Compositio Math. 143 (2007) 1613–1622
doi:10.1112/S0010437X07003028

Quasiregular self-mappings of manifolds and word hyperbolic groups

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Dedicated to Seppo Rickman and Jussi Väisälä

Abstract

An extension of a result of Sela shows that if Γ is a torsion-free word hyperbolic group, then the only homomorphisms $\Gamma \to \Gamma$ with finite-index image are the automorphisms. It follows from this result and properties of quasiregular mappings, that if M is a closed Riemannian n-manifold with negative sectional curvature ($n \neq 4$), then every quasiregular mapping $f : M \to M$ is a homeomorphism. In the constant-curvature case the dimension restriction is not necessary and Mostow rigidity implies that f is homotopic to an isometry. This is to be contrasted with the fact that every such manifold admits a non-homeomorphic light open self-mapping. We present similar results for more general quotients of hyperbolic space and quasiregular mappings between them. For instance, we establish that besides covering projections there are no π_1-injective proper quasiregular mappings $f : M \to N$ between hyperbolic 3-manifolds M and N with non-elementary fundamental group.

1. Introduction

Roughly speaking, a quasiregular mapping is a (possibly) branched covering map with bounded distortion. These include, for instance, piecewise linear maps between 'fat' triangulations of manifolds and maps preserving measurable conformal structures. The theory of quasiregular mappings, founded by Resetnyak and Martio–Rickman–Väisälä in the 1970s, seeks to establish the analogue in higher dimensions of the geometric aspects of the theory of analytic and conformal mappings in the plane, see [IM96, Res89, Ric93] and the references therein.

Here we study the existence or otherwise of branched (not locally injective) quasiregular maps between manifolds of negative curvature. We show, as a particular case of a more general result, that a non-trivially-branched quasiregular mapping $f : M \to N$ between closed hyperbolic manifolds can never induce an injection on fundamental groups. We also strengthen the result of [MMP] that the only *uniformly* quasiregular automorphisms of closed hyperbolic manifolds are the obvious ones (i.e. uniformly quasiconformal mappings isotopic to periodic isometries). In the present article we prove that such manifolds admit no non-obvious quasiregular self-mappings at all: in fact we show that there are no discrete open self-maps whatsoever which are not homeomorphisms isotopic to an isometry. We also prove a number of related results, including an extension of the above theorem to convex co-compact manifolds and a generalization concerning open mappings between closed

Received 22 September 2006, accepted in final form 21 February 2007, published online 9 November 2007.
2000 Mathematics Subject Classification 20F67, 30C65, 53A35.
Keywords: hyperbolic group, Hopfian, quasiregular, hyperbolic manifold, open mapping.
This material is based upon work supported by the National Science Foundation under Grants No. 0200752 and 0457291, the N.Z. Marsden Fund and the NZIMA. The first author is supported by a Royal Society Wolfson Merit Award. He was also the Forder Lecturer during the completion of this work and would like to thank the London Mathematical Society and the New Zealand Mathematical Society for their support.

negatively curved manifolds of dimension $n \neq 4$. We further discuss the case of complete finite-volume hyperbolic manifolds. The proofs of these latter results rely on an analysis of the self-maps of word hyperbolic and relatively hyperbolic groups which is of independent interest.

1.1 Context

To set the context, we recall the content of [MMP]. We use [Ric93] as a basic reference to the theory of quasiregular mappings.

The Lichnerowicz conjecture [Lic64], formulated around 1964, asserts that the only compact Riemannian n-manifold which admits a non-compact conformal automorphism group is the n-sphere. This was proved in 1973 by Lelong-Ferrand [Lel73]. Her argument applies beyond the smooth setting: the only n-manifold which admits a non-compact group G of uniformly quasiconformal homeomorphisms is (up to quasiconformal equivalence) the n-sphere. This statement subsumes the Lichnerowicz conjecture because any uniformly quasiconformal group G admits a bounded measurable conformal structure in which G acts conformally [Gro81, Tuk86]. (Of course, it only makes sense to talk of quasiconformal homeomorphisms when one has a quasiconformal structure on the manifold, but Sullivan proved that, with the possible exception of $n = 4$, every n-manifold admits such a structure (see [TV81]).)

