to a general hyperplane section of X, then the dimension of the stabilizer of X_a for the action of G is bigger than the dimension of the stabilizer for the action of G on a general hyperplane section. In summary we have the following remarks.

REMARK 1.1. Fix an integral variety $X \subset \mathbf{P}^n$ and assume that the general hyperplane sections of X are projectively equivalent. If Y is a hyperplane section of X not projectively equivalent to a general one, then the dimension of the stabilizer of Y for the action of $\operatorname{Aut}(\mathbf{P}^{n-1})$ is bigger than the dimension of the stabilizer of a general hyperplane section.

REMARK 1.2. In 1.1 consider in particular the case $n = 2$. Let X be a reduced plane curve. Since any zero-dimensional subscheme Z of $\mathbf{P}^1$ with $\operatorname{card}(Z_{\mathrm{red}}) > 2$ is stabilized by a finite number of projective transformations, every unreduced section Y of X has $\operatorname{card}(Y_{\mathrm{red}}) \leq 2$.

2. *Proof of* 0.2 *for* $n = 2$: Let C be an integral plane curve such that the general linear sections of C are projectively equivalent. Assume $d := \deg(C) \geq 4$.

(i) Here we assume C smooth. By a theorem of Lluis (see [19], Proposition 5) C is not strange. Fix a general $x \in C$ and let q be the order of contact at x of C with its tangent line T_xC. By 1.2 $\operatorname{card}\big((C \cap T_xC)_{\mathrm{red}}\big) \leq 2$. By [14] (or see [18]) the Gauss map of C is purely inseparable. Hence T_xC is tangent to C only at x. Thus $q + 1 \geq \deg(C)$. Hence we may assume $q > 2$. Thus ([12]) $\operatorname{char}(\mathbf{K}) > 0$ and q is a power of $p := \operatorname{char}(\mathbf{K})$. Such curves are classified in [12]; we obtain $\deg(C) = q + 1$ and C projectively equivalent to the Fermat example (a1).

(ii) From now on in this section we will assume C singular. Let P be a singular point of C. In part (ii) and part (iii) we assume that either C is not strange or C is strange but

P is not the strange point of C. By 1.2 C must have multiplicity $d-1$ at P (hence C is rational). By Bézout's theorem P is the only singular point of C. Let q be the order of contact at a generic $x \in C$ of C with its tangent line T_xC at x. By 1.2 the general tangent line of C intersects C at e points with either $e = 1$ or $e = 2$, and we have either $d = q$, $e = 1$, or $d = 2q$, $e = 2$ or $d = q+1$, $e = 2$. By [12], q is a power of p. If $d \neq q+1$ the map from the conormal variety of C to the dual curve of C has degree d. Hence by the proof of [12], Corollary 7.16, C is a strange curve.

(iii) Here we add the assumption that $d \neq q+1$ to the assumptions of part (ii). Let o be the strange point of C. For degree reasons $o \notin C$. Choose homogeneous coordinates x_0, x_1, x_2 with $o = (1,0,0)$ and with $(0,1,0)$ as point of multiplicity $d-1$ for C. Let f be the equation of C.

(iii1) First assume $d = q$, $e = 1$. By the choice of o and [6], § 3, $f = x_0^q + P(x_1, x_2)$ with $\deg(P) = q$. Since C has multiplicity $q-1$ at $(0,1,0)$, x_1 appears only in degree 1. Hence $f = x_0^q + ax_2^q + bx_1x_2^{q-1}$ with $b \neq 0$. Rescaling x_1 we may assume $b = 1$. If $a = 0$ we have example (a2) given by equation (2). Assume $a \neq 0$. Rescaling x_2 we may assume $a = 1$, i.e. C given by:

$$x_0^q + x_2^q + x_1x_2^{q-1} = 0 \tag{5}$$

Substituting x_0 with $x_0 + x_2$ in equation (2), we obtain (5); hence C is again of type (a2).

