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LOCAL TOPOLOGICAL PROPERTIES OF MAPS AND OPEN EXTENSIONS OF MAPS

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1. Introduction. A *σ-discrete* set in a topological space is a set which is a countable union of discrete closed subsets. A mapping $f : X \to Y$ from a topological space X into a topological space Y is said to be σ-discrete (countable) if each fibre $f^{-1}(y)$, $y \in Y$ is σ-discrete (countable). In 1936, Alexandroff showed that every open map of a bounded multiplicity between Hausdorff spaces is a local homeomorphism on a dense open subset of the domain [**2**]. (The original result was stated in a less general situation (see [**11**, 2.3])). Similarly, Kolmogoroff proved in 1941 that each countable open mapping between compact metric spaces is a local homeomorphism on a dense subset of the domain [**6**]. In 1967, Pasynkov generalized this result of Kolmogoroff to show that every open σ-discrete mapping between Tychonoff spaces with locally Čech complete domain is a local homeomorphism on a dense open subset of the domain [**7**]. The same year, Proizvolov obtained the same result for finite-to-one open mappings between Tychonoff spaces with Čech complete range [**9**]. Here we show that the device of open extensions of maps [**3**; **4**] can be used to weaken the hypothesis of openness of the mapping in all these results to a certain extent. Precise statements of improved theorems are given in Sections 2 and 3, and Section 4 is devoted to examples.

Notation and Terminology. All maps are assumed to be continuous. Let $f : X \to Y$ be a map from a topological space X into a topological space Y. A point $x \in X$ and its image $f(x) \in Y$ are called *singular points* of X and Y, respectively, if there is an open set U of X containing x such that $f(U)$ is not a neighbourhood of $f(x)$. Throughout the paper S and T will always denote the sets of singular points of X and Y respectively. The *branch set* B_f of f is the set of all points of X where f fails to be a local homeomorphism. The *multiplicity* of f at x, $N(x, f)$, is the number of points in $f^{-1}f(x)$ if it is finite, $+\infty$ otherwise. The multiplicity of f on X, $N(f)$, is the supremum of $N(x, f)$, $x \in X$.

The sets of real numbers, integers and natural numbers with usual topologies will be denoted by **R**, **Z** and **N** respectively. Interior of a set A in a topological space B will be denoted by $\text{int}_B\, A$.

2. Quotient open extensions. The method discussed in this section for obtaining open extensions of maps was first constructed in [**3**]. For the sake of completeness and to acquaint the reader with notation and terminology, we

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include a brief description of the construction. For more details and properties see [**3**; **5**].

For each singular point x of X, let Y_x be a copy of Y. Let $W = X \oplus (\bigoplus Y_x)$, where the second disjoint topological sum is taken over all singular points of X. By identifying each singular point $x \in X$ (with X throught of as a subset of W) with its image $f(x)$ (as a point of $Y_x \subset W$), we arrive at a quotient space X^* of W. The inclusion map $i_X : X \to W$ composes with the quotient map $q : W \to X^*$, to give an embedding of X into X^*. Hence X^* may be considered as a superspace of X.

Let f_1 from W onto Y be the function whose restriction to X is f and whose restriction to Y_x is the identity map of Y_x onto Y. The unique function $f^* : X^* \to Y$ satisfying $f^* \circ q = f_1$ is an open extension of f. Since f is continuous, the function f^* is continuous.

Throughout the paper symbols X^*, f^*, q will have the same meaning as in this section. We may remind the reader that a Tychonoff space is said to be *complete in the sense of Čech* if it is a G_δ in its Stone-Čech compactification.

2.1. LEMMA. *If X and Y are Tychonoff spaces which are locally complete in the sense of Čech and if singular points of X do not accumulate, then X^* is locally complete in the sense of Čech.*

Proof. Since X and Y are Tychonoff spaces, so is X^* [**3**, 1.3]. Now, let $p \in X^*$ be any point. The following cases arise.

Case I. The set $q^{-1}(p) = \{x, f(x)\}$. Then there are open neighbourhoods U (in X) of x and V (in Y_x) of $f(x)$ which are complete in the sense of Čech. Further, since completeness in the sense of Čech is open hereditary, by hypothesis on singular points U may be so chosen that it contains no other singular point. Then $q(U \cup V)$ is an open neighbourhood of p which is complete in the sense of Čech.