In the light of the Lelong-Ferrand theorem, it is natural to ask which *non-injective* mappings preserve some bounded measurable conformal structure on a closed n-manifold; such maps are called *rational*. The iterates of a rational map will form a non-compact semigroup of *uniformly quasiregular mappings* (more briefly, *uqr*-maps). There are a number of different types of such mappings in the literature [IM96, May97, Pel99] and a focus of study has been investigating the analogies between the dynamics associated with iterating these maps and the classical Fatou–Julia theory of iteration of rational maps of $\hat{\mathbb{C}}$ (see [HMM04, IM96, MM03]). In [MMP] the authors studied the following.

The Lichnerowicz Problem. Classify those compact n-manifolds, $n \geqslant 3$, that admit a non-injective rational map.

It was shown in [MMP] that the only n-manifolds which admit an unbranched (that is locally injective) but not globally injective *uqr*-map are those quasiconformally homeomorphic to the n-dimensional euclidean space forms. Further, each such map is quasiconformally conjugate to a conformal map. It was also proved in [MMP] that there are no branched quasiregular self-maps of euclidean space forms. Hyperbolic space forms admit no *uqr*-maps either, but to see this one must appeal to two deep facts. First, any manifold that admits a *uqr*-map is *qr*-elliptic, that is, there is a non-constant quasiregular mapping $\mathbb{R}^n \to M^n$; this is proved using non-compactness and a version of the Zalcman rescaling lemma. Secondly, a difficult result of Varopolous *et al.* [VSC92], answering a question of Gromov [Gro81], shows that the fundamental group of a *qr*-elliptic manifold has polynomial word growth. The proof is completed by the elementary observation that the fundamental group of a hyperbolic manifold must have exponential word growth.

Motivated by this discussion, we wish to show that a closed hyperbolic manifold admits no non-trivial quasiregular self-maps whatsoever.

2. Proper open surjections and π_1

The following lemma of Walsh [Wal76, (4.1)] and Smale [Sma57] will turn out to be quite important in what follows and we give a proof of it as it pertains to our study. (Walsh and Smale were working in greater generality but the proof is the same.) Recall that a map is *proper* if the preimage of a compact set is compact.

LEMMA 2.1. *Let M_1 and M_2 be connected manifolds (possibly with boundary). If a map $f : M_1 \to M_2$ is proper, open and surjective, then the index of $f_*\pi_1 M_1$ in $\pi_1 M_2$ is finite.*

Proof. Let $\tilde{f} : M_1 \to \tilde{M}_2$ be a lift of f to the connected cover $p : \tilde{M}_2 \to M_2$ corresponding to $f_*\pi_1 M_1$. We claim that $\tilde{f}$ is surjective. Since f is open and $\tilde{M}_2$ is connected, it is enough to show that the image of $\tilde{f}$ is closed. Suppose that $y \in \tilde{M}_2$ is in the complement of the image and fix a compact neighbourhood U of $p(y)$ such that $p^{-1}(U)$ is a disjoint union of copies U_i of U with $p|_{U_i}$ a homeomorphism to U; let U_0 be the copy containing y. Since $f^{-1}(U)$ is compact, $\tilde{f}(f^{-1}(U))$ is closed and $U_0 \setminus \tilde{f}(f^{-1}(U))$ is a neighbourhood of y.

For $x \in M_2$, we have that $f^{-1}(x) = \tilde{f}^{-1}(p^{-1}(x))$ is compact. Thus, since $\tilde{f}$ is surjective, $p^{-1}(x)$ is compact, hence finite. □

The same argument applies with minor modification in the case that M_i are orbifolds, but we only use this in a minor way in the next section. Further it is also noted in [Sma57, Theorem 3] that under the hypotheses of the lemma the map f induces a surjection on rational homology. Finally, to lend context to what follows, we note a main result of Walsh's paper [Wal76, Corollary 5.15.3], which is essentially a converse to Lemma 2.1.