(iii2) Now we check that example (a2) satisfies the assumptions of 0.2; we will see explicitly the projective equivalence class of a section of example (a2) and check that for a general line L even the sets $L \cap \big(C \cup (\{x_2 = 0\} \cap C)\big)$ are projectively equivalent. The last assertion will be used to check the example of class (a5) for $n = 3$. Taking $C \cap \{x_0 = a_1x_1 + a_2x_2\}$, we see that we have to check that the q roots $\{z^i(w, w')\}$ of the polynomials $wz^q + z + w'$, $w \neq 0$, are projectively equivalent. Up to a dilatation $z \longrightarrow \lambda z$, we see that the roots of each of these polynomials are projectively equivalent to the roots of another polynomial in the family and with $w = 1$. Then with a translation $z \longrightarrow z+t$ we prove that the roots of these polynomials are projectively equivalent to the roots of $z^q + z + 1$, as wanted.

(iii3) Assume $e = 2$, $d = 2q$. By [6], § 3, and the choice of coordinates we have

$$f = x_0^{2q} + x_0^q(ax_1x_2^{q-1} + bx_2^q) + (cx_1x_2^{2q-1} + sx_2^{2q}) \tag{6}$$

for some $a, b, c, d \in \mathbf{K}$. We distinguish several subcases.

(iii3a) Assume $abcs \neq 0$ and $d > 4$. Taking c_ix_i instead of x_i for suitable c_i, we may assume $a = b = 1$ and $c = s$, i.e.

$$f = x_0^{2q} + x_0^q(x_1x_2^{q-1} + x_2^q) + c(x_1x_2^{2q-1} + x_2^{2q}) \tag{7}$$

with $c \neq 0$. Intersect C with the line $\{x_0 = s'x_1 + s''x_2\}$, and put $z := x_1/x_2$; consider the polynomial

$$P(u, v) : u^2z^{2q} + uz^{q+1} + 3uz^q + cz + v \tag{8}$$

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and let $\{z_i(u,v)\}$ the set of its roots. We see that our problem is equivalent to prove that for no $c \neq 0$, c fixed, the sets $\{z_i(u,v)\}$, u, v such that $P(u,v)$ has distinct roots, are all projectively equivalent. We may take $v = 0$ and consider only the subgroup G' of $PGL(2)$ formed by the transformations fixing 0, i.e. sending z into $z/(a'z + a'')$ with $a'' \neq 0$. We have to show that varying the transformations in G' we do not find a general $P(u,0)$. Apply $g \in G'$ with $g(z) = z/(a'z + a'')$ to $P(u,0)/z$ and impose that you find $j\big(P(m,0)/z\big)$ for some $j \neq 0$ and some m. Recall that we assumed $d > 4$. Looking at the coefficient of z we find $a' = 0$. Then, looking at the coefficient of z^2 we find $a'' = 1$, as wanted.

(iii3b) Assume again $d > 4$. The same proof applies if in (6) we have $cs \neq 0$ and $ab = 0$. If in (6) $c = 0$ and $s \neq 0$, then $a = 0$ and we may assume $a = s = 1$. Then the same method works verbatim (if $d > 4$). If $s = 0$ and $c \neq 0$, then $b \neq 0$; we reduce to the case $b = c = 1$ and repeat the same words.

(iii3c) The same method handles the case $d = 4$.

(iv) Here we assume $d = q + 1$. By the definition of q, the fact that C has a unique singular point, and [15], Theorem 2, (or [5]), C is strange. Let $o = (1,0,0)$ be the strange point of C and $P = (0,1,0)$ the singular point of multiplicity q of C. For degree reasons $o \in C_{\text{reg}}$. In this coordinate system the equation f of C is of the form:

$$f = x_0^q(ax_1 + bx_2) + cx_2^{q+1} + gx_1x_2^q \tag{9}$$

We distinguish two subcases. First assume that $g = 0$ in equation (9). Since C is irreducible, $a \neq 0$ in equation (9). Hence, changing the variable x_1, we see that C is in the class (a3). Now assume $g \neq 0$. Substituting x_1 with $x_1 + rx_2$ we reduce to the case $c = 0$. Since **K** is algebraically closed, changing x_2 by a factor we may assume $b = 1$. Substituting in the equation