Case II. The set $q^{-1}(p)$ is a singleton. We leave it for the reader to complete the proof in this case.

2.2. LEMMA. *If singular points of X do not accumulate, then* $\text{int}_X B_f = \emptyset$ *implies* $\text{int}_{X^*} B_{f^*} = \emptyset$. *Further, if in addition no singular point is open in X, then* $\text{int}_{X^*} B_{f^*} = \emptyset$ *implies* $\text{int}_X B_f = \emptyset$.

Proof. First we show that $B_{f^*} \subseteq q(B_f)$. Since for each $x \in S$, $q(Y_x)\backslash\{q(x)\}$ is open in X^* and since f^* restricted to each $q(Y_x)\backslash\{q(x)\}$ is topologically equivalent to the identity map on $Y\backslash\{f(x)\}$, $B_{f^*} \subseteq q(X)$. Now, let $p \in B_{f^*}$. Then f^* fails to be locally one-one at p. If $q^{-1}(p) = \{x, f(x)\}$, then $x \in B_f$ and so $p \in q(B_f)$. If $q^{-1}(p)$ is a singleton, then $q^{-1}(p) \in X$ and f fails to be locally one-one at $q^{-1}(p)$, for otherwise by hypothesis on singular points, f^* will be locally one-one at p. So, $q^{-1}(p) \in B_f$ and consequently $p \in q(B_f)$. Thus $B_{f^*} \subseteq q(B_f) \subseteq q(X)$. Now, suppose $\text{int}_X B_f = \emptyset$. To show $\text{int}_{X^*} B_{f^*} = \emptyset$, assume not, and let $V = \text{int}_{X^*} B_{f^*}$. Then $q^{-1}(V) \cap X$ is a nonempty open subset of X which is contained in B_f and thus contradicts the fact that $\mathrm{int}_X B_f = \emptyset$.

Suppose no point of S is open in X. We first show that $q(B_f) \subset B_{f^*}$. Let $x \in B_f$. If f fails to be locally one-one at x, then f^* fails to be locally one-one at $q(x)$ and so $q(x) \in B_{f^*}$. In the other case, if $x \in S$, let U be a neighbourhood of $q(x)$ in X^*. There exist open neighbourhoods U_1 of x in X and V_1 of $f(x)$ in Y_x such that $q(U_1 \cup V_1) \subset U$, and U_1 contains no other singular point. By continuity of f, there is a neighbourhood U_2 of x such that $f(U_2) \subset V_1$ (here V_1 is regarded as a subset of Y). Since $\{x\}$ is not open in X, there is a point $x_1 \in U_1 \cap U_2$ such that $x \neq x_1$. Then $q(x_1)$ and $q(f(x_1))$ (here $f(x_1)$ is considered as a point of Y_x) are distinct points of $q(U_1 \cup V_1)$ and $f^*(q(x_1)) = f^*(q(f(x_1)))$. Thus f^* fails to be one-one on U and hence $q(x) \in B_{f^*}$. Now suppose $\mathrm{int}_{X^*} B_{f^*} = \emptyset$. To show $\mathrm{int}_X B_f = \emptyset$, assume not. Then by hypothesis on the set S, $\mathrm{int}_X B_f \backslash S$ is a nonempty open subset of B_f. Consequently, $q(\mathrm{int}_X B_f \backslash S)$ is a nonempty open subset of B_{f^*}. This contradicts the fact that $\mathrm{int}_{X^*} B_{f^*} = \emptyset$.

2.3. THEOREM. *Let $f : X \to Y$ be a map from a Hausdorff space X into a Hausdorff space Y such that f is open except at finitely many points. If no singular point is open in X and if $N(f) < \infty$, then* $\mathrm{int}_X B_f = \emptyset$.

Proof. Since X and Y are Hausdorff spaces, so is the space X^* [**3**, 1.3]. Since $N(f) < \infty$ and since the set S is finite, $f^* : X^* \to Y$ is an open map such that $N(f^*) < \infty$. By [**11**, Theorem 2.3] $\mathrm{int}_{X^*} B_{f^*} = \emptyset$. The proof is complete in view of Lemma 2.2.