THEOREM 2.2. *If M and N are compact connected PL manifolds and $f : M \to N$ a map with $|f_*\pi_1 M : \pi_1 N| < \infty$, then f is homotopic to a light open mapping.*

Here a *light* mapping is one for which the preimage of every point is totally disconnected, for instance a Cantor set.

3. Quasiregular mappings between hyperbolic manifolds

The two basic topological properties of quasiregular mappings are that they are open and discrete (the preimage of any point is a discrete set). Thus a proper quasiregular mapping is open and has the property that the preimage of any point is a finite set. The branch set of a quasiregular mapping f, denoted B_f, is the closed set of points where f fails to be a local homeomorphism.

Chernavskiĭ [Cer64] and Väisälä [Vai66] proved a theorem that is key to what we wish to do: they proved that a discrete open mapping $f : M \to M$ of an n-manifold has the dimension of its branch set less than or equal to $n-2$, $\dim(B_f) \leqslant n-2$, and further $\dim(f(B_f)) \leqslant n-2$. As a consequence of the Hurewicz and Wallman theorems [HW41], the set B_f does not locally separate M at any point. Thus, $f|M \setminus f^{-1}(f(B_f)) \to M \setminus f(B_f)$ is a covering map, where $M \setminus f^{-1}(f(B_f))$ itself is a connected open manifold, dense in M. Moreover, for each $y \in M \setminus f(B_f)$ the set $f^{-1}(y)$ must contain exactly the same number d (the *degree*) points.

If $\Gamma \subset \mathrm{Isom}^+(\mathbb{H}^n)$ is a discrete non-elementary (*Kleinian*) group [Bea83], we denote its limit set by $\Lambda(\Gamma) \subset \partial\mathbb{H}^n = \overline{\mathbb{R}}^{n-1}$. The orbit space of a Kleinian group Γ is $\mathbb{H}^n/\Gamma$ a hyperbolic orbifold (manifold, should Γ be torsion-free).

In what follows dim refers to the topological dimension while $\dim_H$ refers to the Hausdorff dimension. Also note that we use the term hyperbolic manifold to mean a hyperbolic manifold whose fundamental group is non-elementary (equivalently not virtually abelian).

THEOREM 3.1. *For $i = 1, 2$, let M_i be a hyperbolic n-orbifold with fundamental group Γ_i and limit set Λ_i. Let $f : M_1 \to M_2$ be a proper quasiregular mapping such that $f_* : \Gamma_1 \to \Gamma_2$ has finite kernel. Suppose that one of the following four conditions is satisfied:*

(i) $\dim(\Lambda_1) \geqslant n-2$;

(ii) $\dim(\Lambda_2) \geqslant n-2$;

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(iii) $\dim_H(\Lambda_1) = n-1$;

(iv) $\dim_H(\Lambda_2) = n-1$.

Then f is a finite-sheeted covering map whose lift to $\mathbb{H}^n$ is a quasiconformal homeomorphism with $f(\Lambda_1) = \Lambda_2$.

Proof. The map f has a lift to the universal covering space $\hat{f} : \mathbb{H}^n \to \mathbb{H}^n$ automorphic with respect to the groups Γ_i,

$$\hat{f} \circ \gamma = \phi(\gamma) \circ \hat{f} : \mathbb{H}^n \to \mathbb{H}^n \tag{1}$$

where $\phi = f_*$. The map $\hat{f}$ is quasiregular since the projections $\mathbb{H}^n \to M_i$ are conformal. We claim that (for topological reasons alone)

$$x_n \to \partial\mathbb{H}^n \Rightarrow \hat{f}(x_n) \to \partial\mathbb{H}^n. \tag{2}$$