$$x_0^q(x_1 + x_2)x_1x_2^q = 0 \tag{10}$$

x_1 with $x_1 - x_2$ and x_0 with $x_0 - x_2$ and then substituting x_2 with tx_2, t with $t^{q+1} = -1$, we reduce equation (3) to equation (10), i.e. C is in the class (a3). Now we check that the curve X with (3) as equation satisfies the hypothesis of 0.2. Intersecting X with the line $\{x_0 = \alpha x_1 + \beta x_2\}$ and setting $z := x_1/x_2$, we see that this is equivalent to the following fact. Consider the roots $\{z_i(u,v)\}$ of the polynomial (in z)

$$P(u,v) := uz^{q+1} + vz + 1 \tag{11}$$

We need exactly that for all u, v for which $P(u,v)$ has no multiple roots the sets $\{z_i(u,v)\}$ are projectively equivalent (or by 1.1 that this is true for general u, v). Fix a sufficiently general projective transformation γ, say $\gamma(z) := (z + m)/(sz + w)$ and look at $S(\gamma) := \gamma^*(\{z_i(1,1)\})$. $S(\gamma)$ is of the form $\{z_i(u,v)\}$ if and only if $m + ms^q + s^q w = 0$. Thus we have one condition on 3 parameters m, s, w. Thus the orbit of $S(1)$ contains $L \cap X$ for a general line L, as wanted.

This concludes the proof of 0.2 for $n = 2$. We want to point out that (as obvious from parts (iii) and (iv)) we used the fact that **K** is algebraically closed. Over a finite field of course every class splits.

3. This section is devoted to the proof of 0.2 for $n \geq 3$. At the end of this section is the proof of 0.2 except the rationality assertion for every curve of type (a6). This assertion will be proved (and state again as Proposition 4.2) in the next section.

(A) Fix a curve $X \subset \mathbf{P}^n$, $n \geq 3$, X satisfying the assumptions of 0.2, and with $\deg(X) \geq n+2$. By 1.1 for a general $P \in X$, the general hyperplane sections $H \cap X$ of X with $P \in H$ are projectively equivalent. Fix a general $P \in X$. Let $Z \subset \mathbf{P}^{n-1}$ be the image of X under the projection from P. Since $H \cap X$ is finite, the proof of 1.1 and its framework (applied to the subgroup of $\mathrm{Aut}(H)$ stabilizing P) shows that, if $P \in H \cap M$, H and M sufficiently general there is a projective transformation from H to M mapping $X \cap H$ onto $X \cap M$ and P to P (up to any coherent identification of H with M as explained in § 1). Thus $Z \subset \mathbf{P}^{n-1}$ satisfies the assumption of 0.2. If $n > 3$ we iterate the procedure. Projecting from $n-2$ general points of X we find a plane curve, C, which must be one of the curves of type (ai), $1 \leq i \leq 4$.

(B) In this section we assume that the rational map from X onto C induced by the projections is birational or equivalently that X is not very strange (the equivalence being just the definition 0.1). Then C cannot be smooth by the formula for the arithmetic genus of a plane curve of degree $\deg(X) - n + 2$ and the bound of the arithmetic genus for a curve of $\deg(X)$ in $\mathbf{P}^n$ (see [3] for this bound in positive characteristic). Now we check that X is strange, unless $n = 3$ and X corresponds to case (a5).

(i) Fix $r \geq 3$ and a non degenerate integral curve $D \subset \mathbf{P}^r$, D not strange, such that the general projection of D from one of its points into $\mathbf{P}^{r-1}$ is a strange curve. We want to prove that $r = 3$, D is contained in a smooth quadric surface Q, and the tangent lines to D form one of the rulings of Q. Indeed take 2 general points x, y of D. Since $r > 2$ and D is not strange we check easily that $(T_xD) \cap (T_yD) = \emptyset$. If $m > 3$ a general $P \in D$ is not in the linear span of $T_xD \cup T_yD$; hence the projection of D from P is not strange. Assume $r = 3$, and take a general $z \in D$ with $(T_xD) \cap (T_yD) = (T_xD) \cap (T_zD) = (T_zD) \cap (T_yD) = \emptyset$; let Q be the smooth quadric containing $T_xD \cup T_yD \cup T_zD$; the projection from a point $P \in \mathbf{P}^3$ maps T_xD, T_yD, and T_zD to 3 concurrent lines if and only if $P \in Q$.