2.4. THEOREM. *Let $f : X \to Y$ be a locally σ-discrete mapping from a Tychonoff space X into a Tychonoff space Y such that singular points do not accumulate. If X and Y are locally complete in the sense of Čech and if no singular point is open in X, then* $\mathrm{int}_X B_f = \emptyset$.

Proof. Since X and Y are Tychonoff spaces which are locally complete in the sense of Čech, by Lemma 2.1. X^* is a Tychonoff space which is locally complete in the sense of Čech. Since f is a locally σ-discrete map, $f^* : X^* \to Y$ is an open locally σ-discrete map. By [**8**, Theorem 1] $\mathrm{int}_{X^*} B_{f^*} = \emptyset$. Since no point of S is open in X, by Lemma 2.2 $\mathrm{int}_X B_f = \emptyset$.

2.5. *Remark.* In [**10**] Väisälä proved that every discrete open mapping on a locally compact Hausdorff space is a local homeomorphism on a dense open subset of the domain. In [**11**] Väisälä, apparently unaware of the work of Pasynkov [**8**], showed that every countable open map between locally compact separable metric spaces is a local homeomorphism on a dense open subset of the domain. Since a locally compact Hausdorff space is complete in the sense of Čech, the above theorem includes improved versions of Väisälä's result.

2.6. THEOREM. *Let $f : X \to Y$ be a finite-to-one map of a Tychonoff space X into a Tychonoff space Y and let Y be complete in the sense of Čech. If f is open* *except at finitely many points, and if no singular point is open in* X, *then* $\operatorname{int}_X B_f = \emptyset$.

Proof. Since X and Y are Tychonoff spaces and since the set S is finite, $f^* : X^* \to Y$ is an open finite-to-one map between Tychonoff spaces. The proof follows immediately in view of [**9**, Theorem 2], and Lemma 2.2.

2.7. *Remark.* In the above theorem, as well as in the original theorem of Proizvolov the hypothesis of completeness can be replaced by any condition which ensures that Y is of second category. In particular, if Y is locally complete in the sense of Čech or locally countably compact, the theorem still holds. Moreover, it is sufficient to require that f be locally finite-to-one instead of finite-to-one.

3. Unified open extensions. In [**4**], there is given a method of unifying the domain and range of a mapping so as to yield a meaningful open extension. Usually, the unified extension is not Hausdorff. However, in some special cases it can be modified so as to be Hausdorff or possesses other separation properties. Here we give a brief description of the construction. For more details and properties see [**4**].

Let W denote the disjoint set theoretic union of X and Y. Then the collection $\mathscr{T}$ of all subsets $Q \subseteq W$, which satisfy the following two conditions (i) and (ii), is a topology for W.

(i) The sets $Q \cap X$ and $Q \cap Y$ are open in X and Y respectively.

(ii) The set $Q \cap S = \emptyset$ or else for each $x \in Q \cap S$, the set $Q \cap Y$ contains a neighbourhood of $f(x)$ in Y.

Hereafter, W is always assumed to be endowed with the topology $\mathscr{T}$. Thus the spaces X and Y are embedded in W as closed and open subspaces respectively. The retraction map $r : W \to Y$ from W onto Y defined by $r(z) = z$ for $z \in Y$ and $r(z) = f(z)$ for $z \in X$ is an open continuous extension of f. If f is an open map, then W coincides with the disjoint topological sum of X and Y. In no other case is W a T_1-space. However, in some special cases it is possible to reduce W to satisfy certain separation axioms and other properties by deleting some of its points. In particular, the following holds.

3.1. THEOREM [**4**]. *Suppose singular points of* X *and* Y *do not accumulate and let* $\tilde{X} = W\backslash T$. *Then* $\tilde{f} = r|\tilde{X}$ *is an open map. Further, if* $f|S$ *is one-to-one, then*

(a) *the space* $\tilde{X}$ *is Hausdorff, whenever* X *and* Y *are Hausdorff;*

(b) *the space* $\tilde{X}$ *is locally compact Hausdorff, whenever* X *and* Y *are locally compact Hausdorff spaces.*

3.2. THEOREM. *Suppose singular points of* X *and* Y *do not accumulate,* $f|S$ *is one-to-one and let* $\tilde{X} = W\backslash T$. *If* X *and* Y *are locally compact separable metrizable spaces, so is the space* $\tilde{X}$.