To see this we argue by contradiction, assuming that after passing to a subsequence $\hat{f}(x_n) \to y \in \mathbb{H}^n$ with the x_n all distinct. Writing p_i for the covering projection $\mathbb{H}^n \to M_i$, we would then have $f(p_1(x_n)) = p_2(\hat{f}(x_n)) \to p_2(y) \in M_2$. Since f is proper this would force the sequence $p_1(x_n)$ to lie in a compact subset of M_1 and hence have a subsequence that converged to $z \in M_1$ with $f(z) = p_2(y)$. Choose $x \in p_1^{-1}(z)$ so that $\hat{f}(x) = y$. Then $\gamma_n(x_n) \to x$ for a sequence of distinct $\gamma_n \in \Gamma_1$. Now $(\hat{f} \circ \gamma_n)(x_n) \to y$. Since $\hat{f}(x_n) \to y$, the automorphic relation $\hat{f} \circ \gamma_n = \phi(\gamma_n) \circ \hat{f}$ implies that $\phi(\gamma_n)(y) \to y$. As ϕ has finite kernel, the set $\{\phi(\gamma_n)\}$ is infinite. This provides the desired contradiction, since Γ_2 is assumed to be discrete.

Next a theorem of Martio and Rickman [MR72, Theorem 5.2, p. 10] (see also [Ric93, Sre73]) allows us to deduce from (2) that $\hat{f}$ extends continuously to $\partial\mathbb{H}^n$ and then, by reflection, extends to a quasiregular map $\overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$. The continuity of the extension guarantees that the automorphic relation (1) holds on $\partial\mathbb{H}^n$.

The branch set of $\hat{f}$, $B_{\hat{f}}$, is closed and, if it is non-empty, (1) implies it clusters to the limit set of Γ_1, so $\Lambda_1 \subset B_{\hat{f}}$. Let $g = \hat{f}|\partial\mathbb{H}^n$. Clearly $B_g = B_f \cap \partial\mathbb{H}^n$ and so

$$\Lambda_1 \subset B_g.$$

Note too that g is a quasiregular map of $\partial\mathbb{H}^n = \overline{\mathbb{R}}^{n-1}$ and so is finite to one. If $\dim(\Lambda_1) \geqslant n-2$, then the co-dimension of B_g in $\partial\mathbb{H}^n = \overline{\mathbb{R}}^{n-1}$ is no more than one and this contradicts the fact that the co-dimension of B_g is at least two. Thus condition (i) implies $B_g = \emptyset$ which in turn implies $B_{\hat{f}} = \emptyset$, $\hat{f}$ is a local homeomorphism of $\overline{\mathbb{R}}^n$ (after reflection) and so a homeomorphism.

We next show that

$$\Lambda_2 = g(\Lambda_1) \subset g(B_g) \subset \hat{f}(B_{\hat{f}}).$$

From Walsh's Lemma 2.1 we see that $H = \phi(\Gamma_1) \subset \Gamma_2$ has finite index. Thus,

$$\Lambda(H) = \Lambda(\Gamma_2).$$

Let $x_0 \in \mathbb{H}^n$ and $x \in \Lambda_1$. Then there is an infinite sequence $\gamma_n \in \Gamma_1$ with $\gamma_n(x_0) \to x$ and by continuity (on $\overline{\mathbb{R}}^n$) and the automorphic relation we have

$$\phi(\gamma_n)(\hat{f}(x_0)) \to \hat{f}(x)$$

and, again, as ϕ has finite kernel and H is discrete, orbits can only accumulate on the limit set. Thus, $\hat{f}(x) \in \Lambda(H) = \Lambda_2$ and $g(\Lambda_1) \subset \Lambda_2$. The converse containment is similar. If $y \in \Lambda_2$, then there are $h_n \in H$ infinite and distinct with $h_n \to y$. As $H = \mathrm{Im}(\phi)$ there are γ_n with $\phi(\gamma_n) = h_n$ and the γ_n are infinite and so forth. Then if condition (ii) holds $\dim(g(B_g)) \geqslant \dim(\Lambda_2) \geqslant n-2$ and so $g(B_g)$ has co-dimension at most one in $\overline{\mathbb{R}}^{n-1}$, which is a contradiction unless $B_g = \emptyset$, as before.