(ii) Now assume $r = 3$ and $D \subset Q$, D as above. Furthermore we assume that D satisfies the assumptions of 0.2. Let q be the multiplicity of intersection of D at a general $x \in D$ with T_xD. By 0.2 for the case $n = 2$, D is of type (q,a) on Q for some $a > 0$ and $q > 2$. The projection of D contracts two lines of Q to two points in the plane; one of these points is the strange point of C (which have multiplicity $q - 1$); the other point is a point of multiplicity $a - 1$ for C. At the beginning of step (ii) in § 2 we checked C has at most the strange point as singular point not of multiplicity $\deg(C) - 1$. Thus we have either $a = 1$ or $a = 2$. Until part (iii) we will assume $a = 1$. Thus here $\deg(D) = q + 1$, $\deg(C) = q$.

(ii1) Vice-versa, take D integral, of type $(q, 1)$ on Q and such that one of the rulings of $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$ is formed by the tangent lines to D; we want to prove that D satisfies the assumptions of 0.2. Fix a general $x \in D$. The plane curve, C, obtained projecting D from x satisfies all the conditions we need to prove that C is in the class (a2). In case (a2) we checked also (in the same part of § 2) that, with the coordinates chosen, for all general lines $L \subset \mathbf{P}^2$, the sets $L \cap \big(C \cup (\{x_2 = 0\} \cap C)\big)$ are projectively equivalent; in these coordinates the line $\{x_2 = 0\}$ is the line containing the strange point and the singular point of C. This condition is equivalent to the fact that for all general planes H with $x \in H$ the sets $D \cap H$ are equivalent up to the action of $\mathrm{Aut}(H \cap Q)$ (with the same non canonical choices as at the beginning of § 1); hence they are projectively equivalent. Consider again 1.1 and its proof. If we take x, y general in D and H containing both x and y, we see that all the general hyperplane sections of D are projectively equivalent. Since the example (a5) is of this type, we checked that example (a5) satisfies the assumptions of 0.2.

(ii2) Now we check the uniqueness of examples (a5). On $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$ choose homogeneous coordinates w_0, w_1 and z_0, z_1 on the two factors. By the definitions of D and q, D has bihomogeneous equation of type $A(z_0, z_1)w_0^q = B(z_0, z_1)w_1^q$, with $\deg(A) = \deg(B) = 1$. Since D is integral, A and B are linearly independent and, up to a change of coordinates, we may assume $A = z_1$ and $B = z_0$.

(iii) Here we handle the case $a = 2$ which arose in part (ii). Thus D is a curve of type $(q, 2)$ on a smooth quadric Q. Since $a = 2 < 3$, D has at most double points. Since the image C of the projection of D from a general $x \in D$ is rational by the classification proved in § 2, D must be rational. Since $p_a(D) = q - 1$ by the adjunction formula, D has double points. Hence C has double points which are not the strange point of C (if C is strange), contradicting the classification of § 2.

(iv) Now we fix $n > 3$, X as in 0.2, X not very strange, and whose projection from $n-3$ general points of X gives the curve D of type (a5). We want to find a contradiction. Thus we may (and will) assume $n = 4$. Thus $\deg(X) = q + 2$. Let $T \subset \mathbf{P}^4$ be the tangent developable of X and let τ be the multiplicity of T at a general point of X. Since the tangent developable Q of D has degree 2, we see that $\deg(T) = 2 + \tau$. Take a general hyperplane H and set $Y := T \cap H$. Y is an integral curve of $H \cong \mathbf{P}^3$ with degree $\tau + 2$ and with at least $\deg(X)$ points with multiplicity τ. By the bound of the genus for curves in H we see that $\tau = 1$, i.e. T is a minimal degree rational scroll. Since Q is smooth, T is not a cone. Hence $T \cong F_1$ and in a base h, f, of $\mathrm{Pic}(T)$ with $h^2 = -1, f^2 = 0, h \cdot f = 1$, we have $X \in |qh + (q+1)f|$. We have $q+2 = \deg(X) \geq 6$, i.e. $q \geq 4$. Note that a degree 3 rational normal curve in $\mathbf{P}^3$ is uniquely determined by 6 of its points. This implies that any projective transformation from the hyperplane H to the hyperplane M, sending $X \cap H$ onto $X \cap M$, sends $T \cap H$ onto $T \cap M$. Thus (with the identifications made in § 1) the problem is to see if these families of $q+2$ points on a curve, V, isomorphic to $\mathbf{P}^1$ are equivalent under the action of $\mathrm{Aut}(V) \cong PGL(2)$. Project V from two general points B, B' of V. We get the curve, C, given by (ai) $i = 2$ or 3. In the proof step (iii2) of § 1 the projective transformations needed to prove that the general sections are projectively equivalent have only one common fixed point (at infinity for the equation (11)). Hence, again by the proof of step (iii2) of § 1, the q+2 points (containing B and B') corresponding to general hyperplane sections through B and B' are not equivalent under the action of $\mathrm{Aut}(V)$.