Proof. By the above theorem $\tilde{X}$ is a locally compact Hausdorff space. The spaces X and Y are separable metrizable and hence second countable. Thus their disjoint topological sum $X \oplus Y$ is second countable. The space W being a continuous image of $X \oplus Y$ possesses a countable net [**1**]. Thus $\tilde{X}$ is a locally compact Hausdorff space with a countable net and hence second countable [**1**]. By Uryshon's metrization theorem $\tilde{X}$ is metrizable.

3.3. *Remark*. If the set S is a singleton, then the two open extensions f^* and $\tilde{f}$ coincide. The converse, that the two open extensions f^* and $\tilde{f}$ coincide implies S is a singleton, is also true.

Throughout the paper, the symbols $\tilde{X}$ and $\tilde{f}$ will have the same meaning as in the above paragraphs.

3.4. LEMMA. *If singular points of X do not accumulate, then* $\text{int}_X B_f = \emptyset$ *implies* $\text{int}_{\tilde{X}} B_{\tilde{f}} = \emptyset$. *Further, if in addition no point of S is open in X, then* $\text{int}_{\tilde{X}} B_{\tilde{f}} = \emptyset$ *implies* $\text{int}_X B_f = \emptyset$.

Proof. Since $Y \backslash T$ is open in X and since $\tilde{f}$ restricted to $Y \backslash T$ is a homeomorphism, $B_{\tilde{f}} \subseteq X$. Let $z \in B_{\tilde{f}}$. Then $\tilde{f}$ is not locally one-to-one at z and $z \in X$. If $z \in S$, then $z \in B_f$. If $z \notin S$, then f fails to be locally one-to-one at z. For otherwise, by hypothesis on singular points, $\tilde{f}$ will be locally one-one at z. Thus $B_{\tilde{f}} \subseteq B_f \subseteq X$ and therefore $\text{int}_X B_f = \emptyset$ implies $\text{int}_{\tilde{X}} B_{\tilde{f}} = \emptyset$.

Now, suppose no point of S is open in X. We first show that $B_f \subseteq B_{\tilde{f}}$. Let $x \in B_f$. Then either f fails to be locally one-one at x or $x \in S$. If f fails to be locally one-one at x, then $\tilde{f}$ also fails to be locally one-one at x and hence $x \in B_{\tilde{f}}$. If $x \in S$, by hypothesis $\{x\}$ is not open in X, $\tilde{f}$ is not local y one-one at x. So, $x \in B_{\tilde{f}}$. Thus $B_f \subseteq B_{\tilde{f}}$.

Now, suppose $\text{int}_{\tilde{X}} B_{\tilde{f}} = \emptyset$. To show $\text{int}_X B_f = \emptyset$, assume the contrapositive. Then by hypothesis on the set S, $\text{int}_X B_f \backslash S$ is nonempty and open. But $\text{int}_X B_f \backslash S$ is open in $\tilde{X}$ and $\text{int}_X B_f \backslash S \subseteq B_{\tilde{f}}$. This contradicts the fact that $\text{int}_{\tilde{X}} B_{\tilde{f}} = \emptyset$.

3.5. THEOREM. *Let $f : X \to Y$ be a map of a Hausdorff space X into a Hausdorff space Y such that singular points of X and Y do not accumulate, and $f|S$ is one-to-one. If $N(f) < \infty$ and if no point of S is open in X, then* $\text{int}_X B_f = \emptyset$.

Proof. Since f satisfies the hypothesis of Theorem 3.1, and since X and Y are Hausdorff spaces, the space $\tilde{X}$ is Hausdorff and $\tilde{f} : \tilde{X} \to Y$ is an open map. Since $N(f) < \infty$, $N(\tilde{f}) < \infty$. By [**11**, Theorem 2.3], $\text{int}_{\tilde{X}} B_{\tilde{f}} = \emptyset$. Since no point of S is open in X, by Lemma 3.4, $\text{int}_X B_f = \emptyset$.

3.6. *Remark*. At a first glance Theorem 3.5 may seem superfluous in view of Theorem 2.3. However, this is not so. In Theorem 2.3 the set S is always assumed to be finite while in Theorem 3.5 it may be infinite. There exist continuous real-valued maps of reals which satisfy the hypothesis of Theorem 3.5 and the set S is infinite (see Example 4.5).