Finally, the finite to one quasiregular self-mapping g of the sphere $\overline{\mathbb{R}}^{n-1}$ preserves sets of maximal Hausdorff dimension $n-1$ (see [Ric93]). Thus, since $g(\Lambda_1) = \Lambda_2 \subset g(B_g)$ under either condition (iii) or (iv), we see that $\dim_H(\Lambda_2) = n-1$ and so

$$\dim_H(g(B_g)) = n-1.$$

However, it is known [Ric93] that for such a quasiregular mapping $\dim_H(g(B_g)) < n-1$. This again will force $B_g = \emptyset$ and $\hat{f}$ to be a homeomorphism. □

If the hyperbolic manifolds M_i are closed, then the hypothesis that f is proper is redundant and all four conditions are satisfied. Thus, we have the following.

COROLLARY 3.2. *Let M and N be closed hyperbolic n-manifolds. Then there is no π_1-injective branched quasiregular mapping $f : M \to N$.*

Note next that the above dimension hypothesis on the limit sets is satisfied if one of them separates $\mathbb{H}^n$. This will be the case if, for instance, one of the manifolds has more than one boundary component (other than cusps).

COROLLARY 3.3. *Let M and N be convex co-compact hyperbolic n-manifolds one of which has more than one boundary component. Then there is no proper π_1-injective branched quasiregular mapping $f : M \to N$.*

One may wonder whether, given any hyperbolic manifolds M and N, there can exist proper branched π_1-injective quasiregular mappings $M \to N$. We prove that in dimension three there are none.

THEOREM 3.4. *Let M and N be hyperbolic 3-manifolds with non-elementary fundamental group. Then there is no proper π_1-injective branched quasiregular mapping $f : M \to N$.*

Proof. Let $\Gamma_M \cong \pi_1(M)$ be the non-elementary Kleinian group with $\mathbb{H}^3/\Gamma_M = M$. Supposing that there is such a map f we argue as in the proof of Theorem 3.1 to produce a map $g : \partial\mathbb{H}^3 \to \partial\mathbb{H}^3$ with $B_g \supset \Lambda_M$. We identify $\hat{\mathbb{C}} = \partial\mathbb{H}^3$. In two dimensions rather more is known about the topological structure of quasiregular maps. In particular, the Stoilow factorization theorem (see the survey [Sto98]) asserts that $g = h \circ R$ where $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a homeomorphism and $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is an analytic rational endomorphism. In particular, B_R and hence B_g are finite point sets. This implies that Λ_M is finite (and therefore contains at most two points) and this contradicts our hypothesis that Γ_M is non-elementary. □

Specific applications of the conditions on the Hausdorff dimension of the limit sets can be found when considering geometrically infinite Kleinian groups [BJ97, Tuk84].

It is clear from Theorem 3.1 that we will have to address the question of when a quasiregular map, or more generally an open map, between hyperbolic manifolds induces an injection on fundamental groups.

4. Endomorphisms of hyperbolic groups

The following theorem was proved by Sela [Sel99, Theorem 3.9] (see also [Sel97, Theorem 1.12]) in the course of his work on the Hopf property for word hyperbolic groups [Gro87].

THEOREM 4.1. *Let Γ be a torsion-free hyperbolic group and let $\phi : \Gamma \to \Gamma$ be a homomorphism. Then there is an integer k_0, so that $\ker(\phi^n) = \ker(\phi^{k_0})$ for every $n > k_0$.*

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Sela also proved that torsion-free, freely indecomposable, non-elementary word hyperbolic groups are co-Hopfian [Sel97]. In particular, co-compact lattices of n-dimensional hyperbolic space are co-Hopfian. Groves recently extended Sela's results from the hyperbolic setting to toral relatively hyperbolic groups [Gro05]. Every geometrically-finite subgroup of $\mathrm{SO}(n,1)$ has a subgroup of finite index that lies in this class.