(v) Here we assume that X is strange. Let q be the multiplicity of intersection of X at a general $x \in X$ with T_xX. By construction q is the corresponding multiplicity for C and the Gauss map is purely inseparable. Let $\mu \geq 0$ be the multiplicity of the strange point of X. By construction the multiplicity, m, of C at its strange point is $\mu + (n-2)(q-1)$. Thus we cannot meet, after $n-2$ projections, a curve of type (ai) with $1 \leq i \leq 3$.

4. First in 4.1 we collect a few facts (relevant to the problem considered in this paper) on very strange curves. Then we give a result (Proposition 4.2) which concludes the proof of 0.2. Then we prove 0.3.

(4.1) Here we give more information about very strange curves, and show where examples relevant to our problem are written down. Fix a non degenerate very strange curve $C \subset \mathbf{P}^n$; set $d = \deg(C) > 2$. By [19], Lemma 4, C is strange. Hence by a theorem of Lluis (see [19], Proposition 5), C is singular. By [20], § 2, with the possible exceptions of very particular curves of degree 11, 12, 13, or 24 and over $\mathbf{K}$ with $p := \mathrm{char}(\mathbf{K})$ very small (conjecturally, they do not exist), even when $n \geq 5$ either a general secant line to C is a multisecant line or a general plane spanned by 3 points of C contains other points of C. Assume that $d > 22$ and that the general secant line to C is not a multisecant line for C. Then Theorem 0.1 of [4] states that $d = 2^k$, the monodromy group G for C is isomorphic $AGL(k, 2)$ (the affine group over the field F_2) and this isomorphism respects the action of G on the general hyperplane sections (which is thus a k-dimensional affine space over F_2); furthermore the same theorem states that if $n \geq 5$, then $\mathrm{char}(\mathbf{K}) = 2$. In the same paper there are several examples (when $\mathrm{char}(\mathbf{K}) = 2$) of such curves are all projectively equivalent. In [4], Proposition 2.3 it is indeed proved that for any such C with degree d all the general hyperplane sections are projectively equivalent; if $d = 2^{n-1}$ it is known ([4], Proposition 2.3, part (c)) that the normalization of C has genus 0 or 1 and that if it has genus 0, C is projectively equivalent to the curve with affine parametrization $f\colon \mathbf{A}^1 \longrightarrow \mathbf{A}^n$ given by $f(t) = (t^{a(0)}, t^{a(1)}, \ldots, t^{a(n-1)})$ with $a(i) = 2^i$ for every i.

PROPOSITION 4.2. *Let $C \subset \mathbf{P}^n$ be an integral non degenerate curve such that all its general hyperplane sections are projectively equivalent, C neither the Fermat plane curve (a1) given by (1) nor the space curve (a5) given by (4). Then C is singular and rational.*

PROOF. The fact that C must be singular follows from the list in 0.2 and the properties of very strange curves explained in 4.1. If $n = 2$, C is rational by the list in 0.2 and the purely inseparability of the projection of C from its strange point onto $\mathbf{P}^1$ for case (a4). If $n > 2$ and C is not very strange, the rationality of C follows from the fact (proved in part (A) of § 3) that a projection of C from $n-2$ general points of C is birational to C and it is a plane curve which satisfies the assumptions of 0.2. Thus we may assume $n > 2$ and that C is very strange. Let o be the strange point of C. To prove that C is rational it is sufficient to check that:

(a) the projection of C from o is purely inseparable onto its image Z;
(b) Z is a rational normal curve in $\mathbf{P}^{n-1}$.