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4. Examples. First we give some simple examples of continuous real-valued maps of reals. These examples will illustrate the facts that (1) maps which are open except at finitely many points occur more frequently than open maps, and (2) the class of maps which are open except at countably many points such that singular points do not accumulate is much larger than the class of maps which are open except at finitely many points. The another set of examples show that the hypotheses in results of Section 2 and 3 are not superfluous. Lastly, we give an example which suggests the cases when the open extensions $f^* : X^* \to Y$ and $\tilde{f} : \tilde{X} \to Y$ are local homeomorphisms.

Functions open except at finitely many points.

4.1. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$ or $f(x) = |x|$. Then f is open everywhere except at the origin. Since translations are homeomorphisms for each $r \in \mathbf{R}$ there exists a continuous $g : \mathbf{R} \to \mathbf{R}$ such that r is the only singular point of g.

4.2. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x$ if $-\infty < x \leqq 1$ and $f(x) = |x - 2|$ if $1 \leqq x < \infty$. Then f is an at most three-to-one map of $\mathbf{R}$ onto $\mathbf{R}$ such that f is open everywhere except at the points $x = 1$ and $x = 2$, i.e., $S = \{1, 2\}$. In fact, if k is any positive integer, then there exists an at most $(k + 1)$-to-one continuous map $h : \mathbf{R} \to \mathbf{R}$ such that the set of singular points of h is precisely $\{1, 2, \ldots, k\}$. Moreover, the functions with these properties may be chosen to be C^∞ maps.

4.3. Every nonconstant polynomial $P : \mathbf{R} \to \mathbf{R}$ is either an open map or is open except at finitely many points.

Functions open except at countably many points.

4.4. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous periodic function of finite period T such that f has at most finitely many singular points in the interval $[0, T]$. Then f is open except at countably many points and singular points do not accumulate.

4.5. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } -\infty < x \leqq 1 \\ -\frac{1}{2}(x - n) + n & \text{if } n \leqq x \leqq n + 1 \text{ and } n \text{ is a positive odd integer} \\ \frac{5}{2}(x - n) + (n - \frac{3}{2}) & \text{if } n \leqq x \leqq n + 1 \text{ and } n \text{ is a positive even integer.} \end{cases}$$

Then f is at most three-to-one continuous map of $\mathbf{R}$ onto $\mathbf{R}$. The set $S = \mathbf{N}$, $T = \{1, 1/2, 3, 5/2, 5, 9/2, 7, \ldots\}$, $f|S$ is one-to-one and is, in fact, defined by

$$f(n) = \begin{cases} n & \text{if } n \text{ is an odd integer} \\ n - \frac{3}{2} & \text{if } n \text{ is an even integer.} \end{cases}$$

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Thus Example 4.5 satisfies the hypothesis of Theorem 3.5 and the set of singular points is infinite.

In Example 4.5 f is not differentiable at every singular point. However, this example may be modified so that f is differentiable and has other desired properties.

General maps.

4.6. If $f : \mathbf{R} \to \mathbf{R}$ is any constant map, then every point is a singular point, i.e., $S = \mathbf{R}$. *There exists no non constant real-valued map of reals such that every point is a singular point.* For, let $f : \mathbf{R} \to \mathbf{R}$ be any non constant map. Then there are points $x, y \in \mathbf{R}$ such that $f(x) \neq f(y)$. For definiteness, assume $f(x) < f(y)$. By the intermediate value theorem, the closed interval $[f[x], f(y)] \subseteq f(\mathbf{R})$. By [**12**, p. 942], there is a countable subset C of $\mathbf{R}$ such that f is open at every point of $\mathbf{R}\backslash f^{-1}(C)$. Since $f(\mathbf{R})$ is uncountable, there exists a point $p \in f(\mathbf{R})$ such that f is open at every point of $f^{-1}(p)$ and so $S \neq \mathbf{R}$.

However, there exists a nonconstant real-valued map of reals such that the set S is dense. For, let $\phi(x) = |x|$ if $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$. Extend ϕ continuously to whole of the real line by defining $\phi(x) = \phi(x + 1)$ for all $x \in \mathbf{R}$. Define

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{4^{n-1}} \phi(4^{n-1}x).$$

Since this series is uniformly convergent, f is continuous on $\mathbf{R}$. It is shown in [**7**, p. 29] that f is nowhere monotonic. That is for any open interval (a, b), there are three points x_1, x_2, x_3 such that $a < x_1 < x_2 < x_3 < b$ and either $f(x_1) < f(x_2)$ and $f(x_2) > f(x_3)$ (or $f(x_1) > f(x_2)$ and $f(x_2) < f(x_3)$). Since f is continuous on the closed interval $[x_1, x_3]$, it assumes maximum (or minimum) value at an interior point x_0 of $[x_1, x_3]$ and hence f is not open at x_0. Thus S is dense in $\mathbf{R}$.