In the current setting we require a variation on what they proved: we must allow freely-decomposable groups, but we need only constrain homomorphisms whose image is of finite index. To this end, we focus on the following invariant of a finitely generated group: the maximum number of non-trivial factors in a free-product decomposition of the group.

Lemma 4.2. *If a finitely generated torsion-free group Γ can be expressed as a free product $\Gamma = A * B$ with A and B non-trivial, then there does not exist an injective homomorphism $\phi : \Gamma \to \Gamma$ with $1 < [\Gamma : \phi(\Gamma)] < \infty$.*

Proof. Recall that the *algebraic rank* of an abstract group Γ is the minimum cardinality among all generating sets for Γ. It follows from the Grushko–Neumann theorem that if G_1 and G_2 are finitely generated groups, then the rank of $G_1 * G_2$ is the sum of the ranks of G_1 and G_2 [Ser80]. Thus, there is an upper bound $n := \lambda(\Gamma)$ on the number of factors in any free decomposition $\Gamma = A_1 * \cdots * A_n$ with the A_i non-trivial. Assume that the displayed decomposition for Γ is maximal in this sense and note that since Γ is torsion-free, the A_i are infinite.

We will be done if we can prove that $\lambda(H) > n$ for every proper subgroup $H \subset \Gamma$ of finite index.

To form a space Y with fundamental group Γ, we take for $i = 1, \dots, n$, a connected space X_i with basepoint v_i and $\pi_1(X_i, v_i) = A_i$, and we identify the v_i to a single vertex v. Let $\widetilde{Y}$ denote the universal covering of Y, let $\overline{Y} = \widetilde{Y}/H$ and consider the covering $p : \overline{Y} \to Y$. The Seifert–van Kampen theorem expresses $H = \pi_1 \overline{Y}$ as the free product of the fundamental groups of the connected components C_{ij} of the $p^{-1}X_i$ together with a finitely-generated free group F which is the fundamental group of the graph $\mathcal{G}$ that has a vertex e_{ij} in each C_{ij} and m edges joining e_{ij} to e_{lk} if $|C_{ij} \cap C_{lk}| = m$.

Each $\pi_1 C_{ij}$ is isomorphic to a subgroup of finite index in A_i and hence is infinite. If for each i, there is only one component in $p^{-1}X_i$ then $\mathcal{G}$ has no vertices of valence 1 and hence F has positive rank. Thus, in all cases, $\lambda(H) > n$. □

Theorem 4.3. *If Γ is a torsion-free non-elementary hyperbolic group, then there is no homomorphism $\phi : \Gamma \to \Gamma$ with $1 < [\Gamma : \phi(\Gamma)] < \infty$.*

Proof. By Lemma 4.2 we may assume that Γ is freely indecomposable and hence co-Hopfian [Sel97]. So if a homomorphism $\phi : \Gamma \to \Gamma$ whose image had finite index greater than one were to exist, then $\ker(\phi) \neq \{1\}$. Let k_0 be the number given by Sela's Theorem 4.1 and set $\Gamma_0 = \phi^{k_0}(\Gamma)$. Then $[\Gamma : \Gamma_0] < \infty$ and the restriction of ϕ to Γ_0 is injective. Given $\gamma \in \Gamma \setminus \{1\}$, as Γ is torsion-free, $\gamma^m \in \Gamma_0 \setminus \{1\}$ for some $m > 0$. Thus, $1 \neq \phi(\gamma^m) = \phi(\gamma)^m$, so $\phi(\gamma) \neq 1$, contradicting the fact that $\ker(\phi) \neq \{1\}$. □

For residually finite groups (such as subgroups of $\mathrm{SO}(n,1)$, our main interest) one may also deduce Theorem 4.3 from Theorem 4.1 by using the following generalization of Malcev's famous observation that finitely generated residually finite groups are Hopfian.