Proof of (a). Let R be a general tangent line to C. We have to check that R intersects $C \backslash \{o\}$ only at one point. Under the projection j of C by $n-2$ general points of C the line R goes to a general tangent line of the plane curve $j(C)$. Thus (a) follows from the list in 0.2, case $n = 2$.

Proof of (b). Let M be a general hyperplane through the strange point o. Since $M \cap C$ is not reduced, by 1.1 there is a positive dimensional subgroup of $\mathrm{Aut}(M)$ stabilizing $M \cap C$. This implies that for a general hyperplane H of the $\mathbf{P}^{n-1}$ containing Z (i.e. for the projection of M from o) $H \cap Z$ is stabilized by a positive dimensional connected subgroup Γ of $\mathrm{Aut}(M)$. Assume by contradiction $\mathrm{card}(H \cap Z) \geq n$. We will check the existence of $S \subseteq H \cap Z$ with $\mathrm{card}(S) = n$ and such that for every $S' \subset S$ with $\mathrm{card}(S') = n - 1$, S' spans H. Since $H \cap Z$ is finite and Γ is connected, the existence of such S contradicts the existence of Γ. Assume that there are no such S. Since $\mathrm{card}(H \cap Z) \geq n$, this implies that Z is very strange and that a general hyperplane section, $Z \cap N$, is such that there is an hyperplane N' of N with $\mathrm{card}(Z \cap N \cap N') = \mathrm{card}(Z \cap N) - 1$, N' spanned by $Z \cap N$. By the irreducibility of Z, for general N any two hyperplanes of N spanned by points of Z contain the same number of points of Z, forcing the existence of S.

Note that now the proof of 0.2 is completed.

Proof of 0.3. Fix $X \subset \mathbf{P}^n$ with X normal, non degenerate, and such that the general zero-dimensional sections of X are projectively equivalent. The first part of 0.3 follows trivially from 0.2. Now we check the second part of 0.3. By 0.2 either $\mathrm{codim}(X) = 1$ or $\mathrm{codim}(X) = 2$.

(i) Here we assume $\mathrm{codim}(X) = 2$. We will find a contradiction. We have $\deg(X) = q + 2$. Taking a general linear section we may assume $n = 4$ and that the general hyperplane section of X is a curve of type (a5).

(i1) Here we check that X is smooth. Assume the existence of $x \in X_{\mathrm{sing}}$. First we assume that not every tangent plane T_yX with $y \in X_{\mathrm{reg}}$ contains x. This implies that for a general hyperplane H with $x \in H$, $Y := X \cap H$ is generically reduced. Since X is normal and $\dim(X) = 2$, X is locally Cohen-Macaulay. Hence Y is reduced. Since X is not a cone with vertex x by assumption, the projection of X from x has 2-dimensional image. Hence by Bertini theorem ([13]) Y is integral. By construction Y is singular. By a standard exact sequence (using that X has no embedded component) we see that Y has the same arithmetic genus of a general hyperplane section (which is smooth and rational), contradiction. Now assume that all the tangent planes T_yX, $y \in X_{\mathrm{reg}}$, contains x. Since a general hyperplane section of X is not projectively normal X cannot be a cone. Thus the projection of X from x has a finite degree, w, and its image is a surface of degree $z > 1$. Let $m \geq 2$ be the multiplicity of X at x. Since a general line D tangent to X has intersection multiplicity q with X, we have $w \geq q$, contradicting the equality: $q + 2 = m + zw$.

(i2) By (i1) we may assume that X is smooth. Since X contains smooth rational curves with self-intersection $q + 2 > 0$, it is rational. Hence $\chi(\mathbf{O}_X) = 1$. Set $L := \mathbf{O}_X(1)$ and $K := K_X$. Hence $L^2 = q + 2$. Since L is very ample with sectional genus 0, by [1], Theorem A, (or see [2]) X is isomorphic to the projection of a smooth rational scroll of degree $q + 2$ in $\mathbf{P}^{q+3}$. In particular $K^2 = 8$. By the adjunction formula we have $K \cdot L = -4 - q$. Since $(q + 2)(q - 8) + 5(q + 4) - 16 + 12 \neq 0$, these numerical data contradict the formula for smooth surfaces in $\mathbf{P}^4$ at page 434 of [11]. (Alternative proof: use that since $h^1(\mathbf{O}_X) = 0$ and the general hyperplane section of X is not linerarly normal, X is not linearly normal; this contradicts an old theorem of Severi proved by M. Dale ([8]) in positive characterstic.)