It seems interesting to determine the set S in this example.

4.7. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x + 1 & \text{if } -\infty < x \leqq -1 \\ 0 & \text{if } -1 \leqq x \leqq 1 \\ x - 1 & \text{if } 1 \leqq x < \infty \end{cases}$$

Then f is continuous and $S = B_f = [-1, 1]$. Thus the hypothesis of f being σ-discrete and finite-to-one can not be omitted in Theorems 2.4 and 2.5 respectively.

4.8. In [**11**, p. 542] Väisälä has given an example of a finite-to-one open map between σ-compact separable metric spaces which is nowhere a local homeomorphism. Thus the hypothesis of completeness cannot be omitted in Theorems 2.4 and 2.6; and also the hypothesis that f is of bounded multiplicity cannot be weakened to even finite-to-one in Theorems 2.3 and 3.5.

4.9. Let Y be the real line with its usual topology and let X be Y with integers as an open discrete set. Let f be the identity mapping of X onto Y. Then every integer in X is open and is a singular point. In this case the open extensions f^* and $\tilde{f}$ are local homeomorphisms. So, $B_{f^*} = B_{\tilde{f}} = \emptyset$ but $\text{int}_X B_f = Z \neq \emptyset$. *This example shows that the restriction of a local homeomorphism need not be a local homeomorphism.* Moreover, this example shows that in some special cases the open extensions f^* and $\tilde{f}$ are local homeomorphisms. Precisely, we have the following.

4.10. THEOREM. *Suppose singular points of X do not accumulate and let each singular point of X be open in X. If $f|(X\backslash S)$ is a local homeomorphism, then f^* is a local homeomorphism. Further, if in addition singular points of Y do not accumulate, then $\tilde{f}$ is also a local homeomorphism.*

In [**11**] ,Väisälä proved as a main result that every countable map between n-manifolds, $1 \leqq n \leqq 3$, is a local homeomorphism on a dense open subset of the domain (In a personal communication, Professor P. T. Church informed the author that he could remove the restriction on n in Väisälä's result.) We ask whether in Väisälä's theorem 'countable map' can be replaced by a 'σ-discrete map'.

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REFERENCES

1. A. V. Arhangel'skiǐ, *An addition theorem for the weight of sets lying in bicompacta*, Dokl. Nauk SSSR *126* (1959), 239–241 (Russian).
2. P. S. Alexandroff, *Uber abzahlbar-fache offene Abbildungen*, Dokl. Akad. Nauk SSSR *4* (1936), 295–299.
3. S. P. Franklin and J. K. Kohli, *On open extensions of maps*, Can. J. Math. *22* (1970), 691–696.
4 J. K. Kohli, *A note on open extensions of maps*, Can. J. Math. *24* (1972), 1139–1144.
5. ——— *Finite-to-one maps and open extensions of maps*, Proc. Amer. Math. Soc. *48* (1975), 464–468.
6. A. N. Kolmogoroff, *Points of local topologicity of enumerably folden open mappings*, Dokl. Adad. Nauk SSSR *30* (1941), 479–481.
7. J. M. H. Olmsted, *Counter examples in analysis* (Holden-Day San Francisco, 1964).
8. B. Pasynkov, *On open mappings*, Soviet Math. Dokl. *8* (1967), 853–856.
9. V. V. Proizvolov, *A generalization of Kolmogoroff's theorem on the points of local homeomorphism and its implications*, Soviet Math. Dokl. *8* (1967), 192–194.
10. J. Väisälä, *Discrete open mappings on manifolds*, Ann. Acad. Sci. Fenn. AI *392* (1966), 1–10.
11. ——— *Local topological properties of countable maps*, Duke Math. J. *159* (1974), 541–546.
12. G. T. Whyburn, *On the interiority of real functions*, Bull. Amer. Math. Soc. *48* (1942), 942–946.

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