Proposition 4.4. *If Γ is finitely-generated, torsion-free and residually finite, and $\phi : \Gamma \to \Gamma$ is a homomorphism for which $[\Gamma : \phi^k(\Gamma)]$ remains bounded as $k \to \infty$, then ϕ is an isomorphism.*

Proof. It is clear that if ϕ is injective then it must be an isomorphism, so it is enough to derive a contradiction from the assumption that $\ker(\phi)$ is non-trivial (hence, infinite since Γ is torsion-free).

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To this end, we consider an element $\gamma \in \ker(\phi)\backslash\{1\}$ that lies in the finite-index subgroup $\bigcap_{k\in\mathbb{N}} \phi^k(\Gamma)$. By residual finiteness, there is a finite quotient $\pi : \Gamma \to Q$ with $\pi(\gamma) \neq 1$. Since $\gamma \in \phi^k(\Gamma)$, for all $k \in \mathbb{N}$ there is some $\gamma_k \in \Gamma$ such that $\pi \circ \phi^n(\gamma_k) = 1$ if $n > k$ and $\pi \circ \phi^k(\gamma_k) = \pi(\gamma) \neq 1$. Thus, we obtain infinitely many distinct maps $\pi \circ \phi^n$ from the finitely generated group Γ to the finite group Q, which is absurd. □

5. Topological rigidity results

The following is an immediate consequence of Theorem 4.3 and Lemma 2.1.

Theorem 5.1. *If M is a connected manifold whose fundamental group is torsion-free, non-elementary and word hyperbolic, then every proper open surjective mapping $f : M \to M$ induces an isomorphism of the fundamental group.*

Remark 5.2. We stated Theorem 5.1 only for manifolds because it is manifolds that are the theme of this section. However, the proof of Walsh's lemma applies in far greater generality and hence the above theorem can be generalized enormously: it suffices to assume that M is a locally-finite cell complex, for example.

If M is closed, then the hypotheses that f is surjective and proper are redundant.

Corollary 5.3. *If M is a closed n-manifold whose fundamental group is torsion-free, non-elementary and word hyperbolic, then every quasiregular mapping $f : M \to M$ is a homeomorphism.*

Proof. Every quasiregular map $f : M \to M$ is open discrete and so finite to one. As above, the induced map on fundamental group is an isomorphism. Thus, f has degree one, $B(f) = \emptyset$ and f is a covering map by [Vai66]. Thus, f is a homeomorphism. □

The Farrell and Jones [FJ89] topological rigidity theorem for non-positively curved manifolds tells us that closed negatively curved manifolds of dimension $n \geqslant 5$ are homeomorphic if their fundamental groups are isomorphic. Perelman's proof of the geometrization conjecture [Per03a, Per02, Per03b] implies that the same result is true in dimension three. If the curvature is strictly negative, the fundamental group of such a manifold is word hyperbolic. Thus, Theorem 5.1 implies the following.

Theorem 5.4. *Let M_1 and M_2 be closed Riemannian n-manifolds ($n \neq 4$) of negative sectional curvature. Suppose there are open maps $f : M_1 \to M_2$ and $g : M_2 \to M_1$. Then M_1 is homeomorphic to M_2.*

Proof. By Theorem 5.1, the compositions $f\circ g : M_2 \to M_2$ and $g\circ f : M_1 \to M_1$ induce isomorphisms on π_1. □

Sela noted a version of this result for degree one maps. Ian Agol (personal communication) has suggested an alternative proof in the hyperbolic case based on the Gromov norm.

The results of Farrell–Jones and Perelman also yield the following refinement.