(ii) Here we assume codim$(X) = 1$, hence deg$(X) = q+1$ by 0.2. The proof of part (i1) works word for word and shows that X is a cone over a smooth hypersurface. Thus taking a general linear section we may assume that X is smooth. The only problem to apply 0.2, induction on n, and the main result of [7], is to prove that *all* (not just sufficiently general ones) smooth hyperplane sections of X are isomorphic as abstract varieties. Using an easy induction, it would be sufficient to prove that *all* smooth curve sections of X are isomorphic (as abstract curves). Let $g = q(q-1)/2$ be the genus of any smooth curve section of X. Since the moduli scheme M_g of curves of genus g is a separated scheme, and since the general curve sections of X correspond to the same point of M_g, all smooth curve sections of X are isomorphic.

5. In this section we make a few remarks on the case dim$(X) > 1$, when we assume that the general hyperplane sections of X are projectively equivalent.

PROPOSITION 5.1. *Let $X \subset \mathbf{P}^n$ be an integral variety with* dim$(X) > 1$. *Assume that the general hyperplane sections of X are projectively equivalent. Then X is uniruled.*

PROOF. Let $X^* \subset \mathbf{P}^{n*}$ be the conormal variety of X. Take a general $H \in X^*$. Set $Y := H \cap X$; Y is reduced. First assume Y integral. By the structure of affine algebraic groups and 1.1, Y is uniruled. Since the union of all such Y cover a Zariski open subset of X, for a general point x of X there is a rational curve C (possibly singular) with $x \in C$. If $\mathbf{K}$ is uncountable, this implies that X is uniruled by the countability of the irreducible components of Hilb(X); if $\mathbf{K}$ is countable we have also to note that we may find such a curve C with bounded degree (not depending on x) by the boundness of X^* and the construction in § 1 which gave 1.1. Now assume Y reducible; let $\{Y_i\}$, $1 \leq i \leq t$, be the irreducible components of Y. By the first part of the proof at least one of these components is uniruled and we conclude easily.

Note that in 5.1 in general X is not separably uniruled (see the example in [7] of the Fermat hypersurface $\{\sum_i (x_i)^{q+1} = 0\}$). See [17] for the notion of reflexivity used in 5.2.

PROPOSITION 5.2. *Let $X \subset \mathbf{P}^n$ be a smooth reflexive surface such that all its general hyperplane sections are projectively equivalent. Then X is a smooth scroll or $\mathbf{P}^2$ embedded as a plane or as a Veronese surface.*

PROOF. By a theorem of Zak, the smoothness and reflexivity assumption, the dual variety X^* is a hypersurface (see [22] or [9] or [10] or [17]). Since X is reflexive, for a general $x \in X$ and a general hyperplane H containing T_xX, $Y := H \cap X$ is singular only at x and at x has as an "ordinary" nodal double point (with an appropriate meaning if char$(\mathbf{K}) = 2$) (see [16], Proposition 3.3, or [17], Theorem 17). First assume that Y is irreducible. Since is must be rational, we have $p_a(Y) = 1$, i.e. X has sectional genus

1. Hence $\mathrm{Aut}^0(Y)$ has dimension one and acts transitively on $\mathrm{Pic}^0(Y)$. Thus no positive dimensional subgroup of $\mathrm{Aut}(H)$ sends Y into itself. As in 1.1, this gives a contradiction. Assume Y reducible. Since it is connected, $Y = A \cup B$ with A and B integral and smooth, $A \cap B = \{x\}$, and A, B intersecting transversally at x. At least one of these components, say B, is rational. Set $K := K_X$ and $L := \mathbf{O}_X(1) = A + B$ (additive notation). Since both A and B move in an algebraic family, we have $A^2 \geq 0$ and $B^2 \geq 0$. since $K \cdot B + B^2 = -2$ (adjuction formula) and $A \cdot B = 1$, we have $(K + L) \cdot B = -1$, i.e. no multiple of $K + L$ is spanned by global sections. We conclude by [1], Theorem A (stated also in [2], § 1).

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