Theorem 5.5. *If M is a closed n-manifold of negative sectional curvature ($n \neq 4$), then every open mapping $f : M \to M$ is homotopic to a homeomorphism.*

6. Quasiregular maps and rigidity of hyperbolic manifolds

We combine Theorem 3.1 with the results of the sort in the last section to prove the results that are the main focus of this article.

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Theorem 6.1. *If M is a convex co-compact hyperbolic n-manifold, then every proper, quasiregular mapping $f : M \to M$ is a homeomorphism.*

Proof. Once again Lemma 2.1 tells us that $f_*(\pi_1 M) \subset \pi_1 M$ is of finite index. Theorem 4.3 then tells us that f_* is an isomorphism, and Theorem 3.1 tells us that f is a homeomorphism. □

We combine the above result with Mostow rigidity to obtain the first item in the following corollary, and include the Euclidean case for comparison.

Corollary 6.2. *Let M be a space form and $f : M \to M$ quasiregular.*

- *If M is hyperbolic, then f is quasiconformal and homotopic to an isometry of finite period.*
- *If M is euclidean, then f is quasiconformally conjugate to a multiplication or f is quasiconformal (i.e. injective).*

In the spherical case, above dimension one, a locally injective map must be injective.

Our results combine to give the following generalization of the Mostow rigidity theorem [Mos68].

Theorem 6.3. *Let M_1 and M_2 be closed hyperbolic n-manifolds. Suppose that there is an open mapping $f : M_1 \to M_2$ and an injection $\phi : \pi_1(M_2) \to \pi_1(M_1)$ with $[\pi_1(M_1) : \phi(\pi_1(M_2))] < \infty$. Then f is homotopic to an isometry.*

One can extend our results for convex co-compact subgroups of $\mathrm{SO}(n, 1)$ to all geometrically finite torsion-free lattices by using the work of Groves [Gro05] in place of Sela's theorem. Walsh's lemma still applies in this context, but one needs an adaptation of Theorem 3.1, which we do not present here. The conclusion is that every quasiregular self-mapping of a finite volume hyperbolic n-manifold is isotopic to an isometry. There are interesting applications for knot groups which we shall report on elsewhere.

Finally we want to make the following observation relating what we have above with Wilson's counterexample to the Whyburn conjecture [Wil73] giving a dichotomy between discrete open and light open mappings of hyperbolic manifolds.

Theorem 6.4. *Let M be a closed hyperbolic n-manifold. Then any discrete open mapping $f : M \to M$ is a homeomorphism isotopic to an isometry. However, there are light open mappings $g : M \to M$ which are not homeomorphisms.*

Proof. If $f : M \to M$ is discrete and open, then f_* is an isomorphism on π_1 and a homeomorphism (as above). Another way to see this is to use Mostow rigidity: lift f to the universal cover $\mathbb{H}^n$, this lifted map is continuous, open and discrete. Using π_1 injectivity as we did earlier, the lifted map extends continuously to $\overline{\mathbb{H}^n}$ and the boundary values are those of an isometry: by rigidity. However, the Whyburn conjecture, known for discrete open maps of the closed ball by [Cer64, Vai66], says that a discrete open mapping of the closed ball which restricts to a homeomorphism on the boundary is a homeomorphism on the interior as well. Thus, f is a covering map of degree one and a homeomorphism. However, the Whyburn conjecture is false in the case of light open mappings and as a consequence it is shown in [Wil73] that every n-manifold M admits a non-homeomorphic light open mapping $f : M \to M$. Wilson's construction actually gives a mapping homotopic to the identity: we already know it must be homotopic to an isometry as f_* induces an isomorphism. □

The above arguments extend to any closed negatively curved n-manifold. The fundamental group acts on the visual boundary of the universal cover as a convergence group [GM87] with every limit point a point of approximation (or conical limit point), and from this one can deduce that the lift of f induces a homeomorphism of the boundary at infinity (cf. [MT88, MT92]), knowing that f induces an isomorphism on π_1.